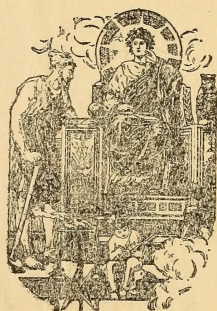


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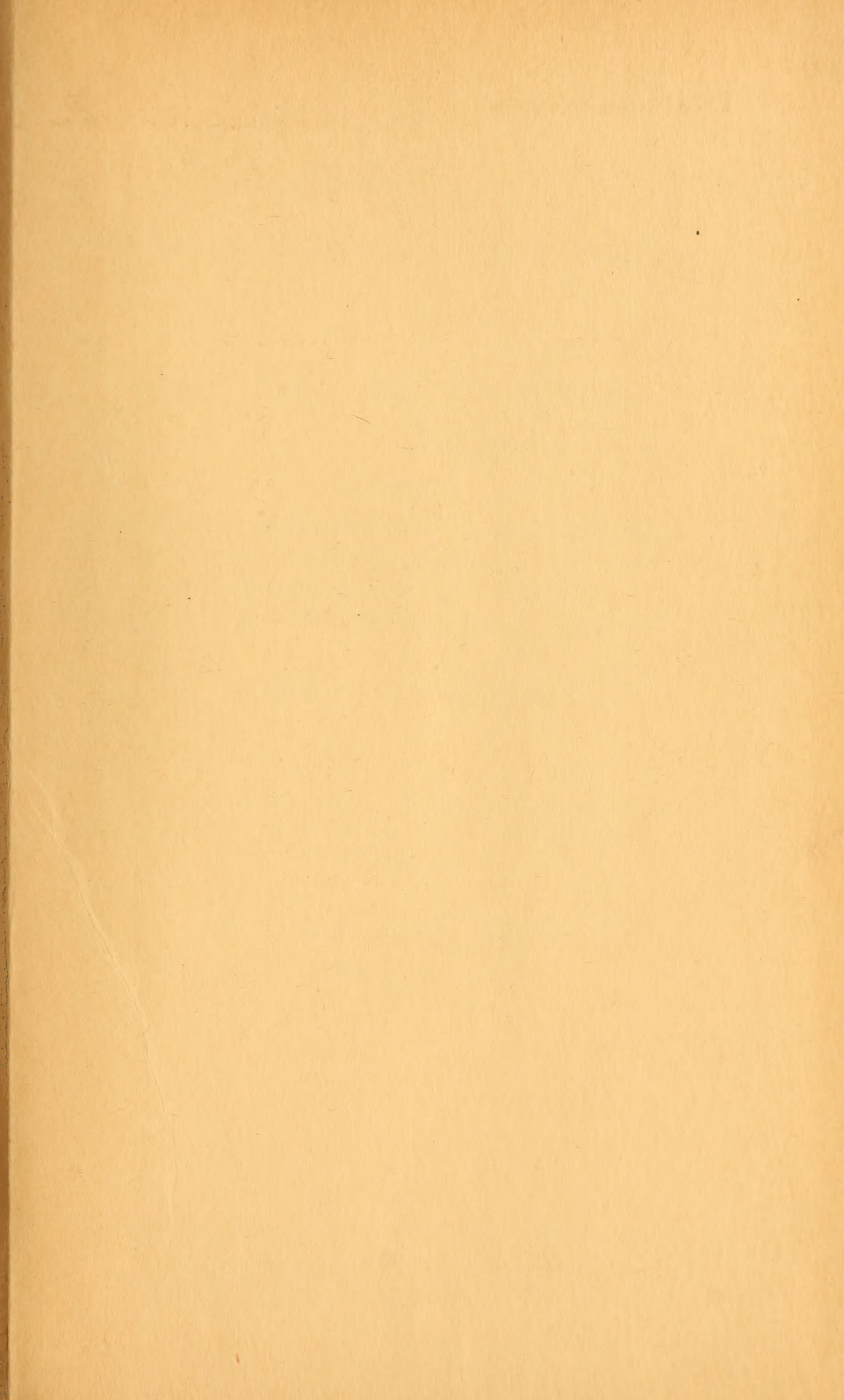
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"Nec araneorum sane textus ideo melior quia ex se fila gignunt, nec noster
vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit. lib. i. cap. l. Not.*

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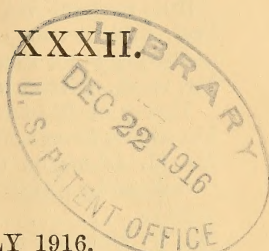
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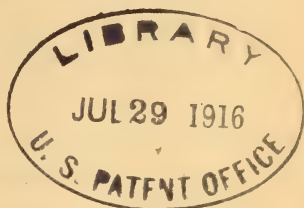
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[SIXTH SERIES.]

JULY 1916.



I. *On the Flow of Compressible Fluid past an Obstacle.*
By Lord RAYLEIGH, O.M., F.R.S.*

IT is well known that according to classical Hydrodynamics a steady stream of frictionless incompressible fluid exercises no resultant force upon an obstacle, such as a rigid sphere, immersed in it. The development of a "resistance" is usually attributed to viscosity, or when there is a sharp edge to the negative pressure which may accompany it (Helmholtz). In either case it would seem that resistance involves something of the nature of a *wake*, extending behind the obstacle to an infinite distance. When the system of disturbed velocities, although it may mathematically extend to infinity, remains as it were attached to the obstacle, there can be no resistance.

The absence of resistance is asserted for an *incompressible* fluid; but it can hardly be supposed that a small degree of compressibility, as in water, would affect the conclusion. On the other hand, high relative velocities, exceeding that of sound in the fluid, must entirely alter the conditions. It seems worth while to examine this question more closely, especially as the first effects of compressibility are amenable to mathematical treatment.

The equation of continuity for a compressible fluid in

* Communicated by the Author.

steady motion is in the usual notation

$$u \frac{d\rho}{dx} + v \frac{d\rho}{dy} + w \frac{d\rho}{dz} + \rho \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) = 0, \quad (1)$$

or, if there be a velocity-potential ϕ ,

$$\frac{d\phi}{dx} \frac{d \log \rho}{dx} + \frac{d\phi}{dy} \frac{d \log \rho}{dy} + \frac{d\phi}{dz} \frac{d \log \rho}{dz} + \nabla^2 \phi = 0. \quad (2)$$

In most cases we may regard the pressure p as a given function of the density ρ , dependent upon the nature of the fluid. The simplest is that of Boyle's law where $p = a^2 \rho$, a being the velocity of sound. The general equation

$$\int \frac{dp}{\rho} = C - \frac{1}{2} q^2, \quad (3)$$

where q is the resultant velocity, so that

$$q^2 = (d\phi/dx)^2 + (d\phi/dy)^2 + (d\phi/dz)^2 \quad (4)$$

reduces in this case to

$$a^2 \log \rho = C - \frac{1}{2} q^2,$$

or

$$a^2 \log (\rho/\rho_0) = -\frac{1}{2} q^2, \quad (5)$$

if ρ_0 correspond to $q=0$. From (2) and (5) we get

$$\nabla^2 \phi = \frac{1}{2a^2} \left\{ \frac{d\phi}{dx} \frac{dq^2}{dx} + \frac{d\phi}{dy} \frac{dq^2}{dy} + \frac{d\phi}{dz} \frac{dq^2}{dz} \right\}. \quad (6)$$

When q^2 is small in comparison with a^2 , this equation may be employed to estimate the effects of compressibility. Taking a known solution for an incompressible fluid, we calculate the value of the right-hand member and by integration obtain a second approximation to the solution in the actual case. The operation may be repeated, and if the integrations can be effected, we obtain a solution in series proceeding by descending powers of a^2 . It may be presumed that this series will be convergent so long as q^2 is less than a^2 .

There is no difficulty in the first steps for obstacles in the form of spheres or cylinders, and I will detail especially the treatment in the latter case. If U , parallel to $\theta=0$, denote the uniform velocity of the stream at a distance, the velocity-potential for the motion of incompressible fluid is known to be

$$\phi = U(r + c^2/r) \cos \theta, \quad (7)$$

the origin of polar coordinates (r, θ) being at the centre

of the cylinder. At the surface of the cylinder $r=c$, $d\phi/dr=0$, for all values of θ .

On the right hand of (6)

$$\frac{d\phi}{dx} \frac{dq^2}{dx} + \frac{d\phi}{dy} \frac{dq^2}{dy} = \frac{d\phi}{dr} \frac{dq^2}{dr} + \frac{1}{r^2} \frac{d\phi}{d\theta} \frac{dq^2}{d\theta}; \quad \cdot \quad \cdot \quad (8)$$

and from (7)

$$\frac{q^2}{U^2} = \frac{1}{U^2} \left\{ \left(\frac{d\phi}{dr} \right)^2 + \frac{1}{r^2} \left(\frac{d\phi}{d\theta} \right)^2 \right\} = 1 + \frac{c^4}{r^4} - \frac{2c^2}{r^2} \cos 2\theta. \quad \cdot \quad (9)$$

Also

$$\frac{1}{U} \frac{d\phi}{dr} = \left(1 - \frac{c^2}{r^2} \right) \cos \theta, \quad \frac{1}{U} \frac{d\phi}{r d\theta} = - \left(1 + \frac{c^2}{r^2} \right) \sin \theta;$$

$$\frac{1}{U^2} \frac{dq^2}{dr} = - \frac{4c^4}{r^5} + \frac{4c^2}{r^3} \cos 2\theta, \quad \frac{1}{U^2} \frac{dq^2}{r d\theta} = \frac{4c^2}{r^3} \sin 2\theta.$$

Accordingly

$$\nabla^2 \phi = \frac{2U^3 c^2}{a^2 r^3} \left\{ - \frac{c^2}{r^2} \left(2 - \frac{c^2}{r^2} \right) \cos \theta + \cos 3\theta \right\}. \quad (10)$$

The terms on the right of (10) are all of the form $r^p \cos n\theta$, so that for the present purpose we have to solve

$$\nabla^2 \phi = \frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + \frac{1}{r^2} \frac{d^2 \phi}{d\theta^2} = r^p \cos n\theta. \quad \cdot \quad (11)$$

If we assume that ϕ varies as $r^m \cos n\theta$, we see that $m=p+2$, and that the complete solution is

$$\phi = \cos n\theta \left\{ A r^n + B r^{-n} + \frac{r^{p+2}}{(p+2)^2 - n^2} \right\}, \quad \cdot \quad (12)$$

A and B being arbitrary constants. In (10) we have to deal with $n=1$ associated with $p=-5$ and -7 , and with $n=3$ associated with $p=-3$. The complete solution as regards terms in $\cos \theta$ and $\cos 3\theta$ is accordingly

$$\begin{aligned} \phi = & (Ar + Br^{-1}) \cos \theta + (Cr^3 + Dr^{-3}) \cos 3\theta \\ & + \frac{2U^3 c^2}{a^2} \left[\cos \theta \left(- \frac{c^2}{4r^3} + \frac{c^4}{24r^5} \right) - \frac{\cos 3\theta}{8r} \right]. \quad \cdot \quad \cdot \quad (13) \end{aligned}$$

The conditions to be satisfied at infinity require that, as in (1), $A=U$, and that $C=0$. We have also to make $d\phi/dr$ vanish when $r=c$. This leads to

$$B = c^2 U + \frac{13U^3 c^2}{12a^2}, \quad D = \frac{U^3 c^4}{12a^2}. \quad \cdot \quad \cdot \quad (14)$$

Thus

$$\phi = U \left\{ r + \frac{c^2}{r} + \frac{13U^2c^2}{12a^2r^3} \right\} \cos \theta + \frac{U^3c^4}{12a^2r^3} \cos 3\theta \\ + \frac{U^3c^2}{a^2} \left\{ \cos \theta \left(-\frac{c^2}{2r^3} + \frac{c^4}{12r^5} \right) - \frac{\cos 3\theta}{4r} \right\} \quad (15)$$

satisfies all the conditions and is the value of ϕ complete to the second approximation.

That the motion determined by (15) gives rise to no resultant force in the direction of the stream is easily verified. The pressure at any point is a function of q^2 , and on the surface of the cylinder $q^2 = c^{-2}(d\phi/d\theta)^2$. Now $(d\phi/d\theta)^2$ involves θ in the forms $\sin^2 \theta$, $\sin^2 3\theta$, $\sin \theta \sin 3\theta$, and none of these are changed by the substitution of $\pi - \theta$ for θ ; the pressures on the cylinder accordingly constitute a balancing system.

There is no particular difficulty in pursuing the approximation so as to include terms involving the square and higher powers of U^2/a^2 . The right-hand member of (6) will continue to include only terms in the cosines of odd multiples of θ with coefficients which are simple powers of r , so that the integration can be effected as in (11), (12). And the general conclusion that there is no resultant force upon the cylinder remains undisturbed.

The corresponding problem for the *sphere* is a little more complicated, but it may be treated upon the same lines with use of Legendre's functions $P_n(\cos \theta)$ in place of cosines of multiples of θ . In terms of the usual polar coordinates (r, θ, ω) , the last of which does not appear, the first approximation, as for an incompressible fluid, is

$$\phi = U \cos \theta \left(r + \frac{c^3}{2r^2} \right) = U \left(r + \frac{c^3}{2r^2} \right) P_1, \quad (16)$$

c denoting the radius of the sphere. As in (8),

$$\Sigma \frac{d\phi}{dx} \frac{dq^2}{dx} = \frac{d\phi}{dr} \frac{dq^2}{dr} + \frac{1}{r^2} \frac{d\phi}{d\theta} \frac{dq^2}{d\theta} = U^3 \left\{ \left(-\frac{36c^6}{5r^7} + \frac{9c^9}{2r^{10}} \right) P_1 \right. \\ \left. + \left(\frac{6c^3}{r^4} - \frac{24c^6}{5r^7} + \frac{3c^9}{2r^{10}} \right) P_3 \right\}, \quad (17)$$

on substitution from (16) of the values of ϕ and q^2 . This gives us the right-hand member of (6).

In the present problem

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d}{d\theta} \right), \quad (18)$$

while P_n satisfies

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP_n}{d\theta} \right) + n(n+1)P_n = 0; \quad (19)$$

so that

$$\nabla^2 \phi = r^p P_n \quad (20)$$

reduces to

$$\frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - \frac{n(n+1)}{r^2} \phi = r^p. \quad (21)$$

The solution, corresponding to the various terms of (17), is thus

$$\phi = \frac{r^{p+2} P_n}{(p+2)(p+3) - n(n+1)}. \quad (22)$$

With use of (22), (6) gives

$$\begin{aligned} \phi = \frac{U^3}{a^2} \left\{ -\frac{c^6 P_1}{5r^5} + \frac{c^9 P_1}{24r^8} - \frac{3c^3 P_3}{10r^2} - \frac{3c^6 P_3}{10r^5} + \frac{3c^9 P_3}{176r^8} \right\} \\ + ArP_1 + Br^{-2}P_1 + Cr^3P_3 + Dr^{-4}P_3, \quad (23) \end{aligned}$$

A, B, C, D being arbitrary constants. The conditions at infinity require $A=U$, $C=0$. The conditions at the surface of the sphere give

$$B = \frac{c^3}{2} \left(U + \frac{2U^3}{3a^2} \right), \quad D = \frac{27c^5 U^3}{55a^2}; \quad (24)$$

and thus ϕ is completely determined to the second approximation.

The P 's which occur in (23) are of *odd* order, and are polynomials in $\mu (= \cos \theta)$ of *odd* degree. Thus $d\phi/dr$ is odd (in μ) and $d\phi/d\theta = \sin \theta \times$ even function of μ . Further,

$$\begin{aligned} q^2 &= \text{even function} + \sin^2 \theta \times \text{even function} = \text{even function}, \\ dq^2/dr &= \text{even function}, \\ dq^2/d\theta &= \sin \theta \times \text{odd function}. \end{aligned}$$

Accordingly

$$\frac{d\phi}{dr} \frac{dq^2}{dr} + \frac{1}{r^2} \frac{d\phi}{d\theta} \frac{dq^2}{d\theta} = \text{odd function of } \mu,$$

and can be resolved into a series of P 's of odd order. Thus not only is there no resultant force discovered in the second

approximation, but this character is preserved however far we may continue the approximations. And since the coefficients of the various P 's are simple polynomials in μ , the integrations present no difficulty in principle.

Thus far we have limited ourselves to Boyle's law, but it may be of interest to make extension to the general adiabatic law, of which Boyle's is a particular case. We have now to suppose

$$p/p_0 = (\rho/\rho_0)^\gamma, \quad . \quad . \quad . \quad (25)$$

making

$$\frac{dp}{d\rho} = \frac{\gamma p_0}{\rho_0} \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = a^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}, \quad . \quad . \quad . \quad (26)$$

if a denote the velocity of sound corresponding to ρ_0 . Then by (3)

$$\frac{a^2}{\gamma-1} \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = C - \frac{1}{2}q^2. \quad . \quad . \quad . \quad (27)$$

If we suppose that ρ_0 corresponds to $q=0$, $C=a^2/(\gamma-1)$, and

$$\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = 1 - \frac{(\gamma-1)q^2}{2a^2}, \quad . \quad . \quad . \quad (28)$$

whence

$$\frac{d \log \rho}{dx} = - \frac{dq^2/dx}{2a^2 - (\gamma-1)q^2}. \quad . \quad . \quad . \quad (29)$$

The use of this in (2) now gives

$$\nabla^2 \phi = \frac{1}{2a^2 - (\gamma-1)q^2} \left\{ \frac{d\phi}{dx} \frac{dq^2}{dx} + \frac{d\phi}{dy} \frac{dq^2}{dy} + \frac{d\phi}{dz} \frac{dq^2}{dz} \right\}, \quad (30)$$

from which we can fall back upon (6) by supposing $\gamma=1$. So far as the first and second approximations, the substitution of (30) for (6) makes no difference at all.

As regards the general question it would appear that so long as the series are convergent there can be no resistance and no wake as the result of compressibility. But when the velocity U of the stream exceeds that of sound, the system of velocities in front of the obstacle expressed by our equations cannot be maintained, as they would be at once swept away down stream. It may be presumed that the passage from the one state of affairs to the other synchronizes with a failure of convergency. For a discussion of what happens when the velocity of sound is exceeded, reference may be made to a former paper*.

* Proc. Roy. Soc. A. vol. lxxxiv. p. 247 (1910); 'Scientific Papers,' vol. v. p. 608.

II. *The Roots of Bessel and Neumann Functions of High Order.* By JOHN R. AIREY, M.A., D.Sc.*

THE general formulæ for the higher roots of $J_n(x)$, $G_n(x)$, &c. have been given by McMahon †. If $m=4n^2$ and $\beta=\frac{\pi}{4}(2n+4s-1)$, and $x_n^{(s)}$ represents the s th root in order of magnitude of the equation $J_n(x)=0$, then

$$x_n^{(s)} = \beta - \frac{m-1}{8\beta} - \frac{4(m-1)(7m-31)}{3(8\beta)^3} - \frac{32(m-1)(83m^2-982m+3779)}{15(8\beta)^5} \dots (1)$$

A similar formula has also been found ‡ for the roots of $Y_n(x)$, where

$$\beta = (2n+4s+1) \frac{\pi}{4} - \arctan \frac{2(\log 2 - \gamma)}{\pi}.$$

When n is large, however, the above expressions are not applicable to the calculation of the earlier roots. In calculating the roots of $J_n(x)$, for example, to two places of decimals, the first root of $J_{100}(x)$ which can be found to this degree of accuracy by (1) is the 50th, and in the case of $J_{1000}(x)$, the first root so obtained is approximately the 1000th.

More suitable formulæ for all the roots of $J_n(x)$ can be derived from a result given by Debye §, of which the first two terms are as follows :

$$J_n(x) = \left(\frac{2}{\pi n \tan \phi} \right)^{\frac{1}{2}} \left[\cos \left\{ n(\tan \phi - \phi) - \frac{\pi}{4} \right\} + \frac{1}{n \tan \phi} \left(\frac{1}{8} + \frac{5}{24} \cot^2 \phi \right) \cos \left\{ n(\tan \phi - \phi) - \frac{3\pi}{4} \right\} + \dots \right], (2)$$

where $\cos \phi = \frac{n}{x}$.

This result can be obtained directly from

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \phi - n\phi) d\phi \quad . \quad . \quad (3)$$

* Communicated by the Author.

† *Annals of Mathematics*, vol. ix. (1895).

‡ *Proc. Phys. Soc.* vol. xxiii. (1911).

§ *Math. Annalen*, 67. Band (1909).

by the method employed by Lord Kelvin* in evaluating the integral

$$u = \int_0^\infty \cos m[x - tf(m)] dm.$$

When x and n are great, the range of integration to be considered in (3) is in the neighbourhood of the places where $x \sin \phi - n\phi$ is stationary with respect to ϕ , i. e. within the small range $\phi_1 - \eta$ and $\phi_1 + \eta$, where

$$x \cos \phi_1 - n = 0, \text{ or } \cos \phi_1 = \frac{n}{x}.$$

By Taylor's theorem, for small values of $\phi - \phi_1$,

$$\begin{aligned} x \sin \phi - n\phi &= x \sin \phi_1 - n\phi_1 - \frac{(\phi - \phi_1)^2}{2!} x \sin \phi_1 \\ &\quad - \frac{(\phi - \phi_1)^3}{3!} x \cos \phi_1 + \frac{(\phi - \phi_1)^4}{4!} x \sin \phi_1 \dots \end{aligned} \quad (4)$$

Substitute

$$(\phi - \phi_1)^2 = \frac{2\mu^2}{x \sin \phi_1} \quad \dots \quad (4')$$

Then

$$d\phi = \left(\frac{2}{x \sin \phi_1} \right)^{\frac{1}{2}} d\mu,$$

and the limits of integration become ultimately $-\infty$ and $+\infty$.

Therefore (3) becomes, when

$$b = \frac{x \cos \phi_1}{6 \left(\frac{x}{2} \sin \phi_1 \right)^{\frac{3}{2}}}, \quad c = \frac{1}{6 (x \sin \phi_1)},$$

and

$$\psi = x \sin \phi_1 - n\phi_1 = n(\tan \phi_1 - \phi_1),$$

$$J_n(x) = \frac{1}{\pi} \left(\frac{2}{x \sin \phi_1} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \cos(\psi - \mu^2 - b\mu^3 + c\mu^4 \dots) d\mu. \quad (5)$$

$$\begin{aligned} &= \frac{\cos \psi}{\pi} \left(\frac{2}{x \sin \phi_1} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \cos(\mu^2 + b\mu^3 - c\mu^4 \dots) d\mu + \\ &\quad \frac{\sin \psi}{\pi} \left(\frac{2}{x \sin \phi_1} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \sin(\mu^2 + b\mu^3 - c\mu^4 \dots) d\mu. \end{aligned} \quad (6)$$

* Lord Kelvin, Proc. Roy. Soc. Feb. 1887; Lord Rayleigh, Phil. Mag. Dec. 1910.

Taking the first integral of (6) and putting

$$\theta = \mu^2 + b\mu^3 - c\mu^4 \dots$$

we find, from

$$\begin{aligned} \int_0^\infty \theta^{x-1} \cos \theta d\theta &= \Gamma(x) \cos \frac{\pi x}{2}, \\ \int_{-\infty}^{+\infty} \cos (\mu^2 + b\mu^3 - c\mu^4 \dots) d\mu &= \Gamma\left(\frac{1}{2}\right) \cos \frac{\pi}{4} + \frac{15b^2}{8} \Gamma\left(\frac{3}{2}\right) \cos \frac{3\pi}{4} \\ &\quad + \frac{3c}{2} \Gamma\left(\frac{3}{2}\right) \cos \frac{3\pi}{4} \dots\dots \\ &= \Gamma\left(\frac{1}{2}\right) \cos \frac{\pi}{4} + \left(\frac{1}{8} + \frac{5}{24} \cot^2 \phi_1\right) \Gamma\left(\frac{3}{2}\right) \cos \frac{3\pi}{4} + \dots \quad (7) \end{aligned}$$

From the second integral, a similar expression is obtained in which the sine occurs in place of the cosine in (7).

Finally, we get from (6),

$$\begin{aligned} J_n(x) &= \frac{1}{\pi} \left(\frac{2}{x \sin \phi_1} \right)^{\frac{1}{2}} \left[\Gamma\left(\frac{1}{2}\right) \cos \left(\psi - \frac{\pi}{4} \right) + \right. \\ &\quad \left. \left(\frac{1}{8} + \frac{5}{24} \cot^2 \phi_1 \right) \cdot \frac{2}{x \sin \phi_1} \cdot \Gamma\left(\frac{3}{2}\right) \cos \left(\psi - \frac{3\pi}{4} \right) + \dots \right]. \quad (8) \end{aligned}$$

The next four terms in the bracket have been calculated, the coefficients of which are more conveniently expressed in decimal form.

$$\text{If } J_n(x) = \frac{1}{\pi} \sum_{s=0}^{s=m} \frac{A_s(\phi_1) \Gamma(s + \frac{1}{2})}{\left(\frac{x}{2} \sin \phi_1 \right)^{s + \frac{1}{2}}} \cos \left\{ \psi - \frac{(2s+1)\pi}{4} \right\}. \quad (9)$$

then the functions $A_s(\phi_1)$ take the following values, k being written for $\cot^2 \phi_1$:

$$\begin{aligned} A_0(\phi_1) &= 1. \\ A_1(\phi_1) &= 0.12500 + 0.20833k. \\ A_2(\phi_1) &= 0.02344 + 0.01215k + 0.11140k^2. \\ A_3(\phi_1) &= 0.00488 + 0.05941k + 0.12310k^2 + 0.06839k^3. \\ A_4(\phi_1) &= 0.00107 + 0.02249k + 0.06521k^2 + 0.10673k^3 \\ &\quad + 0.04443k^4. \\ A_5(\phi_1) &= 0.00024 + 0.00780k + 0.04503k^2 + 0.09702k^3 \\ &\quad + 0.08960k^4 + 0.02986k^5. \end{aligned}$$

A first approximation to the roots of $J_n(x)$ can be found by making the first term in (8) equal to zero

$$\cos \left(x \sin \phi_1 - n\phi_1 - \frac{\pi}{4} \right) = \cos \left[n(\tan \phi_1 - \phi_1) - \frac{\pi}{4} \right] = 0. \quad (10)$$

Hence

$$n(\tan \phi_1 - \phi_1) - \frac{\pi}{4} = \frac{(2p-1)\pi}{2}$$

or

$$n(\tan \phi_1 - \phi_1) = \frac{(4p-1)\pi}{4} \quad . \quad . \quad (11)$$

The errors in the corresponding values of ϕ_1 for the first four roots of $J_1(x)$ are $9' 8''$, $1' 19''$, $0' 25''$, and $0' 11''$.

The roots ρ_p are then given by

$$\rho_p = n \sec \phi_1 \quad . \quad . \quad . \quad (12)$$

Putting

$$n(\tan \phi_1 - \phi_1) = \frac{(4p-1)\pi}{4} + \epsilon, \quad . \quad . \quad . \quad (13)$$

it can be shown without difficulty that, when r is written for $n \tan \phi_1$,

$$\tan \epsilon = \frac{A_1 r - 1.3.5.A_3 r^3 + 1.3.5.7.9.A_5 r^5 - \dots}{A_0 - 1.3.A_2 r^2 + 1.3.5.7.A_4 r^4 - \dots} \quad (14)$$

The quantity ϵ is in general very small and decreases as p , the number of the root, increases. For the first four roots of $J_1(x)$, ϵ has the following values: 0.03596 , 0.01839 , 0.01248 , and 0.00947 .

Any root of $J_n(x)$ can thus be calculated from (13) by successive approximations. Only in extreme cases is it necessary to go beyond the second approximation. In calculating the first root of $J_1(x)$, for example, the following values of ϕ_1 and ϵ were successively obtained:

$$\begin{aligned} \phi_1 &= 74^\circ 43' 10''.4 : \epsilon = 0.03631. \\ &74^\circ 52' 23''.5 : \epsilon = 0.03587. \\ &74^\circ 52' 16''.9, \end{aligned}$$

the last value of ϕ_1 giving 3.8316 as the first root of $J_1(x)$. A number of errors have been discovered in Bourget's Table* of the roots of J_1 to J_5 , seven of these occurring in the values of the roots of J_5 . The first root of $J_5(x)$ is 8.7715 , not 8.780 , and the ninth root 34.989 , not 34.983 . Other methods† give 8.77148 as the first root of $J_5(x)$.

The first five roots of $J_{10}(x)$, $J_{100}(x)$, and $J_{1000}(x)$ have been calculated from the formulæ (12), (13), and (14): the

* *Ann. de l'Ecole Normale*, 3 (1866); Lord Rayleigh, 'Theory of Sound,' vol. i. Table B. p. 330.

† Lord Rayleigh, *Collected Works*, vol. i. p. 190; *Phil. Mag.* June 1916, p. 520.

successive approximations for the roots of $J_{100}(x)$ are as follows :

No. of root.	ϕ_1 .	ρ_p .	ϵ .
$p=1$.	$23^\circ 9' 42''\cdot5$	108·767	0·02755
	$23^\circ 14' 51''\cdot7$	108·837	0·02726
	$23^\circ 14' 48''\cdot5$	108·836	
$p=2$.	$30^\circ 12' 32''\cdot5$	115·715	0·01250
	$30^\circ 13' 48''\cdot5$	115·740	
$p=3$.	$34^\circ 39' 3''\cdot4$	121·561	0·00806
	$34^\circ 39' 38''\cdot2$	121·575	
$p=4$.	$37^\circ 58' 35''\cdot1$	126·861	0·00596
	$37^\circ 58' 55''\cdot3$	126·871	
$p=5$.	$40^\circ 39' 23''\cdot4$	131·817	0·00473
	$40^\circ 39' 36''\cdot5$	131·824	

Roots of $J_{10}(x)$, $J_{100}(x)$, and $J_{1000}(x)$.

	Roots of $J_{10}(x)$.	$J_{100}(x)$.	$J_{1000}(x)$.
1.	14·4755	108·836	1018·62
2.	18·4335	115·740	1032·76
3.	22·0470	121·575	1044·39
4.	25·5095	126·871	1054·74
5.	28·8863	131·824	1064·24

For very large roots of $J_n(x)$, ϕ_1 is nearly equal to 90° and its value is not easily determined from (11). Using seven-figure tables, any root can be found to six significant figures when ϕ_1 is not greater than 85° .

If ϕ_1 exceeds this value, the sine of its complement θ_1 can be calculated from the formula

$$\sin \theta_1 = \kappa + \frac{\kappa^3}{2} + \frac{13\kappa^5}{20} + \frac{61\kappa^7}{80} \dots, \quad (15)$$

where

$$\kappa = \frac{1}{\lambda} = \frac{4n}{\pi(4p-1+2n)},$$

$$\operatorname{cosec} \theta_1 = \lambda - \frac{1}{2\lambda} - \frac{2}{5\lambda^3} - \frac{19}{80\lambda^5} \dots, \quad (16)$$

and

$$\rho_p = n \operatorname{cosec} \theta_1.$$

A more accurate value of $\operatorname{cosec} \theta_1$ can be obtained from

$$\operatorname{cosec} \theta_1 = \mu - \frac{1}{2\mu} - \frac{2}{5\mu^3} - \frac{19}{80\mu^5} \dots, \quad (17)$$

where

$$\mu = \lambda + \frac{1}{8n^2\lambda} + \left(\frac{1}{3n^2} - \frac{25}{384n^4} \right) \cdot \frac{1}{\lambda^3} \dots$$

Taking an extreme case, the second root of $J_{\frac{1}{2}}(x)$ and applying (15), the following values were obtained:—

κ .	θ_1 .	ρ_2 .	ϵ .
0.0553582	$3^\circ 10' 41'' 90$	6.0121	0.02065
0.0551690	$3^\circ 10' 2'' 64$	6.0328	

the second approximation agreeing with the value of ρ_2 from equation (1). For ρ_{50} of $J_1(x)$, (17) gives at once the value 157.8626554.

A simple expression for the roots of $J_n(x)$ when n is large, can be found from (11) and (12).

From (11)

$$\tan \phi_1 = \lambda + \frac{\lambda^3}{5} + \frac{3\lambda^5}{175} \dots,$$

where

$$\lambda = \left[\frac{3(4p-1)\pi}{4n} \right]^{\frac{1}{3}}.$$

Therefore, from (12), since

$$\sec \phi_1 = 1 + \frac{\tan^2 \phi_1}{2} - \frac{\tan^4 \phi_1}{8} \dots,$$

$$\rho_p = n \left[1 + \frac{\lambda^2}{2} + \frac{3\lambda^4}{40} \dots \right]. \dots \dots (18)$$

Even for the small value $n=5$, this gives $\rho_1=8.744\dots$ instead of 8.7715, and for $n=100$, $\rho_1=108.77$, $\rho_2=115.72$, and $\rho_3=121.58$. From (18) it is seen that for very large values of n , $\frac{\rho_1}{n}$ approximates to a ratio of equality.

The equation $\frac{dJ_n(x)}{dx} = 0$ is nearly as important in mathematical and physical problems as $J_n(x) = 0$.
If

$$\lambda = \frac{\pi}{4n} (2n+4p-3) \text{ and } \mu = \lambda - \frac{3}{8n^2\lambda} - \left(\frac{2}{3n^2} - \frac{3}{128n^4} \right) \frac{1}{\lambda^3} \dots,$$

$$\operatorname{cosec} \theta_1 = \mu - \frac{1}{2\mu} - \frac{2}{5\mu^3} - \frac{19}{80\mu^5} \dots$$

and the higher roots of $J_n'(x) = 0$ are given by

$$\rho_p = n \operatorname{cosec} \theta_1.$$

A simple and convenient formula for the roots of $J_n'(x) = 0$

when n is large is

$$\rho_p = n \left[1 + \frac{\lambda^2}{2} + \frac{3\lambda^4}{40} \dots \right],$$

where

$$\lambda = \lambda_1 \left(1 - \frac{3}{8n^2\lambda_1^4} \right)$$

and

$$\lambda_1^3 = \left[\frac{3(4p-3)\pi}{4n} \right].$$

Thus the first root of $J'_{1000}(x) = 0$ is 1008.84.

Lord Rayleigh has given an approximate expression * for the first root of $J'_n(x) = 0$, viz.

$$\rho_1 = n + 0.5134n^{\frac{1}{3}}.$$

The Bessel function $J_{-n}(x)$ of negative order is represented by a formula similar to (9), ψ in this case being equal to $n(\tan \phi_1 - \phi_1 + \pi)$. Hence the first approximations of the roots of $J_{-n}(x)$ can be derived from

$$n(\tan \phi_1 - \phi_1 + \pi) = \frac{(4p-1)\pi}{4}$$

and

$$\rho_p = n \sec \phi_1,$$

ϕ_1 for the first root being found from the least positive value of $\tan \phi_1 - \phi_1$. Thus, for the first root of $J_{-\frac{5}{2}}(x)$, $p=3$ and

$\tan \phi_1 - \phi_1 = \frac{\pi}{10}$. The small quantity ϵ can be determined as before, leading to closer values of ϕ_1 and ρ . The first two approximations of ρ_1 for $J_{-\frac{5}{2}}(x)$ are 1.802 and 1.868.

The Neumann function $G_n(x)$ is given by

$$G_n(x) = -\frac{1}{2} \sum_{s=0}^{s=m} \frac{A_s(\phi_1) \Gamma\left(s + \frac{1}{2}\right)}{\left(\frac{x}{2} \sin \phi_1\right)^{s+\frac{1}{2}}} \sin \left\{ \psi - \frac{(2s+1)\pi}{4} \right\}. \quad (19)$$

Hence it follows that the roots of $G_n(x)$ can be calculated from formulæ similar to (12), (13), and (14). In this case, (13) is replaced by

$$n(\tan \phi_1 - \phi_1) = \left(\frac{4p-3}{4} \right) \pi + \epsilon. \quad (20)$$

A difficulty arises when $p=1$ in this expression, but the first roots of $G_n(x)$ of any order are found from values of

* Phil. Mag. Dec. 1910.

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these functions when the argument and order are nearly equal *. The formula (18) holds for the roots of $G_n(x)$ when n is large, if λ takes the value

$$\lambda = \left[\frac{3(4p-3)\pi}{4n} \right]^{\frac{1}{2}}.$$

The Neumann function $Y_n(x)$ is equal to

$$(\log 2 - \gamma)J_n(x) - G_n(x).$$

Substituting (9) and (19) for $J_n(x)$ and $G_n(x)$, and proceeding as before, we find that

$$\tan [n(\tan \phi_1 - \phi_1)] = \frac{(1 + \epsilon) - \beta(1 - \epsilon)}{(1 - \epsilon) + \beta(1 + \epsilon)}, \quad (21)$$

where

$$\beta = \frac{2(\log 2 - \gamma)}{\pi} = 0.0738043.$$

Therefore

$$n(\tan \phi_1 - \phi_1) = 0.7117276 + (p-1)\pi \quad \dots \quad (22)$$

to a first approximation, $p=1$ corresponding to the first root. The first roots of $Y_n(x)$ are found as in the case of $G_n(x)$. For the second and higher roots, it is not necessary to go beyond the second approximation from (22). Thus, using five-figure logarithms, the first two approximations for the second root of $Y_1(x)$ are 5.330 and 5.3549 †. The first five roots of $G_{100}(x)$ and $Y_{100}(x)$ given below have been calculated from (20) and (21), or by employing the G and Y functions of nearly equal order and argument.

	Roots of $G_{100}(x)$.	$Y_{100}(x)$.
1.	104.380	104.133
2.	112.486	112.325
3.	118.744	118.608
4.	124.275	124.151
5.	129.383	129.266

The approximation for the first root of $Y_{100}(x)$ from (22) gives $\phi_1 = 15^\circ 44' 1''.5$ and $\rho_1 = 103.90$.

Substituting this value of ϕ_1 in (14) to calculate ϵ , it is found that the series in the numerator diverges from the first, $15A_3r^3$ being greater than A_1r . Consequently 103.90 is the nearest value of the first root of $Y_{100}(x)$ by this method.

* Phil. Mag. June 1916, p. 520.

† Proc. Phys. Soc. vol. xxiii. (1911).

III. *On Two Fundamental Problems in the Theory of Elasticity.* By C. E. WEATHERBURN, M.A. (Cantab.), D.Sc. (Sydney); Lecturer in Mathematics and Physics, Ormond College, University of Melbourne (Australia)*.

§ 1. *Introduction.*

THE theory of vector integral equations, which I have treated elsewhere†, finds an important and direct application in the fundamental problems of elastic equilibrium, requiring the determination of the displacement at any point of an elastic body when the value of the surface displacement is known or that of the surface traction. These will be referred to as the first and second boundary problems respectively. In the present paper I generalize the problems by the introduction of a parameter λ , in the manner proposed by Poincaré‡ in his discussion of the problems of the potential theory. Moreover, by the use of vector analysis I construct, from Somigliana's integrals of the equations of equilibrium, dyadics which form the basis of displacement functions whose properties exactly resemble those of ordinary simple and double stratum potentials. The treatment of the fundamental problems in elasticity is thus recast, and will be found to run exactly parallel with that of the potential problems. Corresponding theorems are established concerning the magnitude and the reality of the singular parameter values, and the simplicity of the pole of the solution at each of these. The singular case of the potential problem for the inner region when the value of the normal derivative is known, finds its counterpart in the problem of equilibrium of an elastic body under given surface traction.

The integral equations that arise are vector equations with dyadic kernels and dyadic resolvents. Connected with the resolvent of the dyadic which forms the basis of a double stratum displacement is another, which, in a separate paper§, I prove to be a "Green's dyadic," analogous to the Green's function for Laplace's equation vanishing over the boundary.

Lauricella|| was led, in order to escape the difficulty of

* Communicated by the Author.

† "Vector Integral Equations and Gibbs' Dyadics," *Trans. Camb. Phil. Soc.* vol. xxii. pp. 133-158 (1916).

‡ Cf. "La méthode de Neumann et le problème de Dirichlet," *Acta Math.* Bd. 20 (1896).

§ "Green's Dyadics in the Theory of Elasticity," *Proc. Lond. Math. Soc.* 1916-17.

|| *Atti R. Acc. Lincei* (5), t. 15₂, pp. 75-83 (1906); also *Il Nuovo Cimento* (5), t. 13 (1907). Four notes.

a kernel becoming infinite of the second order, to discard the natural idea of surface traction for that of "pseudo-tension," a concept which has no physical significance. This is equivalent to eliminating the anti-self-conjugate part of the double stratum dyadic. I hope to show that this kernel does not become infinite of too high an order to be treated by the methods of my paper already referred to.

The problems may be attacked from an entirely different point of view, by the use of the Green's functions for Laplace's equation. According to this method * the cubical dilation θ and the molecular rotation \mathbf{R} are regarded for the moment as known, and the displacement \mathbf{D} is found in terms of one or both of them by the aid of the Green's functions for the region occupied by the body. Then, taking the divergence and the curl of the value of \mathbf{D} so found, we deduce, in the case of the first problem, a single integral equation for θ from the solution of which the value of \mathbf{D} is deduced. The treatment of the second problem gives a pair of integral equations, one scalar and one vector, from the former of which θ is obtained in terms of \mathbf{R} ; then this value of θ substituted in the second gives a single vector integral equation for determining \mathbf{R} .

§ 2. *The Equations of Equilibrium.*—Confining our attention to *isotropic* bodies we shall sustain no loss of generality by assuming the bodily forces zero; for it is well known that the equations of equilibrium can always be reduced to the form in which the terms representing these forces do not appear †. The ordinary equations of equilibrium are

$$\nabla^2 u + k \frac{\partial \theta}{\partial x} = 0,$$

$$\nabla^2 v + k \frac{\partial \theta}{\partial y} = 0,$$

$$\nabla^2 w + k \frac{\partial \theta}{\partial z} = 0;$$

while the conditions to be satisfied at the surfaces are expressed by

$$-L = \frac{du}{dn} + k\theta \frac{dx}{dn} + \left(\frac{\partial v}{\partial x} \frac{dy}{dn} - \frac{\partial v}{\partial y} \frac{dx}{dn} \right) + \left(\frac{\partial w}{\partial x} \frac{dz}{dn} - \frac{\partial w}{\partial z} \frac{dx}{dn} \right),$$

* Cf. Boggio, *Atti Lincei*, t. 16₂ (1907), pp. 248-255 and 441-450.

† Cf., e.g., Marcolongo, "Teoria Mat. dello Equilibrio dei Corpi Elastici." Milan (1904), p. 233.

and two similar equations obtained from this by cyclic permutation of the variables. In these u, v, w are the rectangular components of the displacement of the particle at the point (x, y, z) of the body, n the inward drawn normal, θ the cubical dilation, and L, M, N the components of the surface traction at the boundary point considered; while the constant k is given by

$$k = (\Omega^2 - \omega^2)/\omega^2 = (\lambda + \mu)/\mu,$$

Ω being the velocity of propagation of longitudinal waves and ω that of transverse waves in the body. The constants λ, μ are those employed by Lamé*, μ being the rigidity $= \rho\omega^2$. In the equations involving the surface tractions we have supposed the units so chosen that μ is equal to unity; and we shall throughout adhere to this assumption.

If we denote the vector displacement of the point p of the body by \mathbf{D} or $\mathbf{D}(p)$, and the surface traction at the point s by \mathbf{T} or $\mathbf{T}(s)$, the equations for the equilibrium of the particle at p are readily reduced to the single equation

$$\nabla^2 \mathbf{D} + k \text{ grad div } \mathbf{D} = 0, \quad (1)$$

or its equivalent

$$(k+1) \text{ grad div } \mathbf{D} - \text{curl curl } \mathbf{D} = 0; \quad . . . (1')$$

while for the surface-point s

$$-\mathbf{T}(s) = 2 \frac{d}{dn} \mathbf{D} + (k-1) \mathbf{n} \text{ div } \mathbf{D} + \mathbf{n} \times \text{curl } \mathbf{D}, \quad . . . (2)$$

\mathbf{n} being the unit vector in the direction of the inward normal. It will be convenient to introduce the symbol \mathbf{F} or $\mathbf{F}(s)$ to denote the surface traction with its sign changed, *i.e.*

$$\mathbf{F}(s) = -\mathbf{T}(s),$$

whose value is given by the second member of (2). For consistency of notation we shall throughout use the letters p, q to denote points within the region considered but not on the boundary; while $t, s; \mathfrak{A}, \sigma$ will represent boundary points, and dt, ds , &c. the corresponding elements of the boundary. The surface Σ bounding the body separates the finite inner region S from the infinite outer region S' . We shall assume that Σ possesses everywhere a definite tangent plane and two definite principal radii of curvature. Singularities such as points and edges are excluded.

* "Leçons sur la théorie math. de l'élasticité des corps solides." Deuxième Leçon. Paris (1852).

§ 3. *A Particular Integral.*—An unlimited number of particular integrals may be found to the equation (1). In fact, if \mathbf{B} is any biharmonic vector function, it is easily verified that

$$\mathbf{s} = \nabla^2 \mathbf{B} - \frac{k}{1+k} \text{grad div } \mathbf{B}$$

satisfies the equation. Taking $\frac{1}{2} \mathbf{a} r$ as the biharmonic function, where \mathbf{a} is any constant vector and r the “radius vector” measured from a fixed point p , we obtain the particular solution

$$\mathbf{s}_0 = \frac{\mathbf{a}}{r} - \frac{k}{2(1+k)} \text{grad div}(\mathbf{a} r). \quad . \quad . \quad (3)$$

This is the vector equivalent of the set of integrals due to Somigliana*. There is no loss of generality in taking \mathbf{a} as a unit vector. This integral becomes infinite at the pole p to the same order as $1/r$. Hence it cannot be regarded as an actual solution of any physical problem for the elastic body embracing the point p ; but we are able to construct other integrals with \mathbf{s}_0 as a basis, which are finite and continuous throughout the body.

It is important for our argument to notice that $\mathbf{s}_0(pq)$, which has been defined as a function of the variable point q with p as its pole, is symmetrical in p and q . For if \mathbf{r} is the radius vector from p to q

$$\begin{aligned} \mathbf{s}_0(pq) &= \frac{\mathbf{a}}{r} - \frac{k}{2(1+k)} \frac{d}{da} \left(\frac{\mathbf{r}}{r} \right) \\ &= \frac{\mathbf{a}}{r} - \frac{k}{2(1+k)} \left[\frac{\mathbf{a}}{r} - \frac{\mathbf{r}}{r^2} \frac{\mathbf{a} \cdot \mathbf{r}}{r} \right], \end{aligned}$$

which is unaltered if \mathbf{r} changes sign, that is if r is measured from q to p . The expression is therefore symmetrical in p and q , which may be expressed

$$\mathbf{s}_0(pq) = \mathbf{s}_0(qp).$$

In the potential theory $\frac{d}{dn} \left(\frac{1}{r} \right)$ is an integral of Laplace's equation forming the basis of double-stratum potentials. To find an analogous solution of (1) we shall determine by means of (2) the surface traction corresponding to the

* “Sulle equaz. dell' Elasticità,” *Annali di Mat.* (2) t. 16 (1888).

displacement \mathbf{s}_0 . Calculating separately the different terms, we have

$$\begin{aligned}\frac{d}{dn} \text{grad div} (\mathbf{a} r) &= \frac{d}{dn} \left[\mathbf{a} \cdot \text{grad} \left(\frac{\mathbf{r}}{r} \right) \right] \\ &= \frac{d}{dn} \left\{ \mathbf{a} \cdot \left[\text{grad} \left(\frac{1}{r} \right) \mathbf{r} + \frac{\mathbf{I}}{r} \right] \right\} = \mathbf{a} \frac{d}{dn} \left(\frac{1}{r} \right) - \frac{d}{dn} \left(\frac{\mathbf{a} \cdot \mathbf{r} \mathbf{r}}{r^3} \right) \\ &= \mathbf{a} \frac{d}{dn} \left(\frac{1}{r} \right) + \mathbf{a} \cdot \mathbf{n} \text{grad} \left(\frac{1}{r} \right) - \mathbf{a} \cdot \mathbf{r} \left[\mathbf{r} \frac{d}{dn} \left(\frac{1}{r^3} \right) + \frac{\mathbf{n}}{r^3} \right] \\ &= \mathbf{a} \frac{d}{dn} \left(\frac{1}{r} \right) + \mathbf{a} \cdot \mathbf{n} \text{grad} \left(\frac{1}{r} \right) - 3 \frac{d}{dn} \left(\frac{1}{r} \right) \text{grad } r \left(\frac{dr}{da} \right) + \mathbf{n} \mathbf{a} \cdot \text{grad} \frac{1}{r}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dn} \mathbf{s}_0 &= \frac{k+2}{2(1+k)} \mathbf{a} \frac{d}{dn} \left(\frac{1}{r} \right) \\ &\quad - \frac{k}{2(1+k)} \left[\mathbf{a} \cdot \mathbf{n} \text{grad} \frac{1}{r} + \mathbf{n} \mathbf{a} \cdot \text{grad} \frac{1}{r} - 3 \frac{d}{dn} \left(\frac{1}{r} \right) \text{grad } r \left(\frac{dr}{da} \right) \right]. \quad (4)\end{aligned}$$

In the second term of (2) we have

$$\begin{aligned}\text{div } \mathbf{s}_0 &= \mathbf{a} \cdot \text{grad} \frac{1}{r} - \frac{k}{2(1+k)} \nabla^2 (\mathbf{a} \cdot \text{grad } r) \\ &= \frac{d}{da} \left(\frac{1}{r} \right) - \frac{k}{1+k} \frac{d}{da} \left(\frac{1}{r} \right) = \frac{1}{1+k} \frac{d}{da} \left(\frac{1}{r} \right). \quad (5)\end{aligned}$$

Lastly, in the third term

$$\begin{aligned}\mathbf{n} \times \text{curl } \mathbf{s}_0 &= \mathbf{n} \times \text{curl} \frac{\mathbf{a}}{r} \\ &= \mathbf{n} \times \left(\text{grad} \frac{1}{r} \times \mathbf{a} \right) \\ &= \mathbf{n} \cdot \mathbf{a} \text{grad} \frac{1}{r} - \mathbf{n} \cdot \text{grad} \frac{1}{r} \mathbf{a}. \quad (6)\end{aligned}$$

Substituting in (2) the values given by (4), (5), and (6) we find for the value of the surface traction * $\mathbf{F}_0(sp)$ due to the displacement $\mathbf{s}_0(qp)$ whose pole is p ,

$$\begin{aligned}\mathbf{F}_0(sp) &= \frac{1}{1+k} \mathbf{a} \frac{d}{dn} \left(\frac{1}{r} \right) - \frac{1}{1+k} \mathbf{a} \times \left(\mathbf{n} \times \text{grad} \frac{1}{r} \right) \\ &\quad + \frac{3k}{1+k} (\text{grad } r) \frac{d}{dn} \left(\frac{1}{r} \right) \left(\frac{dr}{da} \right). \quad (7)\end{aligned}$$

* It will lead to no confusion if we speak thus of $\mathbf{F} = -\mathbf{T}$ as the "surface traction." The symbol employed will always indicate which quantity is referred to.

§ 4. *Vector Potentials of Elastic Strata.*—If \mathbf{D} and \mathbf{D}' are two solutions of the equation (1) regular within the given region, and \mathbf{F}, \mathbf{F}' the corresponding surface tractions, the bodily forces being supposed zero, Betti's reciprocity theorem* may be put in the form

$$\int \mathbf{F}(s) \cdot \mathbf{D}'(s) ds - \int \mathbf{F}'(s) \cdot \mathbf{D}(s) ds = 0. \quad (8)$$

Take the usual set $\mathbf{i}, \mathbf{j}, \mathbf{k}$, of three unit rectangular vectors, and consider the values of the particular integral \mathbf{s}_0 of the previous section when the vector \mathbf{a} is replaced successively by $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Denote its values in these cases by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$, and the corresponding surface tractions by $\mathbf{F}_1(sp), \mathbf{F}_2(sp), \mathbf{F}_3(sp)$ respectively. Then

$$\left. \begin{aligned} \mathbf{s}_1 &= \frac{\mathbf{i}}{r} - \frac{k}{2(1+k)} \text{grad div}(\mathbf{i}r), \\ \mathbf{s}_2 &= \frac{\mathbf{j}}{r} - \frac{k}{2(1+k)} \text{grad div}(\mathbf{j}r), \\ \mathbf{s}_3 &= \frac{\mathbf{k}}{r} - \frac{k}{2(1+k)} \text{grad div}(\mathbf{k}r). \end{aligned} \right\} \quad (9)$$

We may in the formula (8) replace \mathbf{D}' by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ in succession provided the point p be isolated by (say) a small sphere Z with p as centre. It then follows that

$$\int_{\Sigma+Z} \mathbf{F}(s) \cdot \mathbf{s}_m(ps) ds = \int_{\Sigma+Z} \mathbf{F}_m(sp) \cdot \mathbf{D}(s) ds, \quad (m=1, 2, 3).$$

Multiplying these in order by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and adding, we find a result which may be written

$$\begin{aligned} & \int_{\Sigma+Z} [\mathbf{i} \mathbf{s}_1(ps) + \mathbf{j} \mathbf{s}_2(ps) + \mathbf{k} \mathbf{s}_3(ps)] \cdot \mathbf{F}(s) ds \\ &= \int_{\Sigma+Z} \mathbf{D}(s) \cdot [\mathbf{F}_1(sp) \mathbf{i} + \mathbf{F}_2(sp) \mathbf{j} + \mathbf{F}_3(sp) \mathbf{k}] ds, \quad (10) \end{aligned}$$

where the expressions in square brackets are dyadics. In evaluating the integrals over the small sphere Z , we notice that the first member contributes nothing; for $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ become infinite at p only of the first order, while the area of the small sphere is of the second order. Considering the second member, we take the values of $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ given by (7)

* "Teoria della elasticità," Cap. VI. *Il Nuovo Cimento*, 1872; *Annali di Mat.* (6) 1875.

for the particular values $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of \mathbf{a} . The first term of (7) substituted in (10) gives

$$\frac{1}{1+k} \int_Z \mathbf{D}(s) \cdot \left[\mathbf{i} \frac{d}{dn} \left(\frac{1}{r} \right) \mathbf{i} + \mathbf{j} \left(\frac{1}{r} \right) \mathbf{j} + \dots \right] ds$$

$$= \frac{1}{1+k} \int_Z \mathbf{D}(s) \frac{d}{dn} \left(\frac{1}{r} \right) ds,$$

an expression which becomes equal to $-4\pi\mathbf{D}(p)/(1+k)$ when the radius of Z decreases indefinitely. The next term of (7) contributes to the value of the integral

$$- \frac{1}{1+k} \int_Z \mathbf{D}(s) \cdot \left[\mathbf{i} \times \left(\mathbf{n} \times \text{grad} \frac{1}{r} \right) \mathbf{i} + \mathbf{j} \times \left(\mathbf{n} \times \text{grad} \frac{1}{r} \right) \mathbf{j} + \dots \right] ds,$$

which vanishes along with the radius of the sphere; for since \mathbf{n} has the same direction as \mathbf{r} it follows that $\mathbf{n} \times \text{grad} \frac{1}{r}$ becomes infinite of order less than the second. Hence the integral vanishes in the limit. Lastly, from the third term we derive

$$\frac{3k}{1+k} \int_Z \mathbf{D}(s) \cdot \left[\sum_{ijk} \left\{ \mathbf{i} \cdot (\text{grad } r \text{ grad } r) \mathbf{i} \right\} \right] \frac{d}{dn} \left(\frac{1}{r} \right) ds$$

$$= \frac{3k}{1+k} \int_Z \mathbf{D}(s) \cdot (\text{grad } r \text{ grad } r) \frac{d}{dn} \left(\frac{1}{r} \right) ds.$$

The limit of the value of this expression when Z decreases indefinitely is $-4\pi k\mathbf{D}(p)/(1+k)$.

Substituting in (10) the value thus found for the integral over Z we obtain the result

$$4\pi\mathbf{D}(p) = \int_Z \mathbf{D}(s) \cdot [\mathbf{F}_1(sp) \mathbf{i} + \mathbf{F}_2(sp) \mathbf{j} + \mathbf{F}_3(sp) \mathbf{k}] ds$$

$$- \int_Z [\mathbf{i} s_1(ps) + \mathbf{j} s_2(ps) + \mathbf{k} s_3(ps)] \cdot \mathbf{F}(s) ds, \quad (11)$$

which is the vector equivalent of Somigliana's formulæ* for the components of the displacement at any point of an elastic isotropic body in terms of its surface values and the values of the surface tractions. The formula (11) is analogous to the relation

$$4\pi u(p) = \int \left[u \frac{d}{dn} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{du}{dn} \right] ds,$$

giving the value of an harmonic function u at any point p of

* *Annali di Mat.* 17 (1889).

the region in terms of the boundary values of the function and those of its normal derivative. But this expresses the harmonic function as the difference of two others which are potentials of double and simple strata respectively. It is therefore an obvious suggestion to examine the two integrals in (11) from the point of view of stratum potentials. This has been done by Lauricella * for the case in which the first integral is modified by using only the self-conjugate part of its dyadic. The elimination of the anti-self-conjugate part is the object for which his idea of pseudo-tension is introduced. We shall show, as Lauricella did in the modified case, that the two integrals in (11) possess properties and boundary discontinuities analogous to those enjoyed by the Newtonian and logarithmic potentials of double and simple strata.

§ 5. First, it is evident that each of these integrals satisfies the equation (1) which characterizes a displacement. For each of the vectors $\mathbf{s}_1(ps)$, $\mathbf{s}_2(ps)$, $\mathbf{s}_3(ps)$ as a function of p is a solution of that equation, and so also is $\mathbf{D}(p)$ by hypothesis. Hence the first integral of the second member of (11) must satisfy (1). But as a function of p this is independent of the boundary function $\mathbf{D}(s)$; hence the integral is a solution of (1) whatever be the finite and continuous function $\mathbf{D}(s)$. Indeed it is easily verified that the function $\mathbf{F}_0(sp)$ satisfies (1) for all values of \mathbf{a} . If for brevity we denote the dyadics in (11) by the symbols †

$$\left. \begin{aligned} \Psi(sp) &= \frac{1}{2\pi} [\mathbf{F}_1(sp)\mathbf{i} + \mathbf{F}_2(sp)\mathbf{j} + \mathbf{F}_3(sp)\mathbf{k}] \\ \Phi(pq) &= \frac{1}{2\pi} [\mathbf{i}\mathbf{s}_1(pq) + \mathbf{j}\mathbf{s}_2(pq) + \mathbf{k}\mathbf{s}_3(pq)] \end{aligned} \right\} \dots \quad (12)$$

then the integrals

$$\left. \begin{aligned} \mathbf{W}(p) &= \int \mathbf{v}(s) \cdot \Psi(sp) ds \\ \mathbf{V}(p) &= \int \Phi(ps) \cdot \mathbf{u}(s) ds \end{aligned} \right\} \dots \quad (13)$$

in which $\mathbf{u}(s)$ and $\mathbf{v}(s)$ are finite and continuous vector

* *Il Nuovo Cimento*, loc. cit. pp. 155-174.

† I have chosen the notation to resemble that employed in my earlier papers on the potential theory. Cf. *Quarterly Journ.* vol. xlv. (1914-15). Three papers.

functions of the boundary point s , are solutions of the equation (1) of elastic equilibrium, and are finite and continuous vector functions of the point p of the region bounded by Σ . The function $\mathbf{W}(p)$, whose dyadic $\Psi(sp)$ involves $\frac{d}{dn}\left(\frac{1}{r}\right)$, will be spoken of as the vector potential of a double elastic stratum of moment $\mathbf{v}(s)$; while $\mathbf{V}(p)$, whose dyadic is similarly related to $\frac{1}{r}$, will be called the vector potential of a simple elastic stratum of density $\mathbf{u}(s)$. It will be observed that the density and moment as so defined are vector functions of the position of the point s of the boundary. When no ambiguity is possible these potentials will be called briefly potentials of double and simple strata respectively.

§ 6. *Vector Potential of a Double Elastic Stratum.*—The double stratum vector potential $\mathbf{W}(p)$ defined in the previous section has discontinuities at the boundary exactly resembling those of the ordinary double stratum potential, and expressible in the form

$$\left. \begin{aligned} \mathbf{W}(t^+) &= \mathbf{v}(t) + \int \mathbf{v}(s) \cdot \Psi(st) ds \\ \mathbf{W}(t^-) &= -\mathbf{v}(t) + \int \mathbf{v}(s) \cdot \Psi(st) ds \end{aligned} \right\}, \quad \cdot \cdot \cdot \quad (14)$$

t^+ being a point of the inner region indefinitely close to the boundary point t but not on the boundary, and t^- the corresponding point of the outer region. To prove this we shall consider separately the parts of $\mathbf{W}(p)$ due to the three terms of $\mathbf{F}_0(s)$ given by (7). First the potential

$$\begin{aligned} \mathbf{w}_1(p) &= \frac{1}{1+k} \frac{1}{2\pi} \sum_{i,j,k} \int \mathbf{v}(s) \cdot \left[\mathbf{i} \frac{d}{dn} \left(\frac{1}{r} \right) \mathbf{i} \right] ds \\ &= \frac{1}{2\pi(1+k)} \int \mathbf{v}(s) \frac{d}{dn} \left(\frac{1}{r} \right) ds \end{aligned}$$

is, to a constant factor, that of an ordinary double stratum of moment $\mathbf{v}(s)$. This satisfies the known relations

$$\left. \begin{aligned} \mathbf{w}_1(t^+) &= \frac{1}{1+k} \mathbf{v}(t) + \mathbf{w}_1(t) \\ \mathbf{w}_1(t^-) &= -\frac{1}{1+k} \mathbf{v}(t) + \mathbf{w}_1(t) \end{aligned} \right\} \quad \cdot \cdot \cdot \quad (15)$$

The potential $\mathbf{w}_2(p)$ arising from the second term in (7) is

continuous at the boundary. For it is to a constant multiple equal to

$$\sum_{ijk} i \int \mathbf{v}(s) \cdot \left[i \times \left(\mathbf{n} \times \text{grad} \frac{1}{r} \right) \right] ds = \int \mathbf{v}(s) \times \left[\text{grad} \left(\frac{1}{r} \right) \times \mathbf{n} \right] ds.$$

It is clear from this form of the expression that $\mathbf{w}_2(p)$ is continuous at the boundary. For as p moves up to t along the normal and coincides with it, $\text{grad} \frac{1}{r} dt$ remains a finite vector in the direction of the normal, so that $\text{grad} \left(\frac{1}{r} \right) \times \mathbf{n} dt$ vanishes in the limit. Hence there is no discontinuity due to the element dt , and $\mathbf{w}_2(p)$ is continuous at the boundary.

Lastly, the integral $\mathbf{w}_3(p)$ arising from the third term of (7) is, as in § 4, given by

$$\mathbf{w}_3(p) = \frac{1}{2\pi} \frac{3k}{1+k} \int \mathbf{v}(s) \cdot \left[\text{grad } r \text{ grad } r \right] \frac{d}{dn} \left(\frac{1}{r} \right) ds.$$

This is an integral with discontinuity at the boundary equal to one-third of that of the corresponding ordinary double stratum*; so that

$$\left. \begin{aligned} \mathbf{w}_3(t^+) &= \frac{k}{1+k} \mathbf{v}(t) + \mathbf{w}_3(t) \\ \mathbf{w}_3(t^-) &= -\frac{k}{1+k} \mathbf{v}(t) + \mathbf{w}_3(t) \end{aligned} \right\} \dots \dots (16)$$

Combining the results for the three potentials $\mathbf{w}_1(p)$, $\mathbf{w}_2(p)$, and $\mathbf{w}_3(p)$ we have the discontinuity for $\mathbf{W}(p)$ expressed by (14).

The continuity of the normal derivative of an ordinary double stratum potential has its counterpart in a further property of the vector potential $\mathbf{W}(p)$, viz. that if the function $\mathbf{v}(s)$ is finite and continuous along with its first derivative, the surface traction due to a displacement of the particles represented by $\mathbf{W}(p)$ is continuous at Σ . This may be put more definitely as follows. Imagine a surface Σ' close to Σ , and with its normal everywhere parallel to the normal to Σ . Then the surface traction due to the displacement $\mathbf{W}(p)$ of the body bounded by Σ' remains continuous as Σ' moves up to and passes through Σ . This theorem, which is due to Lauricella †, is true when $\mathbf{v}(t)$ is finite and continuous along with its first derivative.

* Cf. Lauricella, *loc. cit.* pp. 161-164.

† Cf. *Atti Lincei* (5), t. 15 (1906), p. 429; also *Annali di Mat.* (1907), Cap. II. § 6.

§ 7. *Vector Potential of a Simple Elastic Stratum.*—Consider next the simple stratum potential

$$\mathbf{V}(p) = \frac{1}{2\pi} \int \left[\mathbf{i} s_1(ps) + \mathbf{j} s_2(ps) + \mathbf{k} s_3(ps) \right] \cdot \mathbf{u}(s) ds \\ = \int \Phi(ps) \cdot \mathbf{u}(s) ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The dyadic $\Phi(ts)$ is conjugo-symmetric, being both symmetric and self-conjugate. It is symmetric because the function $\mathbf{s}_0(ts)$ is symmetric. It is also self-conjugate; for if expressed in nonion form it has the same coefficient for \mathbf{ij} as for \mathbf{ji} , viz. $-\frac{k}{2(1+k)} \frac{\partial^2 r}{\partial x \partial y}$. The dyadic is therefore conjugo-symmetric and enjoys all the properties established for such a kernel in the last part of my paper first referred to.

We observe that the function $\mathbf{V}(p)$ is continuous at the boundary. This is clear from the form of $\mathbf{s}_0(ts)$ which becomes infinite at $t=s$ only like $1/r$. Further, $\mathbf{V}(p)$ satisfies (1) and may therefore be regarded as the displacement of the point p of a body occupying the region bounded by some surface to be specified. The corresponding surface traction varies with this surface, and will now be shown to be discontinuous* at Σ . Consider a surface Σ' , either within or without Σ , with its normal \mathbf{n}' everywhere parallel to the normal \mathbf{n} to Σ . Let s, σ denote points on Σ' ; t, \mathfrak{s} on Σ . Because the function $\mathbf{s}_0(pq)$ is symmetric we have

$$\mathbf{s}_1(s\mathfrak{s}) = \mathbf{s}_1(\mathfrak{s}s), \quad . \quad . \quad . \quad . \quad . \quad (18)$$

representing a displacement at the point s with \mathfrak{s} as the pole. The surface traction on Σ' due to this displacement is therefore $\mathbf{F}_1'(s\mathfrak{s})$, whose value is obtained from (7) replacing d/dn by d/dn' , so that

$$\mathbf{F}_1'(s\mathfrak{s}) = \frac{\mathbf{i}}{1+k} \frac{d}{dn'} \left(\frac{1}{r} \right) - \frac{1}{1+k} \mathbf{i} \times \left(\mathbf{n}' \times \text{grad} \frac{1}{r} \right) \\ + \frac{3k}{1+k} \frac{d}{dn'} \left(\frac{1}{r} \right) \left(\frac{\partial r}{\partial \xi} \right) \text{grad } r, \quad . \quad . \quad (19)$$

the point $s (= \xi, \eta, \zeta)$ being the current point and \mathfrak{s} the pole from which r is measured. Hence, since in (17) $\mathbf{u}(\mathfrak{s})$ is independent of s , the surface traction at the point s of Σ' due

* Cf. Lauricella, *Il Nuovo Cimento*, loc. cit. pp. 166-174.

to the displacement $\mathbf{V}(p)$ is given by

$$\begin{aligned}\mathbf{F}(s) &= \frac{1}{2\pi} \int \left[\mathbf{i} \mathbf{F}_1'(s\mathfrak{S}) + \mathbf{j} \mathbf{F}_2'(s\mathfrak{S}) + \mathbf{k} \mathbf{F}_3'(s\mathfrak{S}) \right] \cdot \mathbf{u}(\mathfrak{S}) d\mathfrak{S} \\ &= \frac{1}{2\pi} \int \mathbf{u}(\mathfrak{S}) \cdot [\mathbf{F}_1'(s\mathfrak{S})\mathbf{i} + \mathbf{F}_2'(s\mathfrak{S})\mathbf{j} + \mathbf{F}_3'(s\mathfrak{S})\mathbf{k}] d\mathfrak{S} \\ &= \int \mathbf{u}(\mathfrak{S}) \cdot \Psi'(s\mathfrak{S}) d\mathfrak{S}, \quad \dots \dots \dots (20)\end{aligned}$$

the dyadic $\Psi'(s\mathfrak{S})$ being derived from $\Psi(s\mathfrak{S})$ replacing d/dn by d/dn' . This function $\mathbf{F}(s)$ resembles the double stratum potential $\mathbf{W}(p)$ of the previous section, but the dyadic in (20) differs from that in the expression for $\mathbf{W}(s)$ in the order of the variables. The effect of this upon the boundary discontinuity of the expression appears thus. As Σ' moves up to and coincides with Σ , s moves up to and coincides with a point t . But whether s is the point t^+ of the inner region or the point t^- of the outer region, the value of the expression $\mathbf{F}_1'(st)$ given by (19) is equal to $-\mathbf{F}_1'(ts) = -\mathbf{F}_1(ts)$ given by (7); for the change in the order of the variables means a change in the direction of \mathbf{r} . But it is to the element dt in the integration that the discontinuity is due. Hence the discontinuity in the function (20) is opposite in sign to that of $\mathbf{W}(p)$. If, then, $\mathbf{F}(t^+)$ and $\mathbf{F}(t^-)$ denote the surface tractions for the inner and outer regions respectively due to the simple stratum displacement $\mathbf{V}(p)$,

$$\begin{aligned}\mathbf{F}(t^+) &= -\mathbf{u}(t) + \int \mathbf{u}(\mathfrak{S}) \cdot \Psi(t\mathfrak{S}) d\mathfrak{S} \} \\ \mathbf{F}(t^-) &= \mathbf{u}(t) + \int \mathbf{u}(\mathfrak{S}) \cdot \Psi(t\mathfrak{S}) d\mathfrak{S} \} \quad \dots \dots (21)\end{aligned}$$

It should be observed in what relation the formulæ (21) stand to (14). Regarded as equations in $\mathbf{v}(\mathfrak{S})$ and $\mathbf{u}(\mathfrak{S})$ the first of (14) and the second of (21) are integral equations; but they are not associated. Though the order of the variables is different in the two kernels, the unknown occurs as a prefactor in each case. The equation associated to (14 *a*) is obtainable from (21 *b*) by making the kernel the prefactor. The difference is vital because the dyadic is not self-conjugate. Its value is neatly expressed by the formula

$$\begin{aligned}2\pi(1+k)\Psi(sp) &= \mathbf{I} \frac{d}{dn} \left(\frac{1}{r} \right) - \mathbf{I} \times \left[\text{grad} \left(\frac{1}{r} \right) \times \mathbf{n} \right] \\ &\quad + 3k(\text{grad } r \text{ grad } r) \frac{d}{dn} \left(\frac{1}{r} \right).\end{aligned}$$

The first and last terms in this expression are self-conjugate dyadics, but the second is anti-self-conjugate.

§ 8. *The First and Second Boundary Problems.*—I shall consider the fundamental problems of elastic equilibrium in a general form analogous to that proposed by Poincaré for Dirichlet's and Neumann's problems, an arbitrary parameter λ being introduced into the prescribed boundary conditions. We set before ourselves the determination of displacements $\mathbf{W}(p)$ and $\mathbf{V}(p)$ for an elastic isotropic body corresponding respectively to the boundary relations

$$\left. \begin{aligned} \frac{1}{2} [\mathbf{W}(t^+) - \mathbf{W}(t^-)] - \frac{\lambda}{2} [\mathbf{W}(t^+) + \mathbf{W}(t^-)] &= \mathbf{f}(t) \\ \frac{1}{2} [\mathbf{TV}(t^-) - \mathbf{TV}(t^+)] - \frac{\lambda}{2} [\mathbf{TV}(t^-) + \mathbf{TV}(t^+)] &= -\mathbf{f}(t) \end{aligned} \right\} \quad (25)$$

$\mathbf{f}(t)$ being a given piecemeal continuous function of the boundary point, and $\mathbf{TV}(t^+)$ denoting the surface traction at the point t due to the displacement $\mathbf{V}(p)$ for a body occupying the *inner* region, while $\mathbf{TV}(t^-)$ has a similar meaning for a body occupying the *outer* region. The problems (25) will be called the first and second boundary problems respectively. For the parameter values $\lambda = \pm 1$ they relate to the inner and outer regions separately.

Endeavouring to satisfy these by vector potentials

$$\left. \begin{aligned} \mathbf{W}(p) &= \int \mathbf{v}(s) \cdot \Psi(sp) ds \\ \mathbf{V}(p) &= \int \Phi(ps) \cdot \mathbf{u}(s) ds \end{aligned} \right\} \quad (26)$$

due respectively to a double stratum of moment $\mathbf{v}(s)$ and a simple one of density $\mathbf{u}(s)$, we find from the boundary properties of such that $\mathbf{v}(s)$ and $\mathbf{u}(s)$ are solutions of a pair of vector integral equations

$$\left. \begin{aligned} \mathbf{v}(t) - \lambda \int \mathbf{v}(s) \cdot \Psi(st) ds &= \mathbf{f}(t) \\ \mathbf{u}(t) - \lambda \int \chi(ts) \cdot \mathbf{u}(s) ds &= \mathbf{f}(t) \end{aligned} \right\} \quad (27)$$

where

$$\chi(ts) = \Psi_c(ts),$$

that is the dyadic conjugate to $\Psi(ts)$. These integral equations are not associated because $\Psi(ts)$ is not self-conjugate. The kernels of the equations (27),

$$\Psi(st) = \frac{1}{2\pi} [\mathbf{F}_1(st) \mathbf{i} + \mathbf{F}_2(st) \mathbf{j} + \mathbf{F}_3(st) \mathbf{k}]$$

and its conjugate, become infinite at the point $s=t$ but of order less than two, and therefore Fredholm's method of

solution is available. There is no need to eliminate the second term of $\mathbf{F}_0(sp)$ in (7) involving $\mathbf{a} \times \left(\mathbf{n} \times \text{grad} \frac{1}{r} \right)$. This term does not become infinite of the second order as p approaches the boundary point s along the normal. For $\text{grad} \frac{1}{r} = -\mathbf{r}/r^3$; but \mathbf{n} has the same direction as \mathbf{r} , so that $\mathbf{n} \times \text{grad} \frac{1}{r}$ becomes infinite of order less than two. It is this fact that makes the integral $\mathbf{w}_2(p)$ of § 6 continuous at the boundary.

Each of the equations (27) has one and only one solution unless λ is a characteristic number of the kernel of that equation. The solutions are expressible in terms of the resolvent dyadics $\mathbf{H}(ts)$ and $\mathbf{H}'(ts)$ of $\Psi(ts)$ and $\chi(ts)$ respectively, connected with these by equations of the form

$$\mathbf{H}(ts) - \Psi(ts) = \lambda \int \mathbf{H}(t\mathfrak{s}) \cdot \Psi(\mathfrak{s}s) d\mathfrak{s} = \lambda \int \Psi(t\mathfrak{s}) \cdot \mathbf{H}(\mathfrak{s}s) d\mathfrak{s}, \quad (28)$$

and a similar set (28') for $\mathbf{H}'(ts)$ and $\chi(ts)$. In terms of these resolvents the solutions of (27) are

$$\left. \begin{aligned} \mathbf{v}(t) &= \mathbf{f}(t) + \lambda \int \mathbf{f}(s) \cdot \mathbf{H}(st) ds \\ \mathbf{u}(t) &= \mathbf{f}(t) + \lambda \int \mathbf{H}'(ts) \cdot \mathbf{f}(s) ds \end{aligned} \right\}, \quad \dots \quad (29)$$

and these values substituted in (26) give the displacements that satisfy the boundary problems (25), viz.

$$\left. \begin{aligned} \mathbf{W}(p) &= \int \mathbf{f}(s) \cdot \left[\Psi(sp) + \lambda \int \mathbf{H}(s\mathfrak{s}) \cdot \Psi(\mathfrak{s}p) d\mathfrak{s} \right] ds \\ \mathbf{V}(p) &= \int \left[\Phi(ps) + \lambda \int \Phi(p\mathfrak{s}) \cdot \mathbf{H}'(\mathfrak{s}s) d\mathfrak{s} \right] \cdot \mathbf{f}(s) ds \end{aligned} \right\}. \quad (30)$$

§ 9. The resolvents $\mathbf{H}(ts)$ and $\mathbf{H}'(ts)$, in terms of which the solutions have been expressed, are known meromorphic functions of the parameter λ , and either becomes infinite only when λ is a root of its denominator $D(\lambda)$ or $D'(\lambda)$. These singular values of λ depend only on the form of the boundary. The solutions (30) may be written

$$\left. \begin{aligned} \mathbf{W}(p) &= \int \mathbf{f}(s) \cdot \mathbf{H}(sp) ds \\ \mathbf{V}(p) &= \int \Gamma'(ps) \cdot \mathbf{f}(s) ds \end{aligned} \right\} \quad \dots \quad (31)$$

if we define the functions $\mathbf{H}(sp)$ and $\Gamma'(ps)$ by the equations

$$\begin{aligned} \mathbf{H}(sp) &= \Psi(sp) + \lambda \int \mathbf{H}(s\mathfrak{s}) \cdot \Psi(\mathfrak{s}p) d\mathfrak{s}, \\ \Gamma'(ps) &= \Phi(ps) + \lambda \int \Phi(p\mathfrak{s}) \cdot \mathbf{H}'(\mathfrak{s}s) d\mathfrak{s}. \end{aligned}$$

The dyadic $\mathbf{H}(sp)$ so defined is an extension of the resolvent $\mathbf{H}(st)$ obtained from it replacing t by a point p not on the boundary. From the preceding relation it is easily verified that

$$\int \Psi(s\mathfrak{S}) \cdot \mathbf{H}(\mathfrak{S}p) d\mathfrak{S} = \int \mathbf{H}(s\mathfrak{S}) \cdot \Psi(\mathfrak{S}p) d\mathfrak{S},$$

so that $\mathbf{H}(tp)$ is defined by the alternative relations

$$\mathbf{H}(tp) - \Psi(tp) = \lambda \int \mathbf{H}(ts) \cdot \Psi(sp) ds = \lambda \int \Psi(ts) \cdot \mathbf{H}(sp) ds. \quad (32)$$

An exactly similar pair of relations (32') define the dyadic $\mathbf{H}'(tp)$ in terms of $\chi(tp)$. Then the dyadic $\Gamma'(ps)$ defined as above may be extended, replacing s by another point q not on the boundary, the new function $\Gamma'(pq)$ being specified by

$$\Gamma'(pq) = \Phi(pq) + \lambda \int \Phi(ps) \cdot \mathbf{H}'(sq) ds.$$

It is then easily verified that

$$\int \Gamma'(ps) \cdot \chi(sq) ds = \int \Phi(ps) \cdot \mathbf{H}'(sq) ds;$$

and thus the dyadic $\Gamma'(pq)$ is defined by the alternative relations

$$\Gamma'(pq) - \Phi(pq) = \lambda \int \Phi(ps) \cdot \mathbf{H}'(sq) ds = \lambda \int \Gamma'(ps) \cdot \chi(sq) ds, \quad (33)$$

in which q may be replaced by a boundary point t . Similarly, if we define a dyadic $\Gamma(pq)$ by the equation

$$\Gamma(pq) = \Phi(pq) + \lambda \int \Phi(ps) \cdot \mathbf{H}(sq) ds,$$

it satisfies the alternative relations

$$\Gamma(pq) - \Phi(pq) = \lambda \int \Phi(ps) \cdot \mathbf{H}(sq) ds = \lambda \int \Gamma(ps) \cdot \Psi(sq) ds. \quad (33')$$

§ 10. *Singular Parameter Values.*—We may now prove that the characteristic numbers λ_i and λ'_i of the kernels $\Psi(ts)$ and $\chi(ts)$ respectively are real, and in absolute magnitude not less than unity; also that each is only a simple pole of the resolvent involved. For these values the homogeneous equations

$$\left. \begin{aligned} \mathbf{v}(t) &= \lambda_i \int \mathbf{v}(s) \cdot \Psi(st) ds \\ \mathbf{u}(t) &= \lambda'_i \int \chi(ts) \cdot \mathbf{u}(s) ds \end{aligned} \right\} \quad . \quad . \quad . \quad (34)$$

admit each one or more non-zero solutions. Regarded as moment and density of double and simple strata respectively,

these define displacements $\mathbf{W}(p)$ and $\mathbf{V}(p)$ satisfying the homogeneous problems

$$\left. \begin{aligned} \mathbf{W}(t^+) - \mathbf{W}(t^-) &= \lambda_i [\mathbf{W}(t^+) + \mathbf{W}(t^-)] \\ \mathbf{TV}(t^-) - \mathbf{TV}(t^+) &= \lambda_i' [\mathbf{TV}(t^-) + \mathbf{TV}(t^+)] \end{aligned} \right\} \quad (35)$$

The equations (34) are not associated, and the characteristic numbers of $\Psi(ts)$ are in general different from those of $\chi(ts)$. We observe, for the following argument, that if $\mathbf{U}(p)$ is any regular displacement satisfying the equation (1), and $\mathbf{TV}(t)$ the corresponding surface traction, the integral

$$[\mathbf{U}] \equiv - \int \mathbf{U} \cdot \mathbf{TV} dt \quad (36)$$

represents twice the potential energy of the deformed body, and is therefore a positive quantity, vanishing only for a translation or rotation of the body as a whole. Further, if \mathbf{V} is another regular displacement satisfying (1), Betti's theorem gives

$$\int [\mathbf{U} \cdot \mathbf{TV} - \mathbf{V} \cdot \mathbf{TV}] dt = 0 \quad (37)$$

To prove now the *reality* of the characteristic numbers *, consider for example one λ_0' of $\chi(ts)$ for which the second equation (35) admits a solution $\mathbf{V}(p)$. If this parameter value is complex ($= a + ib$), so also is the potential $\mathbf{V}(p)$ ($= \mathbf{U} + i\mathbf{U}_1$). Then, on separating real and imaginary parts in (35), we obtain

$$\begin{aligned} (1-a)\mathbf{TV}^- - (1+a)\mathbf{TV}^+ + b(\mathbf{TV}_1^- + \mathbf{TV}_1^+) &= 0 \\ (1-a)\mathbf{TV}_1^- - (1+a)\mathbf{TV}_1^+ - b(\mathbf{TV}^- + \mathbf{TV}^+) &= 0, \end{aligned}$$

where \mathbf{TV}^- is written for $\mathbf{TV}(t^-)$, and so on. Multiply these equations scalarly first by \mathbf{U}_1 and \mathbf{U} respectively, subtract and integrate over Σ ; then by \mathbf{U} and \mathbf{U}_1 respectively, subtract and integrate as before. Then in virtue of (37) it follows that

$$\left. \begin{aligned} b\{[\mathbf{U}^+] + [\mathbf{U}_1^+] - [\mathbf{U}^-] - [\mathbf{U}_1^-]\} &= 0 \\ -a\{[\mathbf{U}^+] + [\mathbf{U}_1^+] - [\mathbf{U}^-] - [\mathbf{U}_1^-]\} &= [\mathbf{U}^+] + [\mathbf{U}_1^+] + [\mathbf{U}^-] + [\mathbf{U}_1^-] \end{aligned} \right\}, \quad (38)$$

where

$$[\mathbf{U}^-] \equiv \int \mathbf{U} \cdot \mathbf{TV}(t^-) dt,$$

which by (36) is positive, as it applies to the outer region

* The proofs in this § and the next follow closely those of Plemelj for the case of the potential theory. Cf., e. g., "Potentialtheoretische Untersuchungen," § 24.

for which the direction of the inward normal is reversed. Since, then, the second member of the second equation (38) is positive, and cannot vanish except in the degenerate cases already mentioned, the coefficient of a cannot vanish, nor therefore that of b . Hence b itself must be zero, making λ'_0 real. Then $\mathbf{V}(p)$ must be real, giving $\mathbf{U}_1=0$, $\mathbf{U}=\mathbf{V}$, and $a=\lambda'_0$. Thence by (38)

$$\lambda'_0 = \{[\mathbf{V}^-] + [\mathbf{V}^+]\} / \{[\mathbf{V}^-] - [\mathbf{V}^+]\}. \quad (39)$$

Thus the absolute value of λ'_0 is greater than unity, except in the degenerate cases for which one of the expressions $[\mathbf{V}^-]$ or $[\mathbf{V}^+]$ is zero.

Similarly, starting with the first equation (35), we may prove the same result for the characteristic numbers λ_i of $\Psi(ts)$.

§ 11. To prove next that each singular value λ'_0 is a *simple pole* of the resolvent $\mathbf{H}'(ts)$. We have seen that it is a pole, and therefore also a pole of the density $\mathbf{u}(t)$ of the same order. If this order n be >1 , $\mathbf{u}(t)$ may in the neighbourhood of λ'_0 be expressed in the form

$$\mathbf{u}(t) = \mathbf{p}(t)/(\lambda - \lambda'_0)^n + \mathbf{p}_1(t)/(\lambda - \lambda'_0)^{n-1} + \dots,$$

where $\mathbf{p}(t)$ does not vanish identically. Substituting this value in the second of equations (27) written in the form

$$\mathbf{u}(t) - (\lambda - \lambda'_0) \int \chi(ts) \cdot \mathbf{u}(s) ds - \lambda'_0 \int \chi(ts) \cdot \mathbf{u}(s) ds = \mathbf{f}(t),$$

and equating to zero the coefficients of $(\lambda - \lambda'_0)^{-n}$ and $(\lambda - \lambda'_0)^{-n+1}$, we obtain the relations

$$\mathbf{p}(t) - \lambda'_0 \int \chi(ts) \cdot \mathbf{p}(s) ds = 0,$$

$$\mathbf{p}_1(t) - \lambda'_0 \int \chi(ts) \cdot \mathbf{p}_1(s) ds = \int \chi(ts) \cdot \mathbf{p}(s) ds = \mathbf{p}(t)/\lambda'_0.$$

If now we take $\mathbf{p}(t)$ and $\mathbf{p}_1(t)$ as vector densities of simple elastic strata whose potentials are $\mathbf{V}(p)$ and $\mathbf{V}_1(p)$ respectively, these equations are equivalent to

$$\mathbf{TV}^- - \mathbf{TV}^+ - \lambda'_0(\mathbf{TV}^- + \mathbf{TV}^+) = 0,$$

$$\mathbf{TV}_1^- - \mathbf{TV}_1^+ - \lambda'_0(\mathbf{TV}_1^- + \mathbf{TV}_1^+) = (\mathbf{TV}^- - \mathbf{TV}^+)/\lambda'_0.$$

Multiply scalarly the first of these by \mathbf{V}_1 and the second by \mathbf{V} , subtract and integrate over Σ ; then in virtue of (37) and the continuity of a simple elastic stratum potential we deduce

$$[\mathbf{V}^+] + [\mathbf{V}^-] = 0.$$

Each of the expressions $[\mathbf{V}^+]$ and $[\mathbf{V}^-]$ is therefore identically zero. The stratum density $\mathbf{p}(t)$ must therefore vanish, because in the degenerate cases referred to the surface traction is zero, and therefore gives no discontinuity at the boundary. Since then $\mathbf{p}(t)$ is zero for all values of $n > 1$, it follows that the poles of the resolvent $\mathbf{H}'(ts)$ are simple. The same may be proved of the poles λ_i of the resolvent $\mathbf{H}(ts)$.

These resolvents may then, in the neighbourhoods of their poles λ_0 and λ_0' , be expressed in the forms

$$\left. \begin{aligned} \mathbf{H}(ts) &= \mathbf{P}(ts)/(\lambda_0 - \lambda) + \mathbf{K}(ts) \\ \mathbf{H}'(ts) &= \mathbf{P}'(ts)/(\lambda_0' - \lambda) + \mathbf{K}'(ts) \end{aligned} \right\}, \quad \dots \quad (40)$$

where $\mathbf{K}(ts)$ remains finite at λ_0 and $\mathbf{K}'(ts)$ at λ_0' . Since the poles are simple the residues $\mathbf{P}(ts)$ and $\mathbf{P}'(ts)$ are given by *

$$\left. \begin{aligned} \mathbf{P}(ts) &= \sum_{i=1}^k \mathbf{n}_i(t) \mathbf{v}_i(s) \\ \mathbf{P}'(ts) &= \sum_{i=1}^{k'} \mathbf{u}_i(t) \mathbf{m}_i(s) \end{aligned} \right\}, \quad \dots \quad (41)$$

where $\mathbf{n}_i(s)$ and $\mathbf{m}_i(s)$ are the solutions of the homogeneous equations associated with (34), satisfying the orthogonal relations

$$\left. \begin{aligned} \int \mathbf{n}_i(t) \cdot \mathbf{v}_j(t) dt \\ \int \mathbf{m}_i(t) \cdot \mathbf{u}_j(t) dt \end{aligned} \right\} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}, \quad \dots \quad (42)$$

while k and k' are the multiplicities of the roots λ_0 and λ_0' of $D(\lambda)$ and $D'(\lambda)$ respectively.

§ 12. *Solution in the Neighbourhood of a Singular Value of λ .*—The solutions of the boundary problems (25) as expressed by (30) in general become infinite when λ is equal to a singular parameter value λ_0 (say) in the first case and λ_0' in the second. In order that this should not be so it is necessary that the residue of the solution at this pole should vanish, that is

$$\left. \begin{aligned} \int \mathbf{f}(s) \cdot \mathbf{P}(st) ds &= 0 & \text{for } \mathbf{v}(t) \\ \int \mathbf{P}'(ts) \cdot \mathbf{f}(s) ds &= 0 & \text{for } \mathbf{u}(t) \end{aligned} \right\}. \quad \dots \quad (43)$$

These are equivalent, in virtue of (41) and the linear

* Cf. Plemelj, *Monat. für Math. und Physik*, Bd. 15 (1904), S. 127-128.

independence of the functions $\mathbf{n}_i(s)$ and of the others $\mathbf{m}_i(s)$, to the separate conditions

$$\left. \begin{aligned} \int \mathbf{f}(s) \cdot \mathbf{n}_i(s) ds &= 0, \quad i=1, 2, 3, \dots, k \quad \text{for } \mathbf{v}(t) \\ \int \mathbf{f}(s) \cdot \mathbf{m}_i(s) ds &= 0, \quad i=1, 2, 3, \dots, k' \quad \text{for } \mathbf{u}(t) \end{aligned} \right\}, \quad (44)$$

which are the usual necessary and sufficient conditions for the existence of a solution to (27) at a singular parameter value.

If these conditions are not satisfied, we can construct the function

$$\mathbf{f}_1(t) = \mathbf{f}(t) - \int \mathbf{f}(s) \cdot \mathbf{P}(st) ds \quad \dots \quad (45)$$

which satisfies the first of the conditions (43), and similarly the function

$$\mathbf{f}_2(t) = \mathbf{f}(t) - \int \mathbf{P}'(ts) \cdot \mathbf{f}(s) ds \quad \dots \quad (45')$$

which satisfies the second. This is easily verified by means of the values of $\mathbf{P}(st)$ and $\mathbf{P}'(ts)$ given by (41). Thus there do exist solutions to the *modified problems*

$$\left. \begin{aligned} \frac{1}{2} [\mathbf{W}(t^+) - \mathbf{W}(t^-)] - \frac{\lambda}{2} [\mathbf{W}(t^+) + \mathbf{W}(t^-)] &= \mathbf{f}_1(t) \\ \frac{1}{2} [\mathbf{TV}(t^-) - \mathbf{TV}(t^+)] - \frac{\lambda}{2} [\mathbf{TV}(t^-) + \mathbf{TV}(t^+)] &= -\mathbf{f}_2(t) \end{aligned} \right\}, \quad (46)$$

which are regular at the characteristic value λ_0 in the first case, and λ_0' in the second. The solutions of these problems are also expressible in the form (31). The poles of the dyadics $\mathbf{H}(tp)$ and $\mathbf{\Gamma}'(pt)$ are all simple; hence in the neighbourhoods of their respective poles, λ_0 and λ_0' , they may be written

$$\left. \begin{aligned} \mathbf{H}(tp) &= \mathbf{K}(tp) + \mathbf{P}(tp)/(\lambda_0 - \lambda) \\ \mathbf{\Gamma}'(pt) &= \mathbf{G}'(pt) + \lambda_0' \mathbf{Q}'(pt)/(\lambda_0' - \lambda) \end{aligned} \right\}, \quad \dots \quad (47)$$

where $\mathbf{K}(tp)$ and $\mathbf{G}'(pt)$ remain finite at the poles λ_0 and λ_0' respectively. It can then be shown, exactly as I have done in the case of the potential theory*, that the solutions of the modified problems (46) are given by

$$\left. \begin{aligned} \mathbf{W}(p) &= \int \mathbf{f}(s) \cdot \mathbf{K}(sp) ds \\ \mathbf{V}(p) &= \int \mathbf{G}'(ps) \cdot \mathbf{f}(s) ds \end{aligned} \right\}, \quad \dots \quad (48)$$

which are regular at the poles considered.

* Proc. Roy. Soc. Victoria, vol. xxvii. pp. 169-170 (1915).
Phil. Mag. S. 6. Vol. 32. No. 187. July 1916. D

Reverting to the equations (33) we see that the first integral, regarded as a function of p , is of the nature of a simple elastic stratum potential of density $\lambda \mathbf{H}'(tq)$. From this it follows that

$$\begin{aligned}\frac{1}{2}[\mathbf{T}\Gamma'(t^+q) - \mathbf{T}\Gamma'(t^-q)] &= \lambda \mathbf{H}'(tq), \\ -\frac{1}{2}[\mathbf{T}\Gamma'(t^+q) + \mathbf{T}\Gamma'(t^-q)] &= \chi(tq) + \lambda \int \chi(ts) \cdot \mathbf{H}'(sq) ds \\ &= \mathbf{H}'(tq).\end{aligned}$$

Adding and subtracting, we find

$$\left. \begin{aligned}\mathbf{T}\Gamma'(t^-q) &= -(1 + \lambda)\mathbf{H}'(tq), \\ \mathbf{T}\Gamma'(t^+q) &= -(1 - \lambda)\mathbf{H}'(tq).\end{aligned} \right\} \quad . \quad . \quad (49)$$

An exactly similar pair of equations (50) with undashed letters may be deduced from the first form of (33'). From the second integral in (33') which, regarded as a function of q , is of the form of a double stratum potential of moment $\lambda \Gamma(ps)$, we find in a similar manner

$$\left. \begin{aligned}\mathbf{F}(pt^+) &= (1 + \lambda)\Gamma(pt) \\ \mathbf{F}(pt^-) &= (1 - \lambda)\Gamma(pt)\end{aligned} \right\}, \quad . \quad . \quad . \quad (51)$$

showing the nature of the boundary discontinuity of the dyadic $\Gamma(pq)$, which I shall show elsewhere to be the Green's dyadic for the inner region corresponding to zero surface displacement. A similar pair of relations (51') hold for the dyadic $\mathbf{F}'(pq)$. From the preceding results and the equation (47) the following relations may be deduced * :—

$$\left. \begin{aligned}\mathbf{T}\mathbf{G}'(t^-p) &= -(1 + \lambda)\mathbf{K}'(tp) + \mathbf{P}'(tp) \\ \mathbf{T}\mathbf{G}'(t^+p) &= -(1 - \lambda)\mathbf{K}'(tp) - \mathbf{P}'(tp)\end{aligned} \right\}, \quad . \quad . \quad (52)$$

and

$$\left. \begin{aligned}\mathbf{G}(qt^+) &= (1 + \lambda)\mathbf{G}(qt) - \lambda_0 \mathbf{Q}(qt) \\ \mathbf{G}(qt^-) &= (1 - \lambda)\mathbf{G}(qt) + \lambda_0 \mathbf{Q}(qt)\end{aligned} \right\}. \quad . \quad . \quad (53)$$

§ 13. *The First Boundary Problem for one Region only.*—Consider now the boundary problems for the parameter values $\lambda = \pm 1$, which correspond to the inner and outer regions separately. Taking the first boundary problem for the *inner region* \bar{S} , ($\lambda = -1$), the moment $\mathbf{v}(s)$ of the double stratum whose vector potential solves the problem is given by the integral equation

$$\mathbf{v}(t) + \int \mathbf{v}(s) \cdot \Psi(st) ds = \mathbf{f}(t). \quad . \quad . \quad . \quad (54)$$

To prove that $\lambda = -1$ is not singular for the kernel $\Psi(ts)$,

* *Ibid.*, § 2.

it is sufficient to show that the reduced problem is incompatible ; *i. e.* that the homogeneous equation

$$\mathbf{v}(t) + \int \mathbf{v}(s) \cdot \Psi(st) ds = 0 \quad . \quad . \quad . \quad (54')$$

does not admit any solution but zero. Suppose that it does admit one, $\mathbf{v}_1(s)$. Then the potential $\mathbf{W}_1(p)$ of the double stratum of moment $\mathbf{v}_1(s)$ vanishes at the boundary of the inner region, and therefore identically throughout that region, because zero is the only regular solution of (1) vanishing over Σ . The surface traction for the displacement $\mathbf{W}_1(p)$ is therefore zero for the inner region, and being continuous at the boundary is zero for S' also. The value of $\mathbf{W}_1(p)$ must then be identically zero throughout S' . It cannot be the displacement due to a simple translation or rotation, for a double stratum displacement vanishes at infinity. It follows that

$$2\mathbf{v}_1(t) = \mathbf{W}_1(t^+) - \mathbf{W}_1(t^-) = 0.$$

Thus the homogeneous equation (54') does not admit any solution but zero, and $\lambda = -1$ is not a singular value. The equation (54) then gives a unique finite and continuous moment $\mathbf{v}(s)$, which defines a double stratum potential representing the displacement at any point p of the body S , assuming the value $\mathbf{f}(t)$ at the boundary. As a particular case of (31) the solution may be written

$$\mathbf{W}(p) = \int \mathbf{f}(s) \cdot \mathbf{H}_{-1}(sp) ds, \quad . \quad . \quad . \quad (55)$$

where the suffix denotes the value of λ involved. In virtue of (50) this may be given the alternative form

$$\mathbf{W}(p) = -\frac{1}{2} \int \mathbf{f}(t) \cdot \mathbf{T}\Gamma_{-1}(t^+p) dt,$$

the index $+$ having the usual significance.

§ 14. If the body occupies the *infinite outer region* S' the parameter value for the problem is $\lambda = +1$, and the displacement $\mathbf{W}(p)$ of the body, assuming the surface value $-\mathbf{f}(t)$, is expressible as the potential of a double stratum of moment $\mathbf{v}(s)$ given by the integral equation

$$\mathbf{v}(t) - \int \mathbf{v}(s) \cdot \Psi(st) ds = \mathbf{f}(t). \quad . \quad . \quad . \quad (56)$$

Now the parameter value $\lambda = +1$ is singular for the kernel $\Psi(st)$, the homogeneous equation

$$\mathbf{v}(t) = \int \mathbf{v}(s) \cdot \Psi(st) ds \quad . \quad . \quad . \quad (56')$$

admitting certain non-zero solutions. To see this we revert

to formula (11). If instead of p we take a boundary point t , the first member must be replaced by $2\pi\mathbf{D}(t)$, and we may write the equation

$$\mathbf{D}(t) = \int \mathbf{D}(s) \cdot \Psi(st) ds - \int \Phi(ts) \cdot \mathbf{F}(s) ds. \quad (11')$$

If, then, any displacement $\mathbf{D}(p)$ possesses zero surface traction, it satisfies the relation

$$\mathbf{D}(t) = \int \mathbf{D}(s) \cdot \Psi(st) ds,$$

which is identical in form with (56'). But the only regular deformation corresponding to zero surface traction is a translation or rotation of the body as a whole*. Now there are three independent translations, $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and three independent rotations, $\mathbf{i} \times \rho, \mathbf{j} \times \rho, \mathbf{k} \times \rho$, where ρ is the position vector of the particle p referred to the c.m. of the body, and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are taken along the principal axes of the body. These, then, are the six independent solutions of the homogeneous equations (56'). It follows then that the associated homogeneous equation

$$\mathbf{v}(t) = \int \Psi(ts) \cdot \mathbf{v}(s) ds \quad \dots \quad (57)$$

also admits six linearly independent solutions

$$\alpha_i(s), \quad (i=1, 2, \dots, 6);$$

and in order that (56) may possess a solution it is necessary and sufficient that $\mathbf{f}(s)$ be orthogonal to each of the functions $\alpha_i(s)$; that is, it must satisfy the relations

$$\int \mathbf{f}(s) \cdot \alpha_i(s) ds = 0, \quad (i=1, 2, \dots, 6) \quad \dots \quad (58)$$

If these conditions are satisfied (56) admits a solution $\mathbf{v}(s)$ which, when substituted in (26), gives a displacement for the outer region whose value at the boundary is $-\mathbf{f}(t)$.

Even when the conditions (58) are not satisfied the problem may be solved for the boundary value $-\mathbf{f}(t)$ as near as a displacement of the surface as a whole. For by § 12 there exists a solution to the modified problem corresponding to the boundary value $-\mathbf{f}_1(t)$, where

$$\begin{aligned} \mathbf{f}_1(t) &= \mathbf{f}(t) - \int \mathbf{f}(s) \cdot \mathbf{P}_{+1}(st) ds \\ &= \mathbf{f}(t) - \sum_{i=1}^6 \mathbf{v}_i(t) \int \mathbf{f}(s) \cdot \mathbf{n}_i(s) ds \\ &= \mathbf{f}(t) - [A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k} + B_1 \mathbf{i} \times \rho + B_2 \mathbf{j} \times \rho + B_3 \mathbf{k} \times \rho] \\ &= \mathbf{f}(t) - \mathbf{a} - \boldsymbol{\omega} \times \rho, \quad \dots \quad (59) \end{aligned}$$

* Cf., e. g., Marcolongo, *loc. cit.* p. 196.

where \mathbf{a} and $\boldsymbol{\omega}$ are constant vectors. This proves the statement. The solution $\mathbf{W}(p)$ to this problem is given by the double stratum potential

$$\mathbf{W}(p) = \int \mathbf{f}(t) \cdot \mathbf{K}_{+1}(tp) dt,$$

which, in virtue of (52), may be expressed in the alternative form

$$\mathbf{W}(p) = -\frac{1}{2} \int \mathbf{f}(t) \cdot \mathbf{TG}_{+1}(t^{-}p) dt.$$

§ 15. *Second Boundary Problem for one Region only.*— Consider next the second boundary problem requiring the determination of the displacement for a given value of the surface traction. In the case of the *inner region* the particular value of the parameter is $\lambda = +1$, and the required displacement for a given surface traction $\mathbf{f}(t)$ is expressible as the potential of a simple elastic stratum of density $\mathbf{u}(s)$ given by

$$\mathbf{u}(t) - \int \chi(ts) \cdot \mathbf{u}(s) ds = \mathbf{f}(t). \quad . \quad . \quad (60)$$

The value $\lambda = +1$ is, however, a characteristic number of the kernel $\chi(ts)$; for the homogeneous equation

$$\mathbf{u}(t) = \int \chi(ts) \cdot \mathbf{u}(s) ds \quad . \quad . \quad . \quad (61)$$

and its associated

$$\mathbf{u}(t) = \int \mathbf{u}(s) \cdot \chi(st) ds = \int \Psi(st) \cdot \mathbf{u}(s) ds \quad . \quad . \quad (62)$$

admit certain non-zero solutions. The last equation may be written

$$\mathbf{u}(t) = \frac{1}{2\pi} \int [\mathbf{F}_1(st)\mathbf{i} + \mathbf{F}_2(st)\mathbf{j} + \mathbf{F}_3(st)\mathbf{k}] \cdot \mathbf{u}(s) ds,$$

and that this is satisfied by any constant vector \mathbf{a} is easily verified by considering separately the three parts $\mathbf{w}_1(t)$, $\mathbf{w}_2(t)$, and $\mathbf{w}_3(t)$ as in § 6. We thus find

$$\mathbf{w}_1(t) = \mathbf{a}/(1+k), \quad \mathbf{w}_2(t) = 0, \quad \mathbf{w}_3(t) = k\mathbf{a}/(1+k),$$

showing that \mathbf{a} is a solution of (62). Further, if ρ is the position vector of the point p relative to the c.m. of the body S , and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be taken along the principal axes of inertia, it can be similarly shown that $\mathbf{i} \times \rho, \mathbf{j} \times \rho, \mathbf{k} \times \rho$ are also solutions of (62). The only independent solutions of this equation are the six vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{i} \times \rho, \mathbf{j} \times \rho, \mathbf{k} \times \rho$.

Hence, in order that (60) may admit a finite and continuous solution, it is necessary and sufficient that $\mathbf{f}(t)$ be

orthogonal to each of these six vectors. The six conditions thus expressed are equivalent to the two relations

$$\int \mathbf{f}(t) dt = 0, \quad \int \rho(t) \times \mathbf{f}(t) dt = 0, \quad . \quad . \quad . \quad (63)$$

which are the conditions of equilibrium of the body S acted on by the surface forces $\mathbf{f}(t)$. These relations must *à priori* be satisfied if the problem is to admit a solution. Then (60) admits a finite and continuous solution $\mathbf{u}(t)$ which, as density of a simple stratum, defines a potential $\mathbf{V}(p)$ representing the required displacement. This displacement is, by (31), equal to

$$\mathbf{V}(p) = \int \Gamma'_{+1}(pt) \cdot \mathbf{f}(t) dt = \frac{1}{2} \int \Gamma'_{+1}(pt^+) \cdot \mathbf{f}(t) dt. \quad (64)$$

§ 16. If the body occupies the *infinite outer region* S' , the parameter value for the problem is $\lambda = -1$, and the displacement of the body for a given surface traction $-\mathbf{f}(t)$ is expressible as the potential of a simple stratum of density $\mathbf{u}(s)$ given by the integral equation

$$\mathbf{u}(t) + \int \mathbf{u}(s) \cdot \Psi(ts) ds = \mathbf{f}(t). \quad . \quad . \quad . \quad (65)$$

Now the corresponding homogeneous equation

$$\mathbf{u}(t) + \int \mathbf{u}(s) \cdot \Psi(ts) ds = 0 \quad . \quad . \quad . \quad (66)$$

does not admit any solution but zero. For suppose that it admits a solution $\mathbf{u}_1(t)$; then the simple stratum displacement $\mathbf{V}_1(p)$ with this function as density has zero surface traction at the boundary of the outer region. The function $\mathbf{V}_1(p)$ for the region S' must therefore represent one of the degenerate displacements; and since it vanishes at infinity this displacement is identically zero throughout S' . But the function is continuous at Σ , being the potential of a simple stratum. Hence it vanishes over the boundary of the inner region, and therefore also throughout that region. The surface traction at the point t^+ is therefore zero. Thus

$$2\mathbf{u}_1(t) = \mathbf{F}(t^-) - \mathbf{F}(t^+) = 0,$$

which proves the statement.

Since then (66) does not admit any solution but zero, (65) does admit a unique finite and continuous solution $\mathbf{u}(s)$; and this function substituted in (26) gives the required solution of the problem for the outer region S' , which by (31) may be expressed in the form

$$\mathbf{V}(p) = \int \Gamma'_{-1}(pt) \cdot \mathbf{f}(t) dt = \frac{1}{2} \int \Gamma'_{-1}(pt^-) \cdot \mathbf{f}(t) dt.$$

IV. *On the High-Frequency Spectra (L-Series) of the Elements Tantalum-Uranium.* By MANNE SIEGBAHN, Dr. phil., and EINAR FRIMAN, Lic. phil.*

[Plate I.]

Introduction.

IN a former communication † the writers have given an account of some preliminary researches on the high-frequency spectra of the elements gold-uranium. We then followed the line called α , being the one best determined from the measurements by Moseley ‡ of the other elements.

This research contains a somewhat complete representation of the L-series of the heaviest elements (from tantalum to uranium). We have succeeded in finding at least 11 different line-groups. The measurements also indicate some rudimentary groups, the existence of which further investigations may decide.

We have also examined the elements polonium and radium. In the case of polonium, two of the characteristic lines α_1 and β_1 and several others were obtained. It is possible that some of these are due to impurities. With radium, only a very weak α -line could be photographed owing to the small quantity we had at our disposal (0.1 mgr.). These measurements definitely confirm the ordinals of these elements as being 84 and 88 respectively.

Experimental Arrangements.

For these experiments the X-ray spectrometer, seen in Plate I., was used §. The lead-slits, the clockwork with the rock-salt crystal, and the plate-holder are mounted on a marble plate furnished with three set-screws. The width of the slits may be changed arbitrarily. In this case the first slit was 0.1 mm. wide, and the second about 2 mm. The crystal, mounted on a small table, was adjusted by two screws, which respectively displaced the crystal and turned it about a horizontal axis. With the aid of the clockwork the crystal is turned round at a constant speed of 15° in an

* Communicated by the Authors.

† Phil. Mag. xxxi. p. 403 (1916)

‡ Phil. Mag. xxvii. p. 703 (1914).

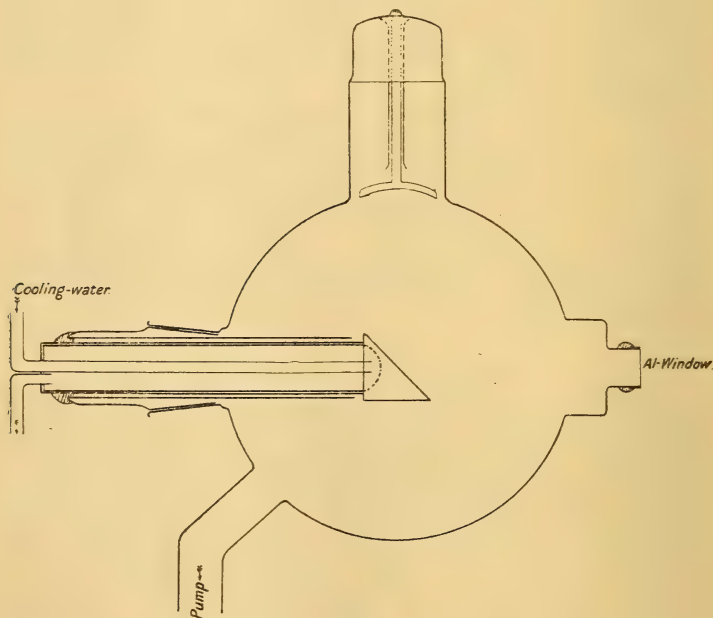
§ Some of the elements were examined with a similar apparatus of wood.

hour. The plate-holder is fastened on the marble plate with two screws, and may, by loosening the screws, be made to approach to or to recede from the crystal. By this means it is possible always to focus the rays on the middle part of the range during examination. The adjustment of the plate-holder perpendicularly to the line: slit-centre-the crystal rotation axis, is made by an optical method.

In order to screen off the photographic plate in the casket as much as possible from the scattered radiation a lead shield extending from the crystal to the casket was placed on the support seen in Pl. I.

The X-ray tube had the form seen in fig. 1. The anti-cathode, consisting of a copper cylinder fastened on with

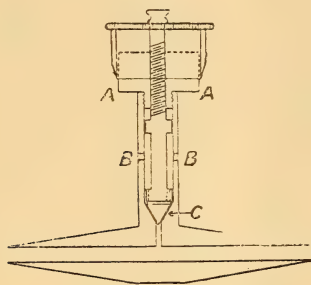
Fig. 1.



putty to the outside of the glass-grinding, was cooled with water. As window, a foil of 0.05 mm. of aluminium was used. The tube was exhausted with a Gaede molecular air-pump. To obtain a constant vacuum it proved advisable to use a valve (fig. 2), placed between the two pumps.

This consists of a micrometric [AA] adjustable cone [C] carefully ground. The air is then constantly streaming through the holes [BB] and the conic opening. The delicate regulation of the vacuum was done through altering the resistance of the air-pump motor.

Fig. 2.



The attachment of the substances on the anticathode has been already described in the case of Au-U*. Of the elements now examined Ta, Ir, and Pt in metal form were soldered on the anticathode, while Os as a salt, Ra as radium bromide, Th as thorium oxide, W and U in powder form were rubbed on it. Po was examined in the form of an electrolytic deposit on a piece of sheet-copper.

Results.

The results of the measurements are given in the Tables I.-XIV. In the first column the notation of most of the lines is to be found, and in the second the relative intensities of the lines under the assumption of the value 10 for the strongest α -line (α_1). The third column contains the glancing angles of reflexion, and the fourth the wavelengths calculated from the formula $\lambda = 2d \sin \phi$, where $\log 2d$ has been taken to be 0.75035. In the last column are the values of $\sqrt{\frac{1}{\lambda}}$. The accuracy of the measurements is estimated to about 0.3 per cent.

* Siegbahn and Friman, *Phil. Mag.* *l. c.*

TABLE I.
Tantalum.

Line.	Rel. Intens.	ϕ .	$\lambda \cdot 10^8$ cm.	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}$.
Cu α_1	10	15° 50' 20''	1·536	0·807
α_2	3	15 44 55	1·528	0·809
α_1	10	15 38 35	1·518	0·812
Cu β_1	4	14 16 30	1·388	0·849
β_4	2	13 48 30	1·343	0·863
β_1	8	13 35 45	1·323	0·869
β_3	3	13 22 55	1·303	0·876
β_2	6	13 9	1·280	0·884
γ_1	3	11 37 55	1·135	0·939
γ_2	1	11 16 50	1·101	0·953
γ_3	1	11 14 40	1·097	0·955

TABLE II.
Tungsten.

Line.	Rel. Intens.	ϕ .	$\lambda \cdot 10^8$ cm.	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}$.
Cu α_2	8	15° 51' 55''	1·539	0·806
Cu α_1	10	15 49 55	1·535	0·807
α_2	3	15 15 25	1·481	0·822
α_1	10	15 9 20	1·471	0·825
Cu β_1	9	14 16 15	1·387	0·849
β_1	3	13 19 5	1·296	0·878
β_1	8	13 7 15	1·278	0·884
β_3	3	12 55 15	1·258	0·892
β_2	6	12 44 20	1·241	0·897
γ_1	3	11 13 15	1·095	0·956
γ_2	1	10 53 35	1·064	0·969
γ_3	1	10 50 5	1·058	0·972

TABLE III.
Osmium.

Line.	Rel. Intens.	ϕ .	$\lambda \cdot 10^8$ cm.	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}$.
Cu α_2	8	15° 51' 20''	1·538	0·806
Cu α_1	10	15 49 20	1·534	0·807
α_2	3	14 22 55	1·398	0·846
α_1	10	14 16 40	1·388	0·849
β_4	3	12 27 15	1·214	0·908
β_1	8	12 15	1·194	0·915
β_3	3	12 3 50	1·176	0·922
β_2	6	11 53 15	1·167	0·926
γ_1	3	10 27 25	1·021	0·990

TABLE IV.

Iridium.

Line.	Rel. Intens.		$\lambda \cdot 10^3 \text{ cm.}$	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}.$
	1	14° 44' 40''	1.432	0.836
Cu β_1	3	14 17 50	1.390	0.848
α_2	3	13 59	1.360	0.858
α_1	10	13 52 40	1.350	0.861
Pt α_2	1	13 34 40	1.322	0.870
Pt α_1	6	13 28 15	1.311	0.873
β_4	3	12 3 45	1.176	0.922
β_1	8	11 50 15	1.154	0.931
β_3	2	11 40	1.138	0.938
β_2	6	11 36 35	1.133	0.940
Pt β_1	4	11 27 30	1.118	0.946
β_5	0	11 17 10	1.101	0.953
Pt β_2	3	11 16	1.100	0.954
γ_1	3	10 7 5	0.989	1.006
γ_2	1	9 50 25	0.962	1.020
γ_3	1	9 46 50	0.956	1.023
γ_4	1	9 22 45	0.917	1.044

TABLE V.

Platinum.

Line.	Rel. Intens.	$\phi.$	$\lambda \cdot 10^3 \text{ cm.}$	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}.$
Ir α_2	0	13° 58'	1.358	0.858
Ir α_1	3	13 52 40''	1.350	0.861
α_2	3	13 35 45	1.323	0.869
α_1	10	13 29 20	1.313	0.873
	2	13 5 35	1.275	0.885
η	2	12 44 50	1.242	0.897
Ir β_1	3	11 52 30	1.158	0.929
β_4	4	11 42 20	1.142	0.936
Ir β_2	2	11 37 45	1.134	0.939
β_1	8	11 28 35	1.120	0.945
β_2	6	11 17 10	1.101	0.953
β_3	1	11 14 50	1.098	0.954
	1	11 5 40	1.083	0.961
β_5	2	10 58 40	1.072	0.966
Ir γ_1	0	10 8 15	0.991	1.005
γ_1	3	9 48	0.958	1.022
γ_2	1	9 32 25	0.933	1.035
γ_3	1	9 0 30	0.929	1.038
γ_4	1	9 11 50	0.900	1.054

TABLE VI.—Gold.

Line.	Rel. Intens.	ϕ .	λ , 10^8 cm.	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}$.
a_2	3	13° 10' 30"	1·283	0·883
a_1	10	13 2 50	1·271	0·887
η	0	12 16 35	1·197	0·914
β_4	1	11 17 40	1·102	0·952
β_1	8	11 3 50	1·080	0·962
β_2, β_3	6 d	10 54 30	1·065	0·969
β_5	2	10 35 50	1·035	0·983
γ_1	3	9 25 35	0·922	1·042
γ_2	1	9 11	0·898	1·055
γ_3	1	9 8 35	0·894	1·057
γ_4	0	8 52 45	0·869	1·073

TABLE VII.—Mercury.

Line.	Rel. Intens.	ϕ .	λ , 10^8 cm.	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}$.
Cu a_2	3	15° 52' 30"	1·539	0·806
Cu a_1	10	14 18 55	1·392	0·847
	3	14 13 40	1·383	0·850
a_2	2	12 50 20	1·251	0·894
a_1	10	12 43 45	1·240	0·898
β_1	8	10 44 45	1·049	0·976
β_2	6	10 40 5	1·042	0·980
γ_1	3	9 9 25	0·896	1·056

TABLE VIII.—Thallium.

Line.	Rel. Intens.	ϕ .	λ , 10^8 cm.	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}$.
a_2	3	12° 28' 15"	1·215	0·907
a_1	10	12 21 40	1·205	0·911
η	0	11 30 55	1·124	0·943
	1	10 42 25	1·046	0·978
β_4	2	10 36 30	1·036	0·982
β_1	8	10 21 15	1·012	0·994
β_2	6	10 17 45	1·006	0·997
β_3	2	10 13	0·998	1·001
β_5	2	9 59 55	0·977	1·011
	0	9 22 45	0·917	1·044
γ_1	3	8 49 55	0·864	1·076
γ_2	1	8 37 40	0·844	1·088
γ_3	1	8 35 10	0·840	1·091
γ_4	1	8 15 25	0·808	1·112

TABLE IX.—Lead.

Line.	Rel. Intens.	ϕ .	$\lambda, 10^8 \text{ cm.}$	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}.$
α_2	3	12° 9' 45"	1·186	0·918
α_1	10	12 3	1·175	0·923
η	0	11 10 30	1·091	0·957
	2	10 28 30	1·023	0·989
β_4	2	10 19 5	1·008	0·996
β_1, β_2	9 d	10 3 35	0·983	1·009
β_3	2	9 54 5	0·968	1·016
	2	9 44 30	0·855	1·081
γ_1	3	8 36 20	0·842	1·090
γ_2	1	8 22 45	0·820	1·104
γ_3	1	8 20 15	0·816	1·107
γ_4	1	8 5 20	0·792	1·124

TABLE X.—Bismuth.

Line.	Rel. Intens.	ϕ .	$\lambda, 10^8 \text{ cm.}$	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}.$
	0	12° 0' 25"	1·171	0·924
α_2	3	11 49 5	1·153	0·931
α_1	10	11 43 25	1·144	0·935
η	1	10 50 35	1·059	0·972
	2	10 9 25	0·992	1·004
β_4	2	9 59 55	0·977	1·012
β_2	6	9 45 35	0·954	1·024
β_1	8	9 43 10	0·950	1·026
β_3	2	9 34 50	0·937	1·033
β_5	2	9 26 20	0·923	1·041
γ_1	3	8 16 40	0·810	1·111
γ_2	1	8 6 45	0·794	1·122
γ_3	2	8 4 15	0·790	1·125
γ_4	0	7 46 45	0·762	1·146

TABLE XI.—Polonium.

Line.	ϕ .	$\lambda, 10^8 \text{ cm.}$	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}.$
	12° 43' 40"	1·240	0·898
Pb α_1	12 3 35	1·176	0·922
	11 45 35	1·147	0·934
α_1	11 21 35	1·109	0·950
	10 46 50	1·053	0·975
	10 41	1·043	0·979
Pb β_1	10 3 10	0·982	1·009
	9 48 50	0·959	1·021
β_1	9 24 45	0·920	1·042
	9 9	0·895	1·057
	8 56 45	0·875	1·069
	8 50 40	0·865	1·075

TABLE XII.

Radium.

Line.	ϕ .	$\lambda \cdot 10^8 \text{ cm.}$	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}.$
α_1	10° 20' 30"	1.010	0.995

TABLE XIII.

Thorium.

Line.	Rel. Intens.	ϕ .	$\lambda \cdot 10^8 \text{ cm.}$	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}.$
α_2	3	9° 54' 50"	0.969	1.016
α_1	10	9 47 25	0.957	1.022
	2	8 28 58	0.830	1.098
β_2	6	8 7 45	0.797	1.121
β_1	8	7 49 20	0.766	1.143
β_3	2	7 44 30	0.758	1.149
γ_1	4	6 40 30	0.654	1.236
$\gamma_{2,3}$	1	6 28 35	0.635	1.255

TABLE XIV.

Uranium.

Line.	Rel. Intens.	ϕ .	$\lambda \cdot 10^8 \text{ cm.}$	$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}.$
α_2	3	9° 25' 35"	0.922	1.042
α_1	10	9 19	0.911	1.048
	2	8 1 40	0.786	1.128
β_2	7	7 43 15	0.756	1.150
	2	7 38 25	0.748	1.156
	2	7 23 45	0.724	1.175
β_1	7	7 20 50	0.720	1.179
β_3	1	7 14 50	0.710	1.187
γ_1	4	6 16 25	0.615	1.275
	0	6 11 25	0.607	1.284
$\gamma_{2,3}$	0	6 4 25	0.596	1.296

The wave-lengths of the 11 certain line-groups are put together in Table XV., and in fig. 3 the values of the ratios of $\frac{1}{\lambda}$ to the ordinals are graphically represented.

TABLE XV.
 λ . 10^8 cm.

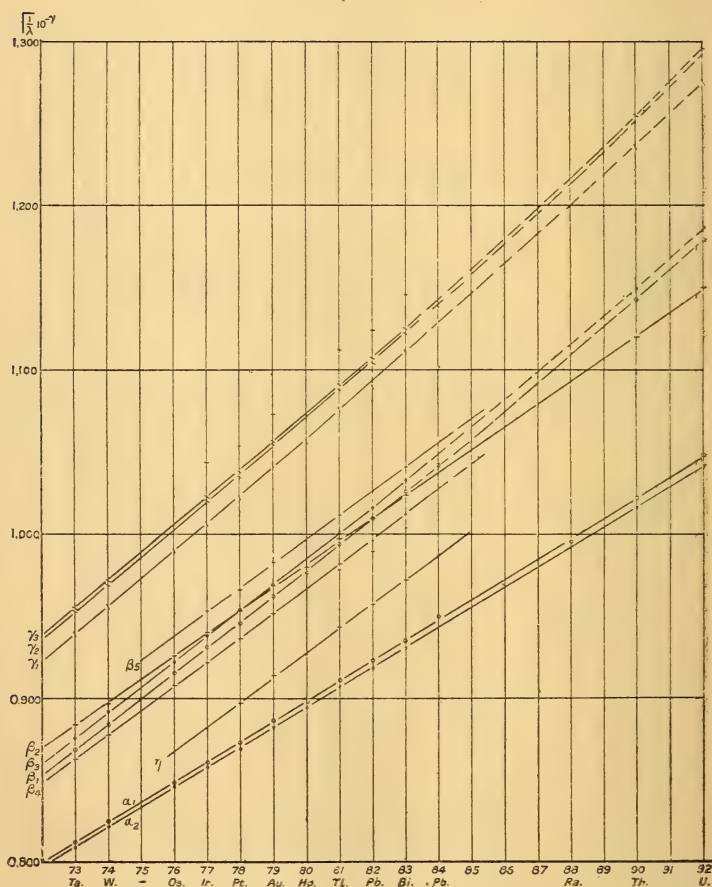
Element.	a_2 .	a_1 .	η .	β_4 .	β_1 .	β_2 .	β_3 .	β_5 .	γ_1 .	γ_2 .	γ_3 .
73 Ta	1.528	1.518	...	1.343	1.323	1.280	1.303	...	1.135	1.101	1.097
74 W	1.481	1.471	...	1.296	1.278	1.241	1.258	...	1.095	1.064	1.058
76 Os	1.398	1.388	...	1.214	1.194	1.167	1.176	...	1.021		
77 Ir	1.360	1.350	...	1.176	1.154	1.133	1.138	1.101	0.989	0.962	0.956
78 Pt	1.323	1.313	1.242	1.142	1.120	1.101	1.098	1.072	0.958	0.933	0.929
79 Au	1.283	1.271	1.197	1.102	1.080	1.065	1.065	1.035	0.922	0.898	0.894
80 Hg	1.251	1.240	1.049	1.042	0.896		
81 Tl	1.215	1.205	1.124	1.036	1.012	1.006	0.998	0.977	0.864	0.844	0.840
82 Pb	1.186	1.175	1.091	1.008	0.983	0.983	0.968	...	0.842	0.820	0.816
83 Bi	1.153	1.144	1.059	0.977	0.950	0.954	0.937	0.923	0.810	0.794	0.790
84 Po	1.109	0.920						
88 Ra	1.010									
90 Th	0.969	0.957	0.766	0.797	0.758	...	0.654	0.635	
92 U	0.922	0.911	0.720	0.756	0.710	...	0.615	0.596	
α	0.0121	0.0123	0.0140					
N_0	6.09	6.97	9.92					

As the graphical representation shows, some of the groups (α_2 , α_1 , β_2) form right lines over the whole range, while others (β_1 , β_3 , γ_1 , γ_2 , γ_3) are slightly curved upwards. In the first case a Moseley relation,

$$\sqrt{\nu} = a(N - N_0),$$

is sufficient. The values of a and N_0 for these groups are

Fig. 3.



given in Table XVI., N_0 seeming to be an integer*. Fig. 3 shows, further, that α_2 and α_1 are nearly parallel, likewise β_1 , β_3 , and γ_1 , γ_2 , γ_3 . The two groups of strongest β -lines, β_1 and β_2 , intersect one another in the case of Pb.

* Rydberg, Phil. Mag. xxviii. p. 144 (1914).

A comparison with the values of Moseley for tantalum-gold shows that his α -, β -, and γ -lines correspond to our α_1 , β_1 , and β_2 . His values, however, are practically all about 1 per cent. greater than ours.

In their excellent research on the soft γ -rays from radium B Rutherford and Andrade * have determined the spectrum of these rays after a similar method to the writers'. They have also shown that lead, the characteristic radiation of which was excited by the β -rays from an emanation-tube, gave the same characteristic lines as radium B. In this way the isotopy of radium B and lead was directly experimentally proved. As Rutherford and Andrade, in the case of radium B, have found a great many lines, a comparison with the lines found by the writers for lead may be of interest. This is given in Table XVI. The agreement is, as seen, very good, especially for the strongest lines.

TABLE XVI.

Radium B. Soft γ -ray spectrum (Rutherford and Andrade).		Lead. X-ray spectrum (Siegbahn and Friman).	
Angle of reflexion.	Intensity.	Angle of reflexion.	Intensity.
$^{\circ}$ 12 16	m	$^{\circ}$ 12 9.8	3
12 3	s	12 3	10
11 42	m		
11 17	f	11 10.5	0
11 0	f		
10 48	f		
10 32	m	10 28.5	2
10 18	m	10 19.1	2
10 3	s	10 3.6	9
		9 54.1	2
9 45	m	9 44.5	2
9 23	f		
8 43	m		
8 34	m	8 36.3	3
		8 22.8	1
8 16	m	8 20.3	1
8 6	m	8 5.3	1

For an evaluation of the numerical results, further investigations of a greater range are necessary. Such researches are at present going on in this laboratory.

Physical Laboratory,
University of Lund.
Feb. 10, 1916.

* Phil. Mag. xxvii. p. 854 (1914).

V. *Electric Discharge in a Transverse Magnetic Field.*
 By Prof. D. N. MALLIK, B.A., Sc.D., F.R.S.E., and
 A. B. DAS, M.Sc.*

[Plate II.]

1. **I**N a paper in the Philosophical Magazine, Oct. 1908, one of us (D. N. M.) considered the behaviour of an electric discharge in a tube of De La Rive's pattern, under gradually decreasing pressure. It was there shown that, starting with a pressure inside the tube equal to that of the atmosphere, the discharge is at first in the nature of a shower of sparks, filling the whole tube; that these various streams combine into a single band or single thick stream, when the pressure is sufficiently low; that only when the discharge is of this kind is there rotation under the action of a magnetic field, as in the usual De La Rive apparatus. It was also noted that, as the pressure is further reduced, the band is gradually changed into a shower again and the rotation ceases.

2. In a subsequent paper (Oct. 1912) a theoretical explanation of the various phenomena was attempted, and it was shown that they are connected with the fact that the relative number of positive and negative ions present in the tube varied at these different stages.

3. The object of the present paper is to discuss, in detail, the behaviour of the discharge when the rotatory stage is passed. It results both from theory and experiment that the phenomena are dependent on (1) the voltage of the primary of the induction-coil, (2) the nature of the coil, (3) the difference of potential between the electrodes of the discharge-tube, (4) their distance, (5) and (6) the pressure and the nature of the gas or vapour in the tube, and (7) the nature of the electrodes.

It has been our object to investigate the manner in which the phenomena are severally dependent on them.

4. That the variation in the character of the discharge depends on the induction-coil used to produce the discharge and the distance between the electrodes in the discharge-tube is clearly seen from the photographs (Pl. II. figs. 1, 2, 3, 4).

* Communicated by the Authors. Paper based on brief notes communicated to the second and the third All-India Science Congresses (1915 and 1916).

Fig. 1. "Band" discharge which rotates when the electro-magnet is excited. (Pressure 55 millimetres.)

Fig. 2. "Glow" discharge at pressure 2 millimetres. (The induction-coil (A) used in these cases gives a spark-length in atmospheric air of 29 millimetres.)

Figs. 3 & 4 are photographs of the discharge at the same pressures as in 1, 2 respectively, with the induction-coil (B) giving a spark-length in atmospheric air of 13.4 millimetres. The discharge-tube (No. 1) is the same in both cases. In the latter group, no band discharge appears, and consequently the rotatory effect is absent throughout the entire range. With a smaller tube (No. 2), however, and the induction-coil (B) the discharge shows the various types referred to on page 50.

5. It is *a priori* evident that there must be a definite relation between pressure in the tube, the voltage of the induction-coil, and the length of rotatory discharge. What the nature of this relation should be would appear from the following considerations.

If n is the number of corpuscles per unit length (along x) per unit area of cross-section of a discharge, then the equation of continuity (J. J. Thomson, 'Conduction of Electricity through Gases') is

$$\frac{\partial n}{\partial t} + \frac{\partial(nu)}{\partial x} = \frac{nu}{\lambda} \left[f\left(\frac{Xe}{p}\right) - \beta \right],$$

where u is the velocity of a corpuscle ;

λ = mean free path ;

f is a function, which determines the number of collisions resulting in ionization, of the mean kinetic energy $Xe\lambda$ of an ion given to it by the electrical field of intensity X ;

β is the fraction of collisions resulting in recombination ;

p = pressure ;

e = charge on an ion (+ and -).

Now, for steady rotation both $\frac{dn}{dt}$ and $\frac{\partial nu}{\partial x}$ must vanish, and we must have

$$f\left(\frac{Xe}{p}\right) - \beta = 0 \quad \text{or} \quad f\left(\frac{Xe}{p}\right) - \beta = 0.$$

$$\therefore \frac{Xe}{p} = \beta', \text{ say.}$$

If we assume β' to be of the form $\frac{a}{p} + b$, the X, p curve is a straight line, and if a is small, $\frac{X}{p} = \text{const.}$

As X and probably also f' and β depend on the distance between the electrodes and the voltage of the induction-coil, there must be an exact relation between pressure &c. determining the condition under which the discharge becomes rotatory. This exact quantitative relation is under investigation.

6. Starting now from the stage at which the discharge is rotatory, we reach, when the pressure is gradually reduced, a stage at which the character of the discharge changes as in figs. 5, 6, 7.

7. Comparing the figs. 5 and 6, which give the initial and the final stages of a discharge at the same pressure, we observe that the illumination is very slight when the discharge is first passed, but that after a time it becomes much more marked. The effect is evidently due to the fact that when the discharge is first passed there is only slight ionization, but that, as the discharge is continued, ionization increases and the consequent illumination. 7 corresponds to a lower pressure than 6.

8. When this stage is reached, the ring-end of the discharge is found to spread over a finite length of the ring electrode, instead of being confined practically to one point in it.

9. As the pressure is further reduced, there is a further spreading out as in fig. 7.

10. If at these stages the electromagnet is excited, there is *dispersal of the streams* constituting the discharge, which is, at the same time, *twisted about the Faraday dark space* on either side of it.

Figs. 8 and 9 show the effect of the magnetic field on fig. 6, which is a photograph of the discharge before the electromagnet is excited.

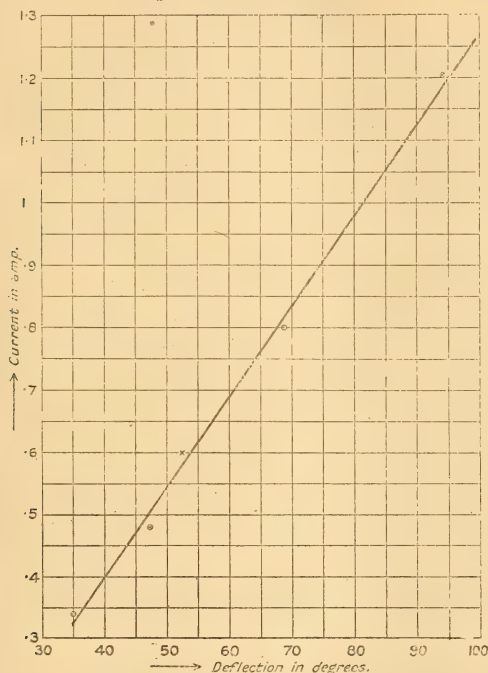
Similarly figs. 10 and 11 correspond to fig. 7 of the non-magnetic field. In 8 and 11 the disk is the cathode, and in 9 and 10 the ring is the cathode.

11. If we admit that the twist is due to an angular displacement of ions about the axis of the electromagnet, this behaviour of the Faraday dark space [10] must be due to the fact that it is a region practically devoid of ions.

12. Fig. 12*a*, curve I. gives the amount of twist for different current strengths in the coil of the electromagnet. It is thus easily seen that since the magnetic force is

proportional to current strength, within the limits of the actual experiment, the twist is also proportional to the intensity of

Fig. 12 a. Curve I.



[*Nature of the discharge.*—When the induction-coil is started, nearly the whole space is dark, and then a bright band with a dark space near the cathode gradually appears.]

the magnetic force. Fig. 12 b, curve II. gives the amount of twist for different current strengths at a lower pressure. It is seen that the corresponding twists are greater in this case than those at a higher pressure.

13. An approximate theory of the magnetic action of the excited magnetic field on these discharges can be worked out as follows :—

Using cylindrical coordinates, z, ρ, θ (where z is measured from the Faraday dark space [11]), the equations of motion of an ion may be written, if m is its mass,

$$m\ddot{z} + A\dot{z} = Ze + He\rho\dot{\theta}, \quad . \quad . \quad . \quad (1)$$

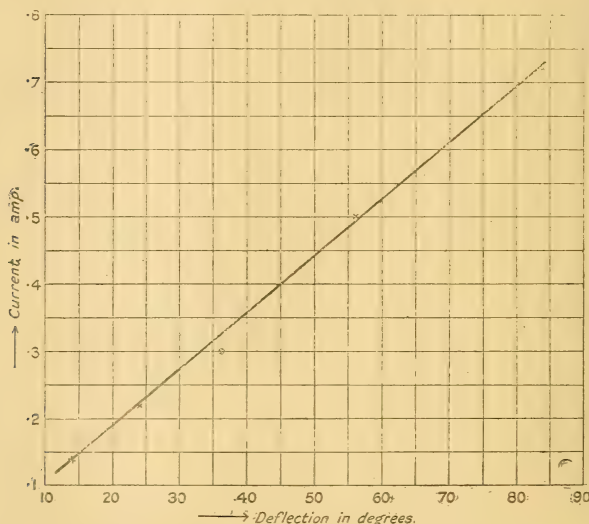
$$m(\ddot{\rho} - \rho\dot{\theta}^2) + A\dot{\rho} = R, \quad . \quad . \quad . \quad (2)$$

$$m\frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) + A\rho\dot{\theta} + B\theta = He\dot{z}, \quad . \quad . \quad . \quad (3)$$

where A =coeff. of viscosity, B a coeff. to be determined,

Z =electric force in the direction of z and R in the direction of ρ , while H =magnetic force, which we know is mainly in the direction of ρ (Phil. Mag. Jan. 1908).

Fig. 12 b. Curve II.



[*Nature of the discharge.*—Showery, with very fine striæ. It takes a certain time for the striæ and illumination to develop after the induction-coil is started.]

Now, considering the equation (3) (to which, alone, we shall confine ourselves), if we have N negative and n positive ions per unit length, in any stream (masses m_1 and m_2 respectively), we have, taking moments about the axis and summing up,

$$\int (m_1 N + m_2 n) ds \frac{d}{dt} (\rho^2 \dot{\theta}) + \int (A_1 N + A_2 n) \rho^2 ds \dot{\theta} + \int B \rho (N + n) ds \theta = \int H \rho e (N + n) ds \dot{z} \quad (4)$$

as the equation of motion of any stream of discharge.

Now we may assume, as in Phil. Mag. Oct. 1912, the action between two streams of lengths ds, ds' to be a repulsion

$$= \frac{e^2 ds ds'}{K r^2} \left[(N - n)^2 - \frac{(nq + Nq')^2}{3V^2} \right] + \frac{\pi \rho}{r^3} (na^2 q - Nb^2 q')^2,$$

where q, q' are the velocities of positive and negative ions, a, b their radii (assumed spherical), r the distance between ds, ds' , K the S.I.C. and V the velocity of light.

The third term of (4) will then be of the form

$$Cf(\alpha)(n-N)^2\theta,$$

the other terms (depending on the velocities) being neglected. Here C is a constant depending on the form of the various streams of discharge, and α the angular coordinate defining the position of the stream, whose equation of motion is given by (4), provided n and N are constant throughout the discharge, observing that in this case, alone, θ will be the same for all points. In any case, if $n=N$, the equation of motion is of the form

$$I\ddot{\theta} + \mu\dot{\theta} = \int H\rho i ds$$

= the couple acting on the discharge due to the magnetic action of the electromagnet,

where
$$I = \int (m_1 + m_2)n ds \rho^2,$$

since $\dot{\rho} = 0$, in the steady state.

But this couple = $\frac{3}{2}Mi$, where M is the total magnetic strength of induced magnetism (Phil. Mag. Oct. 1908).

Therefore we have

$$I\ddot{\theta} + \mu\dot{\theta} = \frac{3}{2}Mi,$$

where
$$\mu = (A_1 + A_2)n \int \rho^2 ds.$$

This is the same equation as was obtained in a previous paper by identifying the discharge (which is in the form of fig. 1) with an electric current.

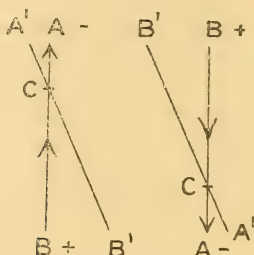
14. If the number of positive particles is small in comparison with that of negative particles, the number of the latter will not necessarily be constant throughout any stream of discharge. In this case, putting $n=0$ and considering the motion of a small element of a discharge, we have, when the steady stage is reached,

$$B\theta = He\dot{z} \quad \text{or} \quad \theta = \frac{He\dot{z}}{B}, \quad . \quad . \quad . \quad (5)$$

where B is a function of ρ, α , defining the position of the element of the discharge considered. This completely explains the twists described in para. 10.

Comparing figs. 8 and 11 with figs. 9 and 10, one might be led to suppose that the twist is independent of the direction

of the discharge. This is not really the case, as is seen from the annexed diagram.



A is the cathode, B is the anode, C the dark space.

Curve II. is also easily explained by the fact that at a lower pressure λ is greater than at a higher pressure.

15. The equation (5) shows that the angular displacement of a corpuscle is proportional to H and is dependent on its distance from the axis as well as on its angular position. The latter explains the dispersal of the rays [10]. The equation (3) also shows that there will be a time-factor; that is, the final effect will be reached only gradually. Experiment bears this out also.

16. When the pressure is further reduced, the discharge becomes striatory (Pl. II. fig. 13). On the introduction of the magnetic field, the Faraday dark space shortens in length, the number of striæ increases, the distance between consecutive striæ decreases, and they also bulge out (fig. 14).

The effect of increasing the magnetic field is to increase the number of striæ and further shorten the length of the Faraday dark space, as well as that between consecutive striæ.

17. On a still further reduction of pressure, all the striæ disappear and the negative glow extends from the cathode to nearly the whole of the tube (fig. 15).

At this stage, the discharge is luminous at first, but the luminosity decreases as the discharge continues to pass, in consequence, evidently, of a gradual recombination of ions.

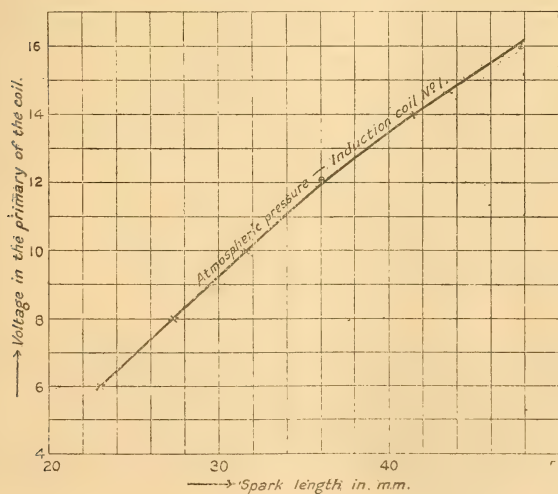
18. When the electromagnet is excited, the striæ make their appearance, and the tube is illuminated with a reddish light (figs. 16 & 17). Fig. 17 shows the effect of increased magnetic field.

19. When the pressure is very low (about $\frac{1}{1000}$ of a millimetre) the negative glow reaches the sides of the vessel and the whole tube becomes phosphorescent, owing evidently to the impact of corpuscles on the sides of the tube.

On the excitation of the magnetic field, the phosphorescent effect becomes concentrated around the cathode, while the anode is surrounded by a red luminosity, the line of demarcation between the two regions being clearly marked.

20. A characteristic variation of the potential difference with pressure during these changes is (as was noted in a previous paper) the most marked feature of these experiments. In order to make a more detailed study of this variation than was attempted previously—through all ranges of pressure—it was necessary, in the first place, to standardize the E.M.F. available with the induction-coils A and B. For this, we observed the spark-distance between two brass spheres, each of radius 3 cm., when the voltage of the primary of the induction-coil was gradually changed, the spring of the interrupter being kept at a constant tension. The result is plotted in curve III., fig. 18, in which ordinates represent spark-lengths in millimetres and the abscissa the voltage of the primaries of the induction-coil A.

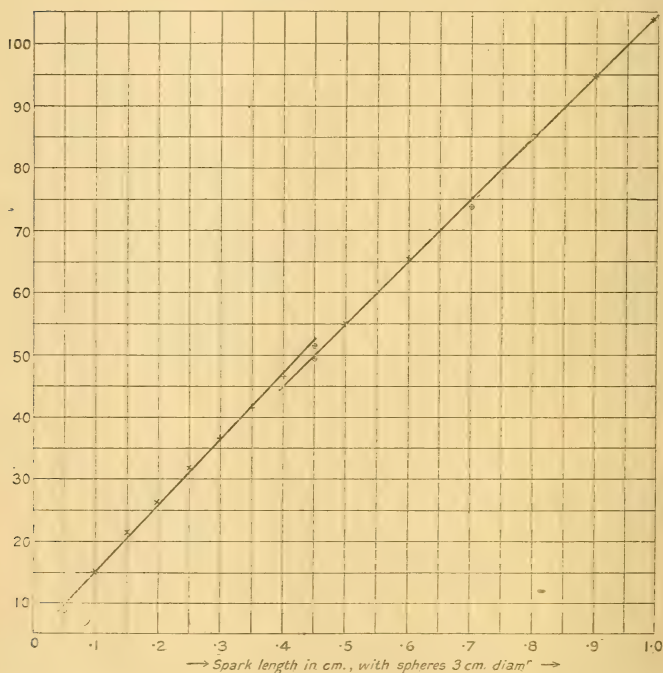
Fig. 18. Curve III.



21. Comparing these with the result given in page 461 of J. J. Thomson's 'Discharge of Electricity through Gases,'

and similarly plotted in curve IV., fig. 19, we come to the conclusion that the voltage of the secondary of an induction coil, under the conditions of these experiments, is fairly accurately measured by the distance of the minimum spark-gap, across which it forces a discharge and is practically proportional to this distance.

Fig. 19. Curve IV.



[Data from Baillie's results—J. J. Thomson's 'Conduction of Electricity through Gases,' 2nd ed. page 461.]

It follows also that the potential difference between the electrodes of the discharge-tube is measurable in terms of the minimum spark-gap with which it is parallel. In this way, the potential difference between the electrodes of the discharge-tube at different pressures of the contained air has been determined.

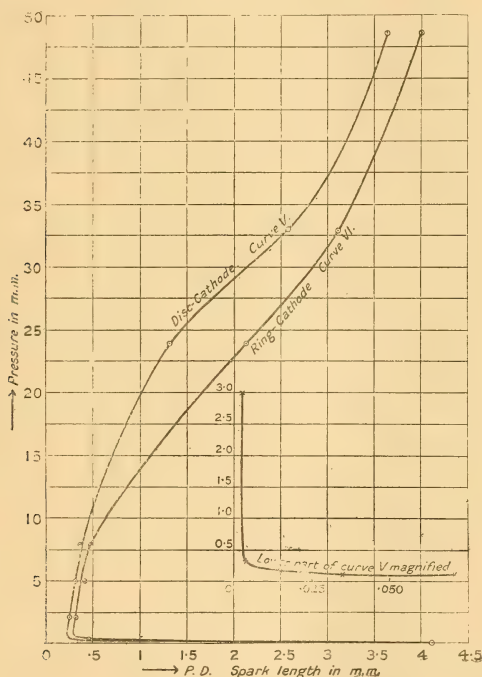
22. Fig. 20, curve V. represents changes in the P.D. with pressure, when the disk is the cathode, the spark-length of the induction-coil in this case being 13.4 mm. in atmospheric air.

Curve VI. represents the relation between these quantities under similar circumstances with the ring as cathode.

These curves correspond to the series of discharges typically represented by figs. 3 and 4. The pressures to which these figures actually relate are, however, comparatively low.

Throughout this series of changes of pressure, the rotatory stage is absent.

Fig. 20. Curves V. & VI.

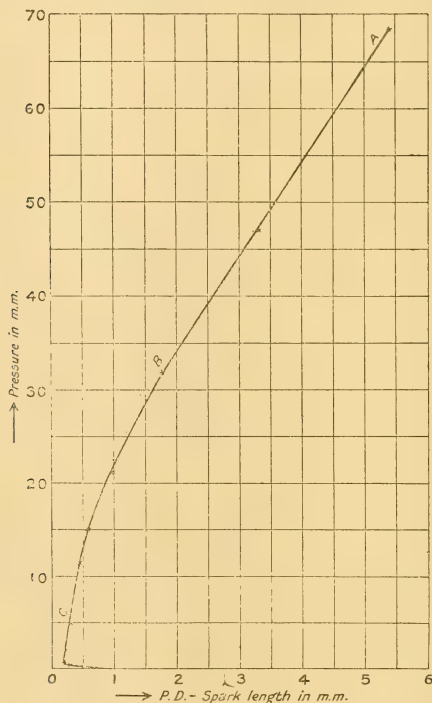


23. In fig. 21, curve VII. represents changes in P.D. with pressure between the electrodes of the same tube with disk as the cathode, the spark-length of the induction-coil in this case being 29 mm. in atmospheric air. It corresponds to the series of discharges typified by figs. 1 and 2.

It will be observed that the latter includes a stage at which discharge has, during a certain range of pressure, the appearance of a band, and that there is rotation, *only when* this state is reached, under the influence of a radial magnetic force.

The straight part A, B of curve VII. corresponds to the rotatory stage of which fig. 1 is the type. The bent part B, C of curve VII. corresponds to the range of pressure over which the discharge undergoes only a twist [10] under the magnetic force.

Fig. 21. Curve VII.



It will be seen that in curves V., VI., the straight portion corresponding to A, B in curve VII., is absent.

24. We conclude, therefore, that only when there is "band" discharge (which alone rotates under the influence of radial magnetic field), is the ratio of potential difference to pressure constant. This is in accordance with art. 5.

It is further worthy of note that the potential difference goes on decreasing as pressure decreases up to a certain point; but after that, it increases with decreasing pressure, so that at very low pressures the potential difference is comparable to what it is at very high pressures.

It follows also from these experiments that the potential difference at any given pressure, although it depends on the

voltage of the induction-coil, is materially affected by the circumstances of the discharge.

25. All these points seem to be capable of explanation on such considerations as the following :—

Let V_0 be the voltage of the induction-coil; then the energy supplied per unit of time by the coil will be proportional to the V_0 , say $i_0 V_0$, where i is the current in the circuit.

Let V be the potential difference between the electrodes; then the energy supplied to the electrodes per unit of time will be proportional to $V = i' V$, say.

Therefore $i_0 V_0 = i' V + \text{energy carried away by the positive and negative ions, thrown off from the electrodes, less the energy carried to the electrodes by positive and negative ions reaching them (per unit of time).}$

But the energy carried off by an ion $= Xe\lambda$.

Therefore,

$$i_0 V_0 = i' V + Xe(Nq'\lambda' + nq\lambda) - E,$$

where n and N are the numbers of positive and negative ions thrown off from the electrodes and occupying unit length of the discharge, and λ, λ' their mean free paths.

In order to find E , we may proceed as follows:

It can be shown that the equations of continuity in a discharge-tube can be written, in the steady state,

$$\left. \begin{aligned} \frac{\partial Nq}{\partial x} &= \alpha Nq' + \gamma nq, \\ -\frac{\partial (nq)}{\partial x} &= \alpha Nq' + \gamma nq, \end{aligned} \right\} \dots \dots \dots (6)$$

where
$$\left. \begin{aligned} \alpha &= \frac{1}{\lambda'} f(Xe\lambda' - \beta') \\ \gamma &= \frac{1}{\lambda} [F(Xe\lambda - \beta)] \end{aligned} \right\}, \dots \dots \dots (7)$$

and n, N the number of positive and negative ions per unit length of discharge, x being measured along the line of discharge.

Therefore, we have

$$Nq' + nq = \text{const.} = \frac{i}{e},$$

where i is the current carried by the discharge.

Again, the energy carried to the cathode by the positive ions may be written equal to

$$\frac{eVx}{d} e^{-kx} (\alpha Nq' + \gamma nq),$$

where k is a coefficient determining the dissipation of energy during the passage of these ions.

Also, the energy carried to the anode by the negative ions may similarly be written equal to

$$\frac{eVx'}{d} e^{-k'x'} (\alpha Nq' + \gamma nq)$$

(J. J. Thomson's 'Conduction of Electricity through Gases,' 2nd ed. p. 490),

where d = distance between the electrodes ;

V = difference of potential between the electrodes assumed to vary uniformly.

$$\therefore E = \frac{eV}{d} \int (\alpha Nq' + \gamma nq) x (e^{-kx} + e^{-k'x'}) dx.$$

But from (5) and (6), if α, γ be regarded as constant,

$$(\alpha N'q' + \gamma nq)(\alpha - \gamma) = \alpha \frac{\partial Nq'}{\partial x} + \gamma \frac{\partial nq}{\partial x};$$

$$\therefore \alpha Nq' + \gamma nq = (\alpha N_1q_1' + \gamma n_1q_1) e^{(\alpha - \gamma)x} \quad \dots \quad (8)$$

if $N = N_1, n = n_1$ at the cathode.

Hence,

$$\begin{aligned} E &= \frac{Ve}{d} (\alpha N_1q_1' + \gamma n_1q_1) \int_0^d x e^{(\alpha - \gamma)x} (e^{-kx} + e^{-k'x'}) dx \\ &= \frac{Ve}{d} (\alpha N_1q_1' + \gamma n_1q_1) \left[\frac{d e^{(\alpha - \gamma - k)d}}{\alpha - \gamma - k} - \frac{e^{(\alpha - \gamma - k)d} - 1}{(\alpha - \gamma - k)^2} + \right. \\ &\quad \left. \text{similar terms in } k' \right] \\ &= X e (\alpha N_1q_1' + \gamma n_1q_1) P, \text{ say.} \end{aligned}$$

If $\alpha = \gamma = 0$, the above equation reduces to $E = 0$. This we may suppose to be the case during the rotatory stage in air [Phil. Mag. Oct. 1912]. Therefore, since in this case $N = n$, and the pressure varies inversely as mean free path, we get the equation

$$i_0 V_0 - i' V = ne X \lambda' \left(q' + q \frac{\lambda}{\lambda'} \right).$$

But $i_0 = i' + i = i' + ne(q + q')$;

$$\therefore V_0 - \frac{i'}{i_0} V = neX\lambda' \frac{q' + q \frac{\lambda}{\lambda'}}{i' + ne(q + q')}.$$

As, moreover, during this stage V is small compared with V_0 , and i' should be small compared with i_0 , we get the simple equation

$$V_0 = \frac{X}{p} \left[\frac{q' + q \frac{\lambda}{\lambda'}}{q + q'} \right] \quad \text{or} \quad \frac{X}{p} = \text{const. nearly.}$$

This, as we have seen, is the case in air (curve VII.); when, however, the pressure is sufficiently reduced, α , γ are no longer zero. In fact, the terms in E become sufficiently effective in making V large, as is found to be the case (art. 24), since α , γ , k , k' are all proportional to pressure, and it is reasonable to suppose $\alpha < \gamma$, remembering that $\alpha \propto \frac{1}{\lambda'}$ and $\gamma \propto \frac{1}{\lambda}$.

26. Although it is not possible to work out completely the theory of this variation of pressure without a knowledge of α , γ , k , k' , we may get some insight into its nature in special cases by proceeding as follows:—

From (8), we have

$$(\alpha N_1 q_1' + \gamma n_1 q_1) e^{(\alpha - \gamma)d} = (\alpha N_2 q_2' + \gamma n_2 q_2) \quad . \quad . \quad (9)$$

if $N = N_2'$, $n = n_2$ at the anode, $q = q_2$ and $q = q'/q_2'$;

$$\text{but} \quad e(N_2 q_2' + n_2 q_2) = i = e(N_1 q_1' + n_1 q_1) \quad . \quad . \quad (10)$$

If, now, $n_2 = 0$,

$$\frac{\alpha i}{\alpha - \gamma} e^{(\gamma - \alpha)d} - \frac{\gamma i}{\alpha - \gamma} = e N_1 q_1'.$$

We have also

$$i_0 V_0 - i' V = X e N_1 q_1' [\lambda' - \alpha P], \text{ since } n_2 = 0;$$

and since $\frac{i'}{i_0}$ is small,

we get X as an exponential function of p .

It is obvious, however, that the above investigation is not capable of giving a complete account of the variation of the potential difference, for we have assumed (1) that the potential varies uniformly from cathode to anode, and (2) that α , γ are constant. As neither of these suppositions can be true always, it is not surprising that the curves obtained are more complicated than those given by theory.

27. It is not without interest to follow the variation of current with pressure and potential difference.

Curves VIII., IX., fig. 22, give the relation between current and pressure in a discharge-tube (No. 1), the induction-coil used giving a spark-length, in air, of 13.4.

Fig. 22. Curves VIII. & IX.

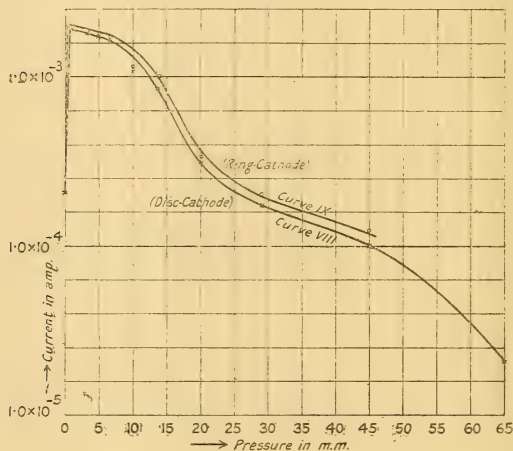
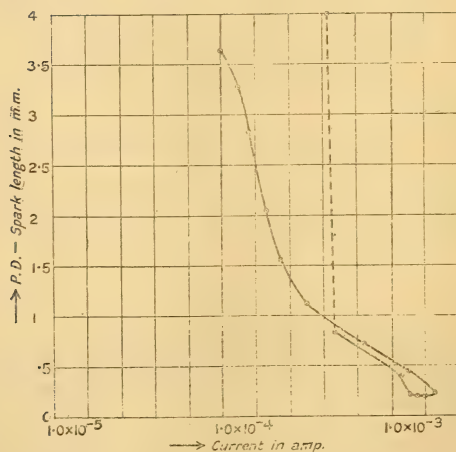


Fig. 23 Curve. X.



The curve X., fig. 23, gives the relation between the current and potential difference.

From (9), (10) we have

$$en_2q_2 = \frac{\alpha i}{\alpha - \gamma} - \left(eN_1q_1' + \frac{\gamma i}{\alpha - \gamma} \right) e^{(\alpha - \gamma)d}.$$

We have also

$$iV_0 = Xe(N_1q_1'\lambda' + n_2q_2\lambda) - E;$$

so that a third relation is required in order that we should know N_1 , n_2 , and i .

According to J. J. Thomson's theory, when there is only one kind of ion in the tube,

$$\sqrt{i} = aV + b.$$

It is our intention to discuss the experimental results in their bearing on this and other theories in a future paper.

Our thanks are due to the authorities of the Presidency College, Calcutta, for affording us facilities for carrying on the above investigations.

VI. Results of Crystal Analysis.

By L. VEGARD, *Dr. phil., University of Christiania* *.

[Plate III.]

§ 1. **I**N a previous paper † I gave an account of the crystalline structure of silver, as determined by the Bragg reflexion method. The X-ray spectrometer was in principle the same as that constructed by Bragg, only differing with regard to details which were mentioned in the paper.

Since then the work has been continued, and some of the results obtained will be given in this paper.

Besides some results concerning the structure of gold and lead, which were announced in a paper read before "Kristiania Videnskapselskap" on November 19, 1915, the present paper will chiefly deal with the more elaborate and complicated case, the determination of the structure of the Zircon group, represented by the minerals zircon (ZrSiO_4), rutile (TiO_2)₂, and tinstone (kassiterite) (SnO_2)₂, which are, as far as I know, the first cases of tetragonal crystals which have yet been analysed ‡.

Not being aware of the fact that the Spinel group recently has been analysed by W. H. Bragg §, I have also made an

* Communicated by the Author.

† L. Vegard, *Phil. Mag.* Jan. 1916.

‡ An account of the analysis of the Zircon group was given in a lecture in *Kristiania Vid. Selsk.* March 10, 1916.

§ W. H. Bragg, *Phil. Mag.* Aug. 1915, p. 305.

analysis of the structure of this group. The following spectra have been measured:—Magnetite (111), (110), and (100); spinel (111), (110), and (100); gahnite (111). The experimental results, as well as the lattice constructed from them, were in close agreement with the results of Bragg; so a more detailed account will be superfluous.

§ 2. *The Structure of Gold and Lead.*

The elements copper, silver, gold, and lead all have crystals which belong to the holohedral class of the cubic system, and from a crystallographic point of view we should expect gold and lead to have a similar space-lattice to that found for copper * and silver †; but still I think an actual determination will be of interest, as several lattices might give the right symmetry.

The gold crystals used for the experiments were kindly lent me by Professor W. C. Brögger. The one specimen had the form of an octahedron, but as it had linear dimensions of the order of only one millimetre, we did not with our instrument detect any reflexion from it. The specimen used had the common form of a thin plate, twinned about its principal face (111).

The crystal plate being quite thin, we only got reflexion from the face (111).

The lead crystal was produced artificially. Several methods were tried—*e. g.*, a gradual cooling of the molten substance, and sublimation of the metal in an electric furnace; but although crystals were formed, they mostly consisted of branches made up of small individual crystals, but we got no crystal face fit for our purpose.

The method which proved most successful was to let lead precipitate on a piece of zinc from a solution of lead-acetate.

In this way we got crystal leaves formed in a similar way to the gold plates with the principal face (111), which gave quite a strong reflexion.

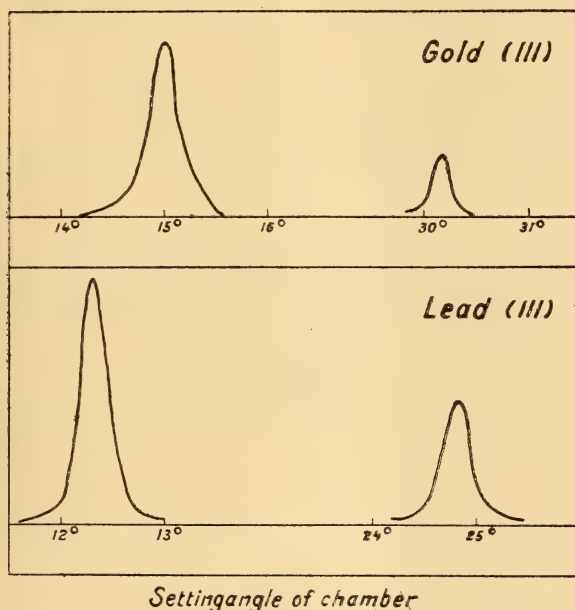
In fig. 1 are given the relative strength and the position of the reflexion maxima for the face (111) of gold and lead as observed with narrow slits (0.4 mm.). The normal variation of intensity with increasing order shows that the "point-planes" parallel to the face (111) are equal and equidistant, and in the simple case of a cubic crystal *with one sort of atoms there can only be one lattice, which satisfies this condition and gives the right glancing angle.*

* W. H. Bragg, *Phil. Mag.* xxviii. (1915) p. 355.

† L. Vegard, *loc. cit.*

For a given lattice we can, as shown by W. H. and W. L. Bragg, calculate the glancing angle (θ) when we know the density (ρ) of the crystal, the atomic weight (A), and the wave-length (λ) of the X-rays.

Fig. 1.



Let the side of the elementary cube be a , and the number of atoms associated with a cube of side a be n , then

$$a^3 = \frac{nA}{\rho N}, \quad \dots \dots \dots (1)$$

where N is the number of atoms in a gram-atom ($N = 61.5 \times 10^{23}$).

Let the spacing of the (111) planes be

$$d_{111} = \epsilon a,$$

then

$$\epsilon a = \frac{\lambda}{2 \cdot \sin \theta_1}, \quad \dots \dots \dots (2)$$

where θ_1 is the glancing angle of the first order.

From (1) and (2) we get

$$\sin \theta_1 = \frac{1}{\epsilon \sqrt[3]{n}} \cdot \frac{\lambda}{2} \sqrt[3]{\frac{\rho N}{A}}.$$

For the simple cube lattice (a) : $n = 1$, $\epsilon = \frac{1}{\sqrt{3}}$.

For the cube-centred lattice (b) : $n = 2$, $\epsilon = \frac{1}{2\sqrt{3}}$.

For the face-centred lattice (c) : $n = 4$, $\epsilon = \frac{1}{\sqrt{3}}$.

A lattice like that of diamond ($n = 8$, $\epsilon = \frac{1}{\sqrt{3}}$) is excluded on account of the normal distribution of intensities.

Calling the glancing angles in the three cases θ_a , θ_b , θ_c , we get

$$\sin \theta_a = \sin \theta_b \sin \theta_c.$$

In Table I. are given the glancing angles for gold and lead calculated for the three lattices, and also the observed values, which are in perfect agreement with the values calculated on the assumption of a *face-centred lattice*.

TABLE I.

	Calculated.			Observed.
	θ_a .	θ_b .	θ_c .	θ_1 .
Gold	11° 55'	19° 08'	7° 28'	7° 26'
Lead	9° 47'	15° 38'	6° 09'	6° 09'·5

Thus it is proved that gold and lead crystals have the same lattice as copper and silver.

§ 3. The Structure of the Zircon Group.

The mineral zircon is a compound with the chemical formula ZrSiO_4 . It may be considered as an addition product of equivalent portions of the two dioxides (ZrO_2 , SiO_2), or as the Zr-salt of an acid of Si corresponding to a formula $\text{Zr}(\text{SiO}_4)$.

The zircon crystals belong to the tetragonal system of the bipyramidal class. Isomorphous with zircon are found a number of substances, of which the following are the best known :

Rutile (TiO_2)₂, kassiterite (SnO_2)₂, and thorite (ThSiO_4), the latter being analogous to zircon.

The determination of the structure of these substances will be of special interest also for the reason that there are a

number of analogous dioxides which show quite a different crystal form; and in some cases the very same dioxide occurs in several forms. Thus the titanium dioxide (TiO_2) is found in three modifications, viz., the two tetragonal forms rutile and anatase, and the rhombic bipyramidal modification brookite.

Generally we should expect that the determination of the crystalline structure of isomeric substances would lead to important results, and especially in cases where we know the energy of transformation from one modification to another; for this energy should equal the difference of potential energy of the system of atoms in the two modifications; and from this equality we may expect to be able to draw valuable information with regard to the law of the forces acting between the atoms, or the forces which constitute the chemical binding.

The present paper will deal only with the determination of the structure of the zircon group; but I hope to be able to extend the investigation also to the other modifications, and to treat the more general atomic problem mentioned above.

The specimens of *zircon* crystals at my disposal had only the faces (110) and (111) well developed. The *rutile* and *kassiterite* crystals, of which we had very fine specimens, had the faces (100), (110), (111), (101), but no face parallel to the base (001)*.

In cases where the crystal has no face parallel to the planes from which the reflexion is to be found, we have generally† been able to find an edge parallel to the plane in question, and the crystal has been mounted with this edge horizontal and the reflexion-plane vertical.

In this way the reflexions from the zircon planes (100), (101), (001), and the plane (001) of rutile have been determined†. In this case, however, we cannot claim quite the same accuracy for the glancing angle and the intensity as when the reflexion is found from a plane crystal surface; but still the accuracy will be sufficient for our present purpose.

§ 4.

The results of observations are given in Table II. and graphically represented in fig. 2. For each of the three minerals the glancing angles and intensity for several orders have been found for the five reflexion-planes (110), (100), (111), (001), (101)†.

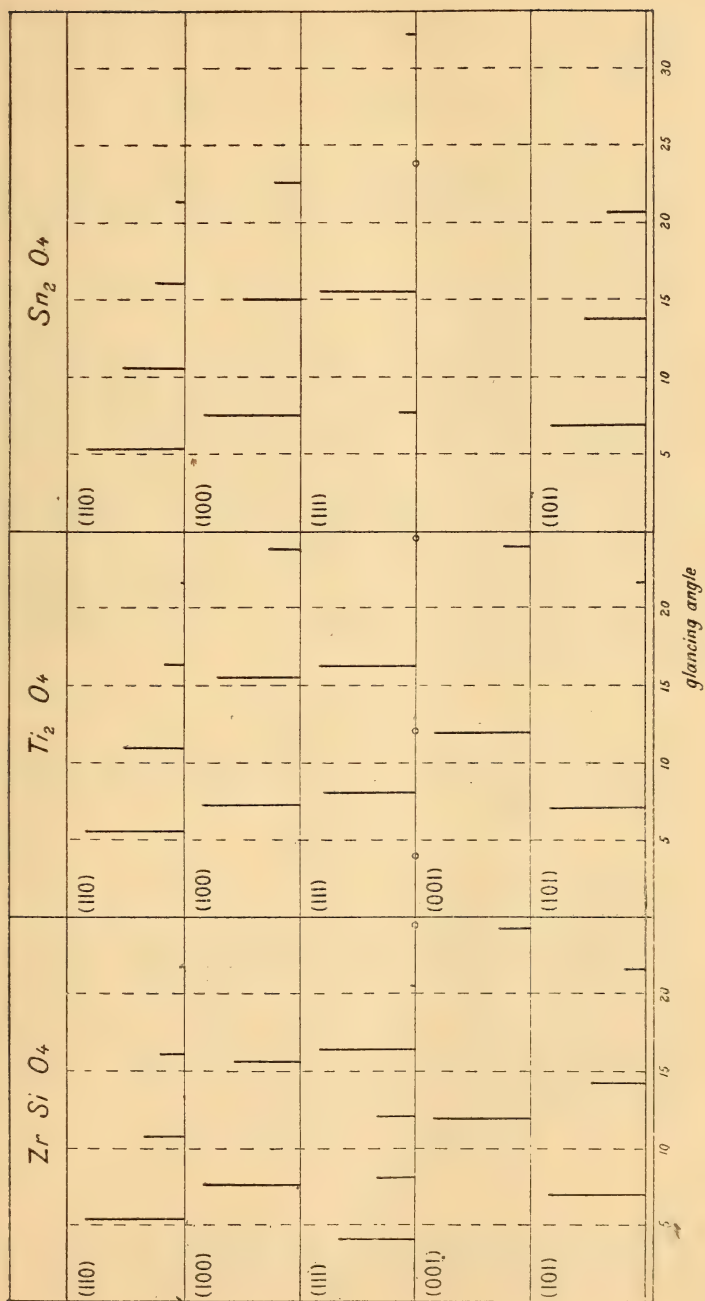
* All crystals were kindly lent me by Professor W. C. Brögger, of the Mineralogical Laboratory.

† With the exception of the (001) plane of Sn_2O_4 .

TABLE II.

Substance.	Face.	Order.						
		1.	2.	3.	4.	5.	6.	8.
Zircon ($ZrSiO_4$)	{ (110)	5° 21' 1° 00	10° 45' 0° 42	16° 13' 0° 26	21° 50' 0° 04			
	{ (100)	7° 43' 1° 00	15° 35' 0° 68					
	{ (111)	4° 1' 0° 80	8° 3' 0° 40	12° 3' 0° 40	16° 15' 1° 00	20° 30' 0° 05	24° 50' 0	
	{ (001)	11° 54' 1° 00	24° 21' 0° 30					
	{ (101)	7° 4' 1° 00	14° 15' 0° 56	21° 40' 0° 08 (?)				
Rutile ($TiO_{2/2}$)	{ (110)	5° 26' 1° 00	10° 55' 0° 62	16° 30' 0° 20	22° 18' 0° 03			
	{ (100)	7° 44' 1° 00	15° 36' 0° 84	23° 47' 0° 35	32° 28' 0° 03			
	{ (111)	— 0	8° 5' 1° 00	— 0	16° 20' 1° 00	0	25° 0' 0	34° 15' 0° 03
	{ (001)	11° 57' 1° 00	24° 28' 0° 28					
	{ (101)	7° 4' 1° 00	14° 14' 0° 06	21° 40' 0° 11	29° 22' 0° 01			
Kassiterite ($SnO_{2/2}$) ...	{ (110)	5° 14' 1° 00	10° 30' 0° 62	15° 51' 0° 30	21° 22' 0° 11			
	{ (100)	7° 28' 1° 00	15° 4' 0° 60	23° 0' 0° 23	31° 20' 0° 04 (?)			
	{ (111)	— 0	7° 39' 0° 17	— 0	15° 26' 1° 00	-- 0	23° 32' 0	32° 12' 0° 13
	{ (001)	—	—	—	—			
	{ (101)	6° 42' 1° 00	13° 30' 0° 64	20° 29' 0° 40	27° 49' 0° 23			

Fig. 2.



We observe the position of the chamber. If the position angle of the chamber for spectra of the order n and i be α_n and α_i , then

$$\cot \theta_i = \frac{n}{i} \operatorname{cosec} \frac{\alpha_n - \alpha_i}{2} - \cot \frac{\alpha_n - \alpha_i}{2} (3)$$

For each face the intensity of the strongest maximum is put equal to 1. Thus it is only the intensities of different orders corresponding to one and the same face which should be comparable.

The intensities of the reflexion are measured in the following way:—

The slit, which the primary beam has to pass before striking the crystal, is made quite narrow, while the slit in front of the chamber is kept open, and the ionization is measured for angles which are near to the glancing angle of the spectrum in question.

The maximum ionization current (when the ionization of the "white radiation" has been subtracted) is taken as a measure of the intensity of reflexion.

In order to make certain that the intensities thus measured correspond to the same strength of the primary beam, the intensity measurements for each face were carried out rapidly and in symmetrical order.

§ 5. Interpretation.

The interpretation is based on the fundamental equation of Bragg, combining the spacing d , the glancing angle θ , and the wave-length λ :

$$n\lambda = 2d \sin \theta_n (4)$$

From the values of θ given in Table II. we can calculate the spacing for any reflexion-face.

In Table III. are given the absolute values of $d_{100} = d_{010}$ and d_{001} , as calculated from (4) ($\lambda = 0.607 \times 10^{-8}$). Column 3 gives the ratios $\frac{d_{001}}{d_{100}}$ and column 4 the ratios c'/a' of the crystallographic axes, as taken from P. Groth, *Chemische Krystallographie*.

We see that in the case of zircon and rutile,

$$\frac{d_{001}}{d_{100}} = \frac{c'}{a'}.$$

In the case of kassiterite the reflexion from (001) has not been found ; but from the similarity between the spectra of

the three substances, there can be no doubt that also for kassiterite the ratio $\frac{d_{001}}{d_{100}}$ is equal to c'/a' .

The elementary cell has a volume

$$d_{100}^2 d_{001} = d_{100}^3 \frac{c'}{a'},$$

and the number of molecules n associated with this volume is

$$n = \frac{N\rho}{M} \frac{c'}{a'} d_{100}^3; \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

ρ is the density, N the number of molecules in a gram-molecule, M is the molecular weight; and in order to preserve the analogy between the three substances, we write the molecular formula for rutile Ti_2O_4 and for kassiterite Sn_2O_4 .

The number n is given in the last column of Table III.; and we see that within the limit of experimental error the number n is equal to $1/8$ for all three substances, or in a rectangular prism with sides $2d_{100}$, $2d_{100}$, and $2d_{001}$ there should be *just one molecule*.

TABLE III.

Substance.	d_{100} .	d_{001} .	$\frac{d_{001}}{d_{100}}$.	$\frac{c'}{a'}$.	n .
ZrSiO ₄	cm. 2.30×10^{-8}	cm. 1.47×10^{-8}	0.640	0.639	0.124
Ti ₂ O ₄	2.26 „	1.46 „	0.646	0.644	0.123
Sn ₂ O ₄	2.335 „	0.123

Before proceeding further in our attempt to arrange the atoms, we shall make a few remarks regarding our elementary lattice, and the formation of compound lattices and some of their properties.

Our elementary lattice will be a prism with one atom in each corner. Let the side of its square base be a , and height c ; then the spacing of a simple elementary lattice would be

$$d_{100} = a, \quad d_{001} = c, \quad d_{110} = \frac{a}{\sqrt{2}},$$

$$d_{101} = \frac{a}{\sqrt{1 + \left(\frac{a}{c}\right)^2}}, \quad d_{111} = \frac{a}{\sqrt{2 + \left(\frac{a}{c}\right)^2}}. \quad . \quad (6a)$$

In describing the compound lattices it may be convenient to refer the lattices to a rectilinear coordinate system. We let the origin coincide with one of the atomic centres, and the z -axis be parallel to the tetragonal axis and the x - and y -axes parallel to the other sides of the lattice.

Any other elementary lattice which may be made to cover the primary one by a simple translatory movement is completely determined with regard to position by giving the coordinates of one of its points. Very often it is most convenient to give the coordinates of the point nearest to the origin, which we shall call the *point of construction*.

Thus a face-centred lattice is made up of four simple lattices with the following construction-points :

$$(000), \quad (a/2, a/2, 0), \quad (0, a/2, c/2), \quad (a/2, 0, c/2),$$

and the spacings :

$$\begin{aligned} d_{100} &= a/2, & d_{001} &= c/2, & d_{110} &= \frac{a}{2\sqrt{2}}, \\ d_{101} &= \frac{a}{2\sqrt{1 + \left(\frac{a}{c}\right)^2}}, & d_{111} &= \frac{a}{\sqrt{2 + \left(\frac{a}{c}\right)^2}}. \end{aligned} \quad (6b)$$

The lattice corresponding to the cube-centred lattice has construction-points

$$(000) \quad \text{and} \quad (a/2, a/2, c/2),$$

and spacings

$$\begin{aligned} d_{100} &= a/2, & d_{101} &= c/2, & d_{110} &= \frac{a}{\sqrt{2}}, \\ d_{101} &= \frac{a}{\sqrt{1 + \left(\frac{a}{c}\right)^2}}, & d_{111} &= \frac{a}{2\sqrt{2 + \left(\frac{a}{c}\right)^2}}. \end{aligned} \quad (6c)$$

A lattice analogous to that of diamond is composed of two parallel face-centred lattices with the construction-points

$$(000) \quad \text{and} \quad (a/4, a/4, c/4),$$

and has the following spacings :

$$\begin{aligned} d_{100} &= a/4, & d_{001} &= c/4, & d_{110} &= \frac{a}{2\sqrt{2}}, \\ d_{101} &= \frac{a}{2\sqrt{1 + \left(\frac{a}{c}\right)^2}}, & d_{111} &= \frac{a}{\sqrt{2 + \left(\frac{a}{c}\right)^2}}. \end{aligned} \quad (6d)$$

Returning to our crystals, we shall first consider the most general type, zircon. In the cell $(2d_{100}) \cdot (2d_{100}) \cdot (2d_{110})$ there should be placed

one atom of Zr, one atom of Si, and four oxygen atoms.

The simplest way in which we might arrange the atoms of Zr and Si in the cell would be to suppose the Zr atoms arranged in a simple elementary lattice, for which we put $a = 2d_{100}$ and $c = 2d_{001}$, and the Si atoms in a similar lattice with construction-point $(a/2, a/2, c/2)$; but such an arrangement would make the spacing of the (100) and (001) planes twice as large as they are actually observed.

The spacings being inversely proportional to $\sin \theta$, we can easily make a comparison with the observed angles and those to be expected for the four types of lattices previously constructed, and we notice at once that the observed spacings have a relation to each other similar to those of equation (6 *d*) corresponding to the diamond type of lattice, and a calculation will show that within the limit of experimental error the ratios expressed in (6 *d*) are satisfied by observations, or

$$\left. \begin{aligned} \sin \theta_{100} : \sin \theta_{001} : \sin \theta_{110} : \sin \theta_{1\bar{1}1} : \sin \theta_{111} \\ = 4 : 4 \frac{a}{c} : 2\sqrt{2} : 2\sqrt{1 + \left(\frac{a}{c}\right)^2} : \sqrt{2 + \left(\frac{a}{c}\right)^2} \\ = 4 : 6.26 : 2.83 : 3.71 : 2.11. \end{aligned} \right\} \quad (7)$$

From equations (6 *d*) we find for the sides of the elementary lattice :

$$a = 4d_{100},$$

$$c = 4d_{001}.$$

Now in a lattice of the diamond type there are 8 points associated with a volume equal to that of the elementary lattice (a^2c), and consequently in a cell $(a/4, a/4, c/4)$ there should be $1/8$, or exactly the number which is found for Zr or Si atoms in the elementary cell $d_{100}^2 d_{001}$.

This leads us to the assumption that in zircon the Zr and Si atoms are arranged in a lattice of the diamond type.

Regarding the relative position of the Zr and Si lattices, it is not necessarily determined from symmetry. The only condition to be fulfilled is that corresponding points of the two lattices must be situated on the tetragonal axis, or the position of one must be derived from the position of the other by a translation parallel to the c -axis. If the construction-point for the Zr atoms has the coordinates (000), the

coordinates for the construction-point of the Si lattice will be $(00l)$, where $0 < l < c$.

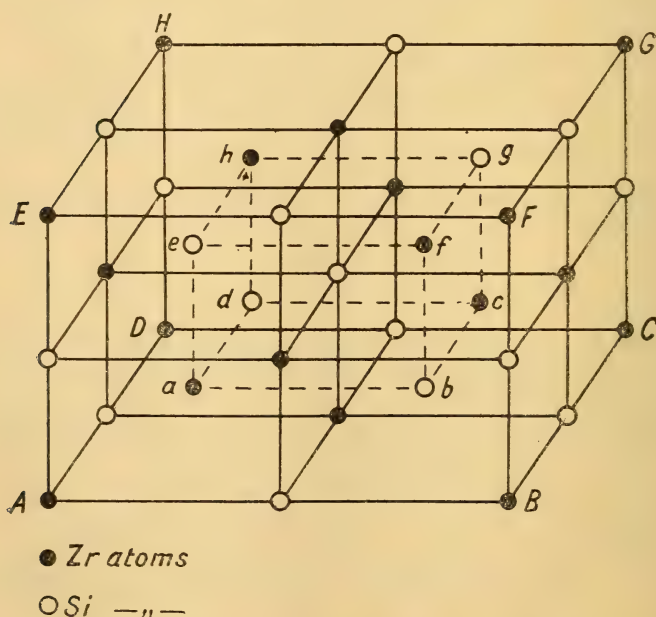
To get further, we must take into account the distribution of intensities for different spectra. An important fact in this connexion is that the face (001) gives a normal distribution of intensities. This fact limits the possible values of l to $c/4$ or $c/2$.

The assumption $l = c/4$, however, leads to consequences in conflict with observations. Thus it cannot explain that the face (101) gives a nearly normal distribution of intensities, and we have as the only possibility :

$$l = c/2.$$

The arrangement of the Zr and Si atoms is shown in fig. 3. We might also come to this lattice in another way. It might be considered as composed of two lattices of the rock-salt type with points of construction (000) and $(a/4, a/4, c/4)$, corresponding to the diamond type.

Fig. 3.



From the isomorphism of the three minerals we should expect the lattices of the metal atoms to be found by substitution of the atoms in the zircon lattice with corresponding

atoms of the other substances. In both cases—for Ti_2O_4 as well as for Sn_2O_4 —the Zr and Si atoms are to be substituted with the same sort of atoms. This will considerably alter the type of the lattice and make it much simpler. In fact, the metal atoms will be arranged in a prism-centred lattice with sides $2d_{100}$, $2d_{100}$, $2d_{001}$; but in order to preserve the analogy with zircon, we shall suppose the lattice formed in the same way from an elementary lattice, $a=4d_{100}$, $c=4d_{001}$.

From the equations (6 *c*) and (6 *d*) we see that the ratios for the spacings (100) (001) (110) (101) should be unaltered; but for the face (111) the spacing would in comparison be four times smaller than in the case of zircon, or the *sine* of the first-order glancing angle four times as large.

As a matter of fact this does not occur.

It is only the spectra corresponding to the orders 1, 3, 5, &c. of zircon which have vanished for the (111) face in the case of TiO_4 and Sn_2O_4 , and the spacings have the following ratios:

$$d_{100} : d_{001} : d_{110} : d_{101} : d_{111} \\ = 1/4 : \frac{c}{4a} : \frac{1}{2\sqrt{2}} : \frac{1}{2\sqrt{1+(\frac{a}{c})^2}} : \frac{1}{2\sqrt{2+(\frac{a}{c})^2}} \quad (8)$$

This apparent discrepancy, however, is not fatal to the correctness of the assumed arrangement of the metal atoms; for as a matter of fact the first-order spectrum for the face (111) in the case of Ti_2O_4 and Sn_2O_4 is *produced entirely by the oxygen atoms*.

§ 6. *The Arrangement of the Oxygen Atoms.*

In the case of zircon the observed spacings for all the reflexion planes considered are just the same as we should get from the lattice of Zr and Si without the oxygen atoms; hence it follows that the arrangement of the oxygen atoms must be determined from the intensity measurements. Only in the case of rutile and kassiterite the maxima of order 1, 3, 5, &c. for the face (111) should be due entirely to the oxygen atoms.

The problem before us is to arrange the oxygen atoms in the Zr-Si lattice—with four atoms to each pair of (Zr-Si) atoms—in such a way that the distribution of intensities of the reflexion maxima is explained and the crystallographic symmetry accounted for.

There are two different types of arrangements according to the view we take with regard to the chemical nature of the substances. The one view is expressed in the formulæ: $\text{Zr}(\text{SiO}_4)$, $\text{Ti}(\text{TiO}_4)$, $\text{Sn}(\text{SnO}_4)$; and if these were the right expressions, the structure—even in the case of rutile and kassiterite—should distinguish between two sorts of metallic atoms.

According to the other view, zircon is considered as a kind of addition product of two dioxides ZrO_2 , SiO_2 ; and if we substitute Zr and Si with either Ti or Sn, we should expect to get a lattice where all Ti or Sn atoms are equal.

In accordance with the first view, we should arrange four atoms of oxygen round each of the Si atoms in accordance with the symmetry of the crystal.

We have to consider the following three arrangements:—

(1) The four O-atoms are placed along the tetragonal axis through the Si atom, and with the O-atoms symmetrically arranged on both sides of the Si atom.

(2) The four O-atoms are arranged in a plane through the Si atom perpendicular to the C-axis with tetragonal symmetry with regard to this axis. There are two different arrangements. If we take the Si atom as origin the O-atoms will be situated in the (xy) plane and will either have the coordinates:

$$(l, l), (-l, l), (l, -l), (-l, -l), \text{ or } (l, 0), (0, l), (0, -l), (-l, 0),$$

where l is a parameter.

(3) The oxygen atoms are arranged on the diagonals AG, BH, &c., fig. 3. Let the construction-points of the two face-centred Si lattices be (000) and $(a/4, a/4, c/4)$, then the construction-points of the oxygen lattices would be:

$$\begin{aligned} &(\epsilon_1 a, \epsilon_1 a, \epsilon_1 c), \quad (-\epsilon_1 a, -\epsilon_1 a, \epsilon_1 c), \quad (-\epsilon_1 a, \epsilon_1 a, -\epsilon_1 c) \\ &(\epsilon_1 a, -\epsilon_1 a, -\epsilon_1 c), \text{ and} \\ &\left(\frac{a}{4} + \epsilon_2 a, \frac{a}{4} + \epsilon_2 a, \frac{c}{4} + \epsilon_2 c\right), \quad \left(\frac{a}{4} - \epsilon_2 a, \frac{a}{4} - \epsilon_2 a, \frac{c}{4} + \epsilon_2 c\right), \\ &\left(\frac{a}{4} - \epsilon_2 a, \frac{a}{4} + \epsilon_2 a, \frac{c}{4} - \epsilon_2 c\right), \quad \left(\frac{a}{4} + \epsilon_2 a, \frac{a}{4} - \epsilon_2 a, \frac{c}{4} - \epsilon_2 c\right). \end{aligned}$$

To get the right spacing for the face (100), ϵ_1 and ϵ_2 must have the same numerical value; but still we have to distinguish between the two cases:

$$(3 a) \quad \epsilon_1 = \epsilon_2,$$

$$(3 b) \quad \epsilon_1 = -\epsilon_2.$$

This arrangement will be clear by noticing that if $a=c$ the oxygen atoms would be arranged in the corners of tetrahedra with the Si atoms at the centres.

None of these arrangements, however, is able to explain the experiments. Thus the types (1), (2), and (3 *a*) would make the second order spectrum of the face (111) vanish, contrary to observations.

The question whether the oxygen atoms might be arranged according to (3 *b*) is of fundamental importance with regard to the properties of the atoms. If the atoms were arranged in this way, *the tetragonal lattice could be considered as a cubic lattice compressed in the direction of one of the principal axes, and there would be nothing in the geometry of the lattice to explain its tetragonal form.*

The compression of the lattice would be due to *symmetry properties of the centres (atoms)*, and under the conditions present the atoms would exert a different force in the direction of the tetragonal axis from that in a direction perpendicular to it.

The arrangement (3 *b*), however, cannot be accepted although it gives a finite value for the second order spectrum of the face (111); but no value of ϵ will satisfactorily explain the actual intensity-distribution observed. Thus in order to account for the disappearance of the first order spectrum of (111) for rutile and kassiterite, we must put $\epsilon = -1/8$. With this value of ϵ the amplitude in the case of rutile would be given by the formula :

$$A_{111} = 1 + 1.38 \cos n \frac{\pi}{2},$$

which would give a ratio of the intensity of the first to that of the second order equal to about 16 : 1, while the spectra actually observed are in the ratio 1 : 1.

In a similar way we can show that we cannot arrange the oxygen atoms round each of the Zr atoms, when the Zr atoms are to take up a central position in groups of four O-atoms.

Let us then try to associate two oxygen atoms with each Si and Zr atom. The O-atoms must be situated on a straight line through the Si or Zr atom considered and at equal distances on both sides of it; but the distance from a Si atom to the two neighbouring O-atoms may not be equal to the corresponding distance for the Zr atom.

Then the determination of the lattice under these conditions would involve the determination of the two distances (parameters) and the orientation of the lines through the

Zr and Si atoms relative to the axis of the crystal and relative to each other.

With regard to the direction of the lines, let us first determine their position with regard to the tetragonal axis, and consider the following three cases:—

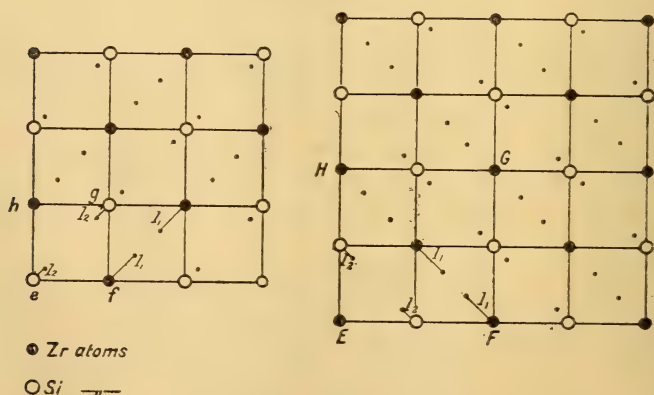
(1) All lines are parallel to the tetragonal axis. This orientation is excluded because for zircon it would make the second order spectrum of face (111) vanish.

(2) The lines belonging to one sort of atoms (Zr say) might be parallel to the tetragonal axis, the lines of the other sort perpendicular to this axis. The latter lines are divided into two groups in such a way that an individual of one group is perpendicular to one of the other. Such an arrangement would not explain the distribution of the intensities of the face (111) for zircon and the disappearance of the first order spectrum of the (111) face of rutile and kassiterite.

(3) All lines are perpendicular to the tetragonal axis. Let us consider the lines through the Zr atoms. The lines through the atoms belonging to one of the face-centred Zr lattices must be perpendicular to the lines through the atoms of the other face-centred lattice. In order to preserve the right spacing for the faces (100) and (111), the lines must be drawn so as to halve the angle between the sides (a) of the square base of the lattice.

The lines through the Si atoms must be arranged in the same way.

Fig. 4.



With regard to the relative position of the lines through the Zr atoms and those through the Si atoms, there are two different possibilities, which will be apparent from fig. 4,

which gives the distribution of points in two consecutive planes (001).

In a plane (001) either all lines may be parallel as shown in the figure, or the lines through Si can be drawn perpendicular to those through the Zr atoms. The latter arrangement is excluded as it would not explain the distribution of intensities of the (111) face.

We shall then consider the arrangement represented in fig. 4, which, as will be seen, with a proper choice of the parameters ϵ_1 and ϵ_2 , will give the right lattice for the *zircon group*. This lattice is composed of 12 face-centred lattices with the construction-points which are given in Table IV.

TABLE IV.

Atom.	x .	y .	z .
Zr	0	0	0
	$-a/4$	$a/4$	$c/4$
Si	$a/2$	0	0
	$a/4$	$a/4$	$c/4$
O associated with Zr..	$\epsilon_1 a$	$\epsilon_1 a$	0
	$-\epsilon_1 a$	$-\epsilon_1 a$	0
	$-a/4 + \epsilon_1 a$	$a/4 - \epsilon_1 a$	$c/4$
	$-a/4 - \epsilon_1 a$	$a/4 + \epsilon_1 a$	$c/4$
O associated with Si...	$a/2 + \epsilon_2 a$	$\epsilon_2 a$	0
	$a/2 - \epsilon_2 a$	$-\epsilon_2 a$	0
	$a/4 + \epsilon_2 a$	$a/4 - \epsilon_2 a$	$c/4$
	$a/4 - \epsilon_2 a$	$a/4 + \epsilon_2 a$	$c/4$

Curiously enough, this lattice does not apparently possess the same symmetry elements as the crystal; thus the planes (100) and (010) are not symmetry planes with respect to the points of the lattice, and the lattice possesses no tetragonal screw axis.

But still the lattice has the properties necessary to explain the symmetry of the crystals.

Let a (100) plane containing Zr and Si atoms divide the lattice in two parts I and II. The points of the mirror image of I do not coincide with equivalent points of II; but they can be brought to coincide by a translatory motion along the three axes ($x=a/4$, $y=a/4$, $z=c/4$).

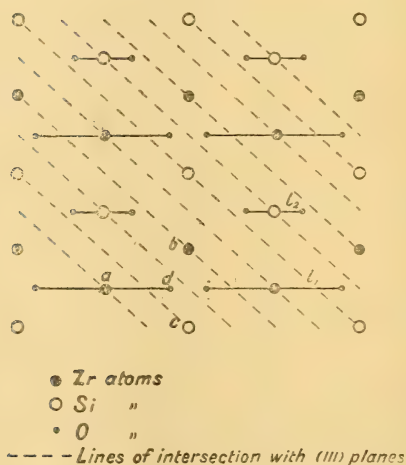
The distance moved being of atomic dimensions, the difference of position of the points II and those of the image of I is not to be detected as long as we regard the properties shown by a large number of atoms.

Thus this asymmetry should not even be felt by ordinary light waves because one wave-length would cover some thousand atoms; first when we come to a wave-length of the order of that of Röntgen rays, the want of symmetry with regard to the face (100) can be detected.

From these considerations we arrive at the following general rule.

Fig. 5.

*Arrangement of atoms in
(110) plane*



In order that a certain space lattice shall explain an element of symmetry ascribed to the finite crystal, it will be sufficient that the lattice possesses a symmetry of the following kind:

The lattice must be brought to coincidence with itself by performing on it the operation characteristic of the symmetry element in question and a translatory movement $x = \epsilon_1 a$, $y = \epsilon_2 b$, and $z = \epsilon_3 c$, where a, b, c are the sides of the elementary lattice, and $\epsilon_1, \epsilon_2, \epsilon_3$ quantities not greater than unity.

§ 7. Calculation of Intensities and Determination of the Parameters.

When the rays are reflected from a face with equivalent and equidistant planes the intensity distribution is said to be normal, and according to Bragg it is characterized by a gradual diminution of intensities with increasing order.

The exact law for this variation is not known. Bragg* finds that the intensities corrected for the temperature effect as derived from the formula of Debye† are approximately inversely proportional to the square of the order number (n). The cause of this variation is yet unknown.

In spite of the fact that the rate of variation will vary from one case to another, it will generally not be difficult to see from the observed intensities whether the spectrum is normal or not. The criterion is not so much the rapidity with which the intensity falls with increasing order, but much more a typical regular form of the intensity curve.

The problem of finding the distribution of intensities in the case that not all the reflexion planes of the face are identical, has been treated by W. H. and W. L. Bragg. The calculation is based on the assumption that the amplitude reflected from a certain point-plane is proportional to the mass associated with unit area of the plane.

In view of the theory of secondary radiation given by Sir J. J. Thomson‡, it would be more natural to suppose the amplitude proportional to the *number of electrons* per unit area, and introducing the atom-model of Rutherford and Bohr we should put the reflecting power of an atom proportional to the atomic number.

As the atomic number for most elements is approximately proportional to the atomic weight, it will make very little difference whether we use atomic weights or atomic numbers; but as it must be the number of electrons and not the gravitational mass which is concerned, we shall introduce the atomic numbers in our calculations.

Let unit area of the reflecting plane be composed of ν_1 atoms of atomic number N_1 , ν_2 atoms of atomic number N_2 , &c., then the number of electrons per unit area (μ) will be

$$\mu = \nu_1 N_1 + \nu_2 N_2 + \dots \dots \dots (9)$$

* W. H. Bragg & W. L. Bragg, 'X-Rays and Crystal Structure,' p. 193.

† P. Debye, *Verh. d. D. Phys. Ges.* xv. 1913; *Ann. d. Phys.* 1914, p. 49.

‡ Sir J. J. Thomson, 'Conduction of Electricity through Gases,' p. 321.

Let the face considered have a spacing d_0 . A length equal to d_0 on the normal to the face will be cut by a number of point-planes (r) with numbers of atoms per unit area $\mu_1, \mu_2 \dots \mu_r$.

We select an arbitrary point (0) on the normal, and call the distances from this point to the (r) planes

$$d_1 d_2 \dots d_i \dots d_r.$$

The intensity of the reflected wave from such a face has been calculated by Bragg in the case of $r=2$ and a general geometrical method is given *.

The general analytical expression for the intensity will be

$$I_n = k_n A_n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

k_n is a factor which W. H. and W. L. Bragg put proportional to the intensities of the normal spectrum, and they give the following values

$$\begin{array}{c} n = | \quad 1 \quad | \quad 2 \quad | \quad 3 \quad | \quad 4 \quad | \quad 5 \\ \hline K_n = | 100 \quad | 20 \quad | 7 \quad | 3 \quad | 1 \end{array}$$

A_n is the calculated amplitude which is given by the formulæ:

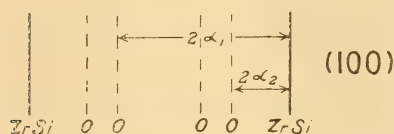
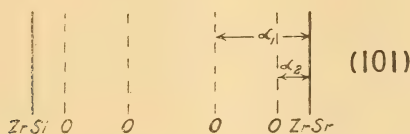
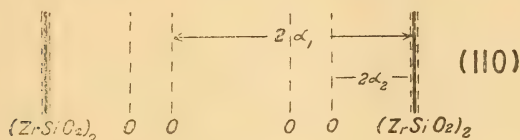
$$\left. \begin{aligned} A_n^2 &= f_1(n)^2 + f_2(n)^2 \\ f_1(n) &= \sum_{i=1}^{i=r} \mu_i \cos n 2\pi \frac{d_i}{d_0} \\ f_2(n) &= \sum_{i=1}^{i=r} \mu_i \sin n 2\pi \frac{d_i}{d_0} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (11)$$

In a great number of cases we can give the point 0 such a position that the r planes are symmetrically arranged with regard to this point, and as $\sin(-\alpha) = -\sin \alpha$, the quantity $f_2(n) = 0$.

The lattice of the zircon group as given in Table IV. will give the spacing shown in fig. 6 for the five faces experimented upon. Of these the face (001) has identical and equidistant planes and should give a normal spectrum which is also in agreement with experiments. The intensities of the spectra of the four other faces should be given by the following expressions for $f_1(n)$ and $f_2(n)$:—

* W. L. Bragg, Proc. Roy. Soc. lxxxix. p. 483 (1914).

Fig. 6.



$$\frac{\alpha}{2}$$

$$\left. \begin{aligned} \text{Face (111)} \quad & \begin{cases} f_1(n) = N_1 + (-1)^n N_2 + (N_1 + N_2 + 4N_3) \cos n \frac{\pi}{2} \\ \quad + 2N_3 (\cos n\alpha_1 + (-1)^n \cos n\alpha_2). \\ f_2(n) = (N_1 - N_2) \sin n \frac{\pi}{2}. \end{cases} \\ (110) \quad & \begin{cases} f_1(n) = (N_1 + N_2 + 2N_3) + N_3 (\cos 2n\alpha_1 + \cos 2n\alpha_2). \\ f_2(n) = 0. \end{cases} \\ (101) \quad & \begin{cases} f_1(n) = (N_1 + N_2) + 2N_3 (\cos n\alpha_1 + \cos n\alpha_2). \\ f_2(n) = 0. \end{cases} \\ (100) \quad & \begin{cases} f_1(n) = (N_1 + N_2) + 2N_3 (\cos 2n\alpha_1 + \cos 2n\alpha_2). \\ f_2(n) = 0. \end{cases} \end{aligned} \right\} \quad (12)$$

N_1, N_2, N_3 are the atomic numbers of Zr, Si, and O respectively, or atoms which may substitute them in the lattice.

The angles α_1 and α_2 are connected to the two parameters ϵ_1 and ϵ_2 (Table IV.) in the following way :

$$\alpha_1 = 4\pi\epsilon_1, \quad \alpha_2 = 4\pi\epsilon_2.$$

§ 8. Zircon.

To get the intensities of zircon we have in equations (9) to put

$$N_1 = 40, N_2 = 14, N_3 = 8;$$

and putting

$$\alpha_1 = \pi - \beta,$$

we get

$$\left. \begin{aligned} (111) \quad & \begin{cases} f_1(n) = 40 + (-1)^n 14 + 86 \cos n \frac{\pi}{2} + (-1)^n 16 (\cos n\beta \\ f_2(n) = 26 \sin n \frac{\pi}{2} \quad \quad \quad + \cos n\alpha_2). \end{cases} \\ (110) \quad & \begin{cases} f_1(n) = 70 + 8 (\cos 2n\beta + \cos 2n\alpha_2). \\ f_2(n) = 0. \end{cases} \\ (101) \quad & \begin{cases} f_1(n) = 54 + 16 ((-1)^n \cos n\beta + \cos n\alpha_2). \\ f_2(n) = 0. \end{cases} \\ (100) \quad & \begin{cases} f_1(n) = 54 + 16 (\cos 2n\beta + \cos 2n\alpha_2). \\ f_2(n) = 0. \end{cases} \end{aligned} \right\} \quad (13)$$

In determining the intensities we shall have to remember that the spectra from the faces (101) and (100) were determined by reflexion from an edge of the crystal. Especially in the case of the (100) face the reflexion to be observed was very weak ; and under these conditions we must expect too low a value for the first order spectrum, because a smaller portion of the primary beam will be reflected into the chamber when the glancing angle is small.

The reflexion from the (101) face was better ; but in this case also the first order spectrum is found too weak in comparison with those of higher order.

The spectra for the faces (111) and (110), however, are very accurately determined ; but we see from the expression for $f_1(n)$ that the spectrum of the face (110) will be very nearly normal, and the position of the oxygen atoms will affect the intensities very little. Still, we notice from fig. 2 that the intensity of the third order spectrum is too large as compared with that of the second order.

The spectrum (111), on the other hand, is very much influenced by the position of the oxygen atoms, and will form our main basis for the determination of α_1 and α_2 .

The values of $f_1(n)$ and $f_2(n)$ for the (111) face are given in Table V. for $n=1, 2, \dots 6$.

TABLE V.

n .	$f_2(n)$.	$f_1(n)$.	I obs.	k .
1	1.63	$1.63 - (\cos \beta + \cos \alpha_2)$	0.80	1.00
2	0	$-2.00 + (\cos 2\beta + \cos 2\alpha_2)$	0.40	0.30
3	-1.63	$1.63 - (\cos 3\beta + \cos 3\alpha_2)$	0.40	0.12
4	0	$8.75 + (\cos 4\beta + \cos 4\alpha_2)$	1.00	0.05
5	-1.63	$1.63 - (\cos 5\beta + \cos 5\alpha_2)$	0.05	0.015
6	0	$-2.00 + (\cos 6\beta + \cos 6\alpha_2)$	0	0.005

In column 4 are given the relative intensities observed and in the last column the relative intensities of a normal spectrum. These values are somewhat different from those given by Bragg, because the spectra of the crystals considered usually gives a normal spectrum with relatively stronger maxima of higher order.

Comparing the values of the last two columns, we see that β and α_2 must be given such values that the intensity of the first order spectrum is diminished and that of the third increased.

Consequently

$$\cos \beta + \cos \alpha_2 = 2 \cos \frac{\beta + \alpha_2}{2} \cos \frac{\beta - \alpha_2}{2} \text{ must be positive (I.)}$$

$$\cos 3\beta + \cos 3\alpha_2 = 2 \cos 3/2(\beta + \alpha_2) \cos 3/2(\beta - \alpha_2) \\ \text{must be negative (II.)}$$

As both β and α_2 must be less than π , condition I. gives that

$$\beta + \alpha_2 < \pi.$$

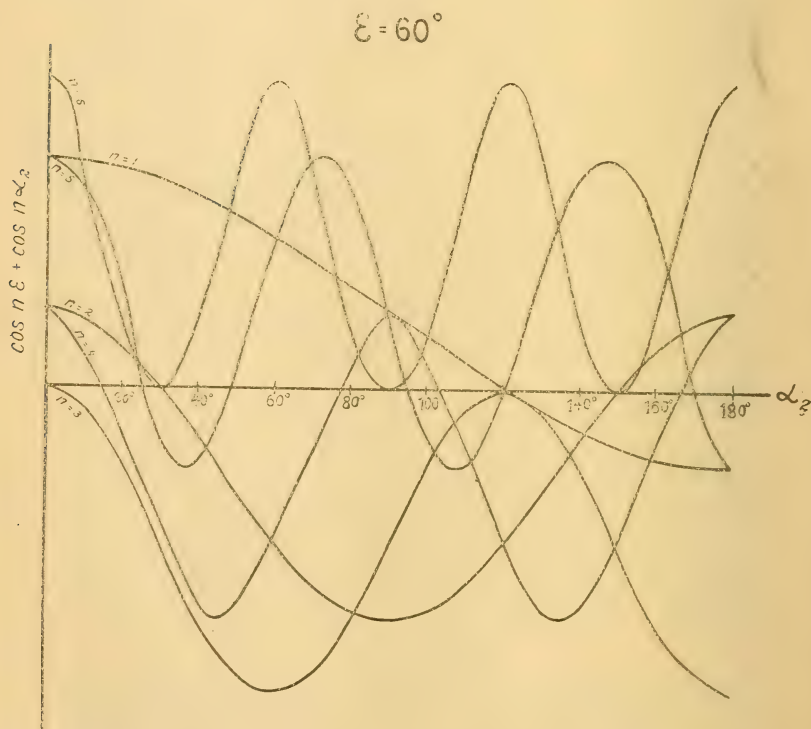
As the atoms must have a distance which is of the order of the length d_{100} , neither α_2 nor β can have a very small value. Then we can suppose $\cos 3/2(\beta - \alpha_2)$ positive, and from condition (II.) we get

$$\frac{\pi}{3} < \beta + \alpha_2 < \pi.$$

Now in order that the oxygen atoms may considerably reduce the intensity of the first order spectrum $\beta + \alpha_2$ must be considerably smaller than π , and we should expect both α_2 and β to be in the first quadrant, somewhere between 25° and 90° .

To determine β and α_2 more accurately we could calculate the intensities for varying values of β and α_2 . We can arrive at fairly good values more quickly by means of a graphical method. We can very quickly draw curves representing $\cos n\beta + \cos n\alpha_2$ for various values of β , α_2 and n . In fig. 7 the curves are given for $\beta = 60^\circ$ and for values of α_2 varying from 0° to 180° , and for $n = 1, 2, 3, 4, 5, 6$.

Fig. 7.



In this way I have found that the following values will give the best agreement with observations :

$$\beta = 30^\circ, \quad \alpha_2 = 60^\circ.$$

In the formula for $f_1(n)$ of the face (111) and also for the

(110) and (100) faces, the values of β and α_2 may be interchanged without altering the value of $f_1(n)$; and consequently we might equally well satisfy these faces with $\alpha_2=30^\circ$ and $\beta=60^\circ$. The face (101), however, does not permit such an interchange, and the fact that this face gives a comparatively strong second order reflexion will make the combination $\beta=60^\circ$, $\alpha_2=30^\circ$ impossible.

With our present knowledge with regard to the laws governing the intensity variations we cannot claim a very great accuracy for the angles α_2 and β . In the present case our experimental material is also somewhat limited—thus, if accurate intensity measurements for the (101) and (100) faces were available, we should probably reduce the possible errors in the values of β and α_2 still more.

In Table VI. are given the values of A^2 and the calculated and observed intensities corresponding to $\beta=30^\circ$ and $\alpha_2=60^\circ$ for the faces (111) and (101).

TABLE VI.

n.	(111)			(101)		
	A^2 .	$I_{cal.}$	$I_{obs.}$	A^2 .	$I_{cal.}$	$I_{obs.}$
1.....	2.73	91	80	9.5	100	100
2.....	4.0	40	40	11.4	36	56
3.....	9.6	38	40	5.6	7	8
4.....	60.0	100	100			
5.....	6.3	3	5			
6.....	4	0.7	0			

The agreement between calculated and observed values is a very good one, and in view of the fact that the intensities of the other faces are also explained, we see that there can be no doubt that the lattice given in Table IV. is the right one, and that we have found very nearly the true values of α_2 and β .

The values found for these angles give the following values for the two parameters of the lattice :

$$\epsilon_1 = \frac{\alpha_1}{4\pi} = \frac{\pi - \beta}{4\pi} = \frac{5}{24},$$

$$\epsilon_2 = \frac{\alpha_2}{4\pi} = \frac{1}{12}.$$

The distances l_1 and l_2 (figs. 4, 5) from the Zr and Si atoms to the oxygen atoms will be

$$l_1 = \epsilon_1 a \sqrt{2} = 2.71 \times 10^{-8} \text{ cm.}$$

$$l_2 = \epsilon_2 a \sqrt{2} = 1.08 \times 10^{-8} \text{ cm.}$$

The distance from the Zr atom to one of the oxygen atoms associated with it is more than twice the corresponding distance for Si, a fact which may be due to the greater affinity between the Si and O atoms.

§ 9. Rutile and Kassiterite.

If the lattices of $(\text{TiO}_2)_2$ and $(\text{SnO}_2)_2$ belong to the type given in Table IV., the spectra should be derived from the formulæ (12) by inserting the corresponding values of the atomic numbers.

In both cases $N_1 = N_2 = N$, and for the (111) face we get

$$(111) \quad \begin{cases} f_1(n) = N(1 + (-1)^n) + 2(N + 2N_3) \cos n \frac{\pi}{2} \\ \quad + 2N_3 (\cos n\alpha_1 + (-1)^n \cos n\alpha_2). \\ f_2(n) = 0. \end{cases}$$

Now the Röntgen ray analysis shows that spectra of uneven order with regard to n disappear; consequently we have for all values of q :

$$f_1(2q-1) = 0 = 2N_3 (\cos (2q-1)\alpha_1 - \cos (2q-1)\alpha_2),$$

which gives

$$\alpha_1 = \alpha_2 = \alpha.$$

This is an important result as it shows that in each of the two minerals all the metal atoms are identical as regards their relation to the oxygen atoms, and it is impossible to consider rutile say—as a titanium—titanate.

The amplitudes for the four cases considered will be

$$\left. \begin{aligned} A_{111} &= N + (-1)^n(N + 2N_3) + 2N_3 \cos 2n\alpha, \\ A_{110} &= N + N_3 + N_3 \cos 2n\alpha, \\ A_{101} &= N + 2N_3 \cos n\alpha, \\ A_{100} &= N + 2N_3 \cos 2n\alpha. \end{aligned} \right\} \dots (14)$$

The expression for A_{111} , A_{110} , and A_{100} will not be altered if we substitute $\pi - \alpha$ for α . Which of these two is the right value can be decided by means of the (101) spectrum.

For $(\text{TiO}_2)_2$ we find as the best value $\alpha = 111^\circ.5$, and for $(\text{SnO}_2)_2$ $\alpha = 112^\circ.5$.

In the case of SnO_2 it is only the (111) and (100) spectra which can be used for the determination of α ; for the (110) spectrum is very nearly normal regardless of the value we give α , and the (101) spectrum is determined from a very small face, which will cause the observed first-order spectrum to be too weak.

In Table VII. are given the values of A^2 and K and the calculated and observed intensities.

The agreement is a very good one; thus the lattice given in Table IV. can explain the typical distribution of intensities observed.

TABLE VII.
 $(\text{TiO}_2)_2, \alpha = 111^\circ.5.$

(111)				(101)			
k .	A^2 .	$I_{\text{cal.}}$	$I_{\text{obs.}}$	k .	A^2 .	$I_{\text{cal.}}$	$I_{\text{obs.}}$
100	2.99	100	100	100	1.02	100	100
20	14.6	98	100	20	0.41	8	6
7	0.14	0.3	0	7	5.15	35	11
3	7.55	7.7	5	3	2.13	6	1
(100)				(110)			
100	0.42	100	100	100	9.32	100	100
20	2.08	98	85	20	14.6	31	50
7	4.00	66	35	7	19.3	14	16
3	0.14	0	0	3	7.55	2.4	3
$(\text{SnO}_2)_2, \alpha = 112^\circ.5.$							
(111)				(100)			
100	2.92	18.5	17	100	5.85	100	100
30	52.5	100	100	30	9.82	51	60
12	0.1	0	0	12	14.75	30	28
5	39.1	12.5	13	5	4.55	4	4

The results of the calculation are collected in Table VIII., giving the parameters ϵ and the distances l (figs. 4, 5) for the three crystals.

TABLE VIII.

Crystal.	α .	ϵ .	l .
ZrSiO_2 { Zr ...	150°	5/24	$2.71^\circ \times 10^{-8}$
	Si ... 60°	1/12	1.08 „
$(\text{TiO}_2)_2$	111° 5	0.155	1.99 „
$(\text{SnO}_2)_2$	112° 5	0.157	2.08 „

Photographs of a model of the zircon lattice are shown in Plate III.

§ 10. *The Molecular Structure of the Lattice.*

The Röntgen-ray analysis has shown that crystals are built up of atomic lattices; and in a number of cases previously treated by W. H. and W. L. Bragg the arrangement of the atomic lattices has left no room for such a thing as a molecule. Suppose, for instance, that in a crystal of rock-salt we fix our attention to a certain Na atom, we cannot from the geometrical arrangement tell which Cl atom is associated with it. This fact, however, does not without further proof necessarily involve any fundamental change in our conception of the chemical binding as taking place between pairs of atoms (Na-Cl).

The atoms might possibly be connected up in pairs in such a way that all requirements of symmetry were fulfilled.

In the case of rock-salt and similar substances there should be *four* pairs of simple cubic lattices, and the lines connecting each pair should be arranged with cubic symmetry; but as we have three equal directions in the crystal and four pairs, such an arrangement *does not seem possible*.

If, however, we regard our lattice for the Zircon group, we notice that each of the Zr or Si atoms is associated with two oxygen atoms; thus the groups SiO_2 and ZrO_2 form a kind of "molecular elements" of the lattice. This is not merely a way of regarding the geometrical arrangement of the atoms; but we have reason to believe that the groups SiO_2 and ZrO_2 form chemically saturated compounds. First of all, the fact that the oxygen atoms are closer to the Si than to the Zr atoms goes to support this view, and, further,

a peculiarity with the geometry of the lattice will make the assumption of molecules almost a necessity.

Let us consider the arrangement of atoms in the (110) plan of zircon, fig. 5. The triangle (abc) has a Zr atom in two of its corners and a Si atom in the third. Now an oxygen atom d belonging to the Zr atom a will—on account of the geometry found for the lattice—have equal distances to the Zr atom b and the Si atom c (fig. 5). Now the chemical-affinity forces must necessarily be different for (Zr-O) and (Si-O), and if such forces were acting between the oxygen atom d and any of the atoms c and b , we cannot explain an equilibrium position of the oxygen atom d which makes the distances ($d-b$) and ($d-c$) equal.

§ 11. *The Structure of Thorite.*

As mentioned in the introduction, the mineral thorite (ThSiO_4) also belongs to the Zircon group. This mineral, however, occurs in the so-called “metamict” form, which indicates a state in which the outer appearance of a crystal is preserved, but the substance itself has in the course of time become isotropic.

Thus the crystals, when examined with polarized light, give no indication of an optical axis, and it seems as if the atomic arrangement in a lattice has become unstable.

Now it would be a matter of interest to see how these crystals behave towards Röntgen rays. Do they give any reflexion? Or does a “metamict” crystal possess any trace of its original lattice?

I have made a series of experiments with a number of different minerals to investigate this point. A full account of these experiments will be given later. In this connexion I shall only mention that the mineral thorite gave no X-ray reflexion at all, although several very fine crystals were tested. Thus the Röntgen-ray analyses have shown that the lattice of thorite was completely broken down, and only the outer form has been preserved to indicate the atomic framework which once existed in the crystal. All symmetry properties, however, go to support the view, that the atoms must have been arranged in a lattice of the zircon type.

§ 12. *Remarks regarding the Intensities of the Normal Spectrum.*

W. H. Bragg and W. L. Bragg have found that the intensities of a normal spectrum gradually diminish with increasing order. As an average they put the relative

intensities equal to 100, 20, 7, 3 for the orders 1, 2, 3, 4. These numbers, however, contain the influence of temperature, which will tend to diminish the intensities of higher order as compared with those of low order. Corrected for temperature effect the intensities should be approximately inversely proportional to the square of the order number. As also apparent from the way in which Bragg has stated the law, it can only be considered as tentative and as a first approximation; and I want here to give some facts which indicate that the intensity law cannot be quite so simple.

In the case of silver we have to deal with well-defined crystals of only one element, and the two faces considered (100) and (111) give both a normal spectrum*; but the intensity falls off much more rapidly for the first than for the second face.

The face (111) gives an abnormally slow rate of fall with increasing order. I have made a careful examination of this point, comparing the first and second order of the two faces, but could only confirm the result first obtained. Also the (111) faces of gold and lead show the same abnormally slow rate of fall. The results of the intensity measurements are given in Table IX.

TABLE IX.

	Face.	Opening of slit.	Order.		
			1.	2.	3.
Silver	(100)	1 mm.	100	20	7
	"	"	100	25	
	"	0.5 mm.	100	19.5	
	(111)	1 mm.	100	51	11
	"	"	100	50	
Gold	"	0.5 mm.	100	35	
	(111)	—	100	35	
	"	0.5 mm.	100	48	
Lead	(111)	—	100	49	

Also the face (110) of the Zircon group gives an abnormally slow rate of fall of the intensities, even when we take into account the effect of the oxygen atoms. Thus in the case of tinstone, where the oxygen atoms can have very little influence, the distribution corrected for the

* Vegard, Phil. Mag. Jan. 1916.

disturbing influence of oxygen should be 100, 62, 30, 8 of a normal spectrum of the orders 1, 2, 3, 4.

As, in the case of silver, the value of θ occurring in the Debye temperature factor should be the same for both faces, the rate of fall of the intensity will be different for face (100) and (111) of silver, even if we correct for temperature in the way given by Debye.

So long as we know so little about the cause of the intensity variation, it will be difficult to say anything definite with regard to the explanation of the observed differences.

It might be possible that the temperature factor is different for different faces even in the case of a cubic crystal. On account of the different elastic properties along different point-planes, such an explanation might not be unlikely. But if it should be impossible to account for the difference of intensity variation by differences of the temperature factor, we should probably have to suppose that the atoms had different reflexion properties in different directions.

Summary.

1. The lattice of *gold* and *lead* has been determined, and found identical with that of copper and silver.

2. The structure of the Zircon group has been completely determined. The Si as well as the Zr atoms are arranged in tetragonal lattices of the *diamond type*. In the case when Si and Zr are replaced by identical atoms Ti or Sn, we get a simple *prism-centred* lattice for the metallic atoms.

3. The tetragonal structure is not *produced by symmetry properties of the atomic centres, but by the tetragonal arrangement of the oxygen atoms*.

4. The lattice has a sort of *molecular structure* with molecules of the type MO_2 , where M is an atom of Si, Zr, Ti, or Sn. The three atoms forming one molecule are situated on a straight line and with M in a central position. This line might be called the molecular axis.

The positions of the oxygen atoms are determined, when we know the directions of the molecular axes and the distance to the central atom M (molecular distance). The fact that the molecular distance is different for different central atoms M, as also certain geometrical relations of the zircon lattice, goes to support the view that the groups MO_2 form chemically bound molecules.

5. For all the minerals considered, the molecular axes are equally arranged and are always perpendicular to the tetragonal axis, which accounts for the fact that the ratio c/a is smaller than unity and almost equal for all minerals.

6. The "metamict" crystal of thorite gave no X-ray reflexion. The lattice is completely destroyed.

7. Experiments on the intensities of the normal spectrum have shown that different faces of the same crystal give a different law of the variation of intensities with increasing order.

I have much pleasure in thanking Professor W. C. Brögger for lending me, from the excellent collection of his laboratory, the crystals necessary for this research, as also for the kind interest he has taken in the work.

I am also very much pleased to thank Mr. H. Schjelderup for his most valuable assistance in making the observations on which the present work is based.

Physical Institute, Christiania.

March 31, 1916.

VII. *On Aerial Waves generated by Impact.* By SUDHANSUKUMAR BANERJI, M.Sc., *Sir Rashbehari Ghosh Research Scholar in the University of Calcutta**.

1. *Introduction.*

HERTZ, in his well-known paper† on the collision of elastic solids, shows that when two bodies impinge on each other with moderate velocities, the elastic distortions are more or less entirely localized over the region of contact, and that the duration of impact, though in itself a very small quantity, is a large multiple of the gravest period of free vibrations of either body. It follows, therefore, that no appreciable vibrations of the solids are set up by the impact, and that all parts of the impinging bodies, except those infinitely close to the point of impact, move as parts of rigid bodies.

In a recent paper‡ Lord Rayleigh has investigated the circumstances of the first appearance of sensible vibrations in the case of two impinging spheres, and his results seem to show that if vibrations are excited at all, the leading term in the radial displacement at the point of contact during the

* Communicated by Prof. C. V. Raman, M.A.

† Hertz's 'Miscellaneous Papers,' English Edition, p. 146. [See also Love's 'Treatise on Elasticity,' Second Edition, p. 195.]

‡ Lord Rayleigh, "On the Production of Vibrations by Forces of Relatively Long Duration with Application to the Theory of Collisions," Phil. Mag. vol. xi. pp. 283-291 (1906). ['Scientific Papers,' vol. v. pp. 292-299.]

early part of the collision is given by an expression of the type

$$\phi = \frac{3}{4} \frac{\sqrt{\pi}}{a_2 n^{\frac{1}{2}}} \cdot \frac{k_2 \dot{a}^{\frac{3}{2}}}{a_2 n^{\frac{1}{2}}} \cos\left(nt + \frac{\pi}{4}\right),$$

where \dot{a} is the relative velocity of impact, a_2 is a certain constant which can be easily calculated from Lamb's theory, and

$$k_2 = \frac{4}{9} \frac{\sqrt{2r} \cdot E}{\rho}, \quad n = \sqrt{\frac{E}{\rho}} \cdot \frac{85\pi}{\sqrt{3r}},$$

r being the radius of the sphere, E the Young's modulus, and ρ the density.

The leading term due to the end of the collision is obtained from this by changing nt to $n(t - \tau)$, τ being the duration of impact.

Also the ratio of the maximum kinetic energy of vibrations to the energy before collision is approximately given by an expression of the type

$$R = \frac{1}{50} \cdot \frac{\dot{a}}{\sqrt{E/\rho}}.$$

Since $\sqrt{E/\rho}$ is the velocity of longitudinal vibrations along a bar of the material of the solids in question, we see that, in general, the expression for ϕ is very small in magnitude, and that R is an exceedingly small ratio.

Lord Rayleigh's results show that under ordinary conditions, that is, unless the spheres are very large in size or the relative velocity of impact is very great, vibrations should not be generated in appreciable degree, and that the energy of the colliding spheres remains translational. Moreover, even if vibrations be excited at all, the pitch of the gravest sound so produced would be very high, in fact almost beyond the range of audibility. For example, in the case of two mahogany balls of 6 cm. diameter, the frequency of the gravest vibrations excited would be about 37,000 per sec. We know, however, from experience that when two spheres, say two billiard-balls, impinge directly upon each other, aerial waves of considerable intensity are generated which are audible as the characteristic sound of impact. The investigation described in the present paper was undertaken to ascertain, both theoretically and experimentally, the origin and characteristics of the sound produced by such impact.

Since, as we have seen, under ordinary conditions vibrations cannot be excited in any perceptible degree, practically the

whole of the sound of impact must be principally due to the impulse given to the fluid medium by the surfaces of the spheres, which undergo a sudden change of velocity as a result of the impact. The only alternative explanation that might be suggested is some kind of action, namely, a sudden compression or rarefaction in the neighbourhood of the region of contact; but this, it seems, can hardly be correct, as the spherical shape of the balls and the smallness of the relative velocity of impact would not readily admit of any specially intense compression or rarefaction being set up in the medium round the region of contact. Probably some kind of local reciprocating motion would be set up in this region, but this would not be of much importance.

The first hypothesis suggested in the preceding paragraph regarding the origin of the sound can be fully tested by an experimental and theoretical investigation of the distribution of intensities in different directions round the colliding spheres, and by studying the manner in which the sound depends (1) on the duration of the impact, (2) on the coefficient of restitution, (3) on the diameter of the balls, and (4) on the relative velocity of impact and possibly other factors also.

2. *Measurement of the Intensity.*

The distribution of intensities in different directions round the colliding spheres is found to possess many remarkable peculiarities which would be very difficult to reconcile with any other hypothesis regarding the origin of the sound. Even by the unaided ear one can perceive that the intensity of the sound is greatest when heard in the direction of movement of the colliding spheres, and is comparatively quite feeble in the plane at right angles to this line. Inside a laboratory the reflexions from the walls of the room give some trouble. The contrast between the intensities in the two directions is therefore best appreciated by the unaided ear when the observations are made in the open air, so as to avoid such reflexions as far as possible. A rough estimate of the ratio of the intensities can be made by varying the distance of the colliding spheres from the observer. So far as could be judged, the sound in the direction of impact appeared at least three or four times more intense in one direction than in the other. Some uncertainty was caused by the difference in the character of the sound from various directions, this difference being so marked that by its aid alone the angle made by the line of collision with the direction of the observer could be judged with fair accuracy. Other remarkable peculiarities were revealed when it was arranged

to obtain a quantitative measurement of the relative intensities in actual experiment. It was then noticed that the intensity practically vanishes on a cone making an angle of about 67° with the line joining the centres.

After many trials an apparatus has been devised which appears to satisfy the necessary conditions of extreme sensitiveness, suitability for quantitative work, and convenience in actual use. This apparatus, which is believed to be of a new type, is based on a ballistic principle. Its construction is quite distinct from that of the phonoscope invented by Dr. Erskine-Murray, or other similar devices in which the motion of a membrane or disk on which the sound-waves are incident deflects a pivoted mirror connected with it. As a matter of fact, a phonoscope of the ordinary type was given the first trial, but proved quite unsuitable for the present work, as the deflexion observed with it was too small and too sudden to be capable of measurement by visual observation, or even for satisfactory photographic registration.

The apparatus finally devised and employed consists of a small mirror attached to a pivoted axle whose free movement is controlled by a fine spiral spring. (In practice the balance-wheel and hair-spring of a watch proved very satisfactory, the mirror being attached radially to the wheel with a little cement.) The sound is received by a horn over the tubular end of which a mica disk is fixed. A sharp metal pointer is fixed normally to the centre of the disk and its end lightly touches the pivoted mirror referred to above, but is not connected with it. The light from a slit illuminated by an arc-lamp is condensed by a lens on the pivoted mirror, the reflected light forming a sharp image of the slit on a distant graduated screen. For the production of impact, the balls are hung side by side by bifilar suspension from a framework which is capable of rotation round a vertical axis. The balls can be made to impinge on each other in a direction making any desired angle with the axis of the horn by simply rotating the framework. This angle can be read off on a graduated circle fixed below the frame. In order to obtain perfect regularity in the sound of the impact and to avoid unnecessary reflexions from closely contiguous bodies, an electromagnetic arrangement was used by which the balls could be automatically dropped on breaking the circuit.

As soon as the balls collide, the sound-wave generated by the impact passes through the horn and impinges on the mica disk. The motion of the pointer attached to the disk gives a kick to the pivoted mirror, which moves off freely until it is brought to rest by the controlling spiral spring.

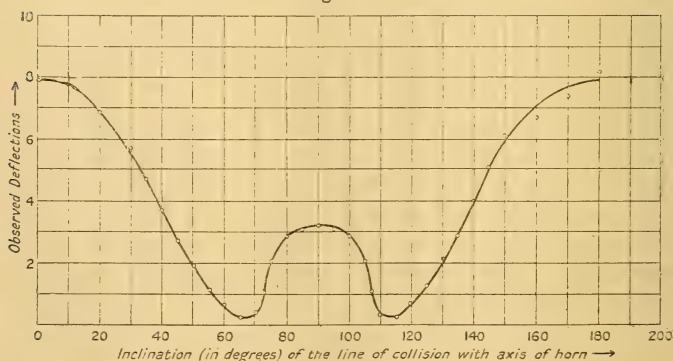
The mirror then comes back to the pointer, which brings it to rest. The deflexion of the spot of light on the distant screen gives a measure of the kick given to the mirror. It is found that the apparatus is extremely sensitive, very faint sounds being sufficient to produce deflexions which can be read off by eye nearly as easily as those of a ballistic galvanometer. Moreover, the behaviour of the mirror is very regular, and its motion perfectly aperiodic. The mode of action of the apparatus described above can be verified by observing the motion of the mirror and the pointer under a low-power microscope.

One valuable feature of the apparatus is that it is quite unaffected by any echoes of the original sound of impact from the walls of the laboratory-room in which the experiments are made. This is because the pointer attached to the mica disk ceases to touch the mirror long before the echoes from the walls arrive at it. The results obtained by its use have been verified by working at different points within the room, and also in rooms of widely differing shape and size.

In order to fully understand the action of the mica disk and pointer, we have to study their forced vibrations under the influence of the sound-pulse. We shall presently come to this point. Meanwhile the results obtained by its use may be described.

Observations have been made of the deflexions shown by the apparatus when two balls impinge directly upon each

Fig. 1.



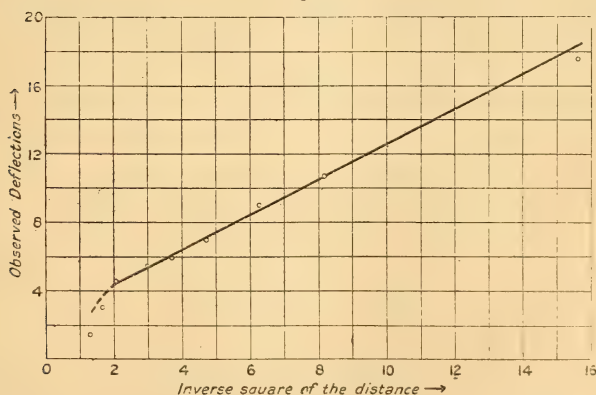
other with a given velocity in different directions with respect to the axis of the receiving horn. The results are recorded in fig. 1. The curve exhibits quite a number of remarkable peculiarities. It shows that the intensity is maximum in the

line joining the centres, and that it gradually diminishes until it practically vanishes at an angle of about 67° with the line joining the centres, when it again increases rather abruptly until it attains a second maximum value at an angle of 90° . The experiment has been tried with pairs of spheres of various materials, viz. (1) billiard-balls, (2) marble, (3) aluminium, and (4) wood, and analogous results are noted in all the cases.

It is not difficult to see, in a general way, that the distribution of intensity shown in fig. 1 is in accordance with the hypothesis as to the origin of the sound with which we started. As a result of the impact the balls undergo very rapid changes of velocity in opposite directions. The case is somewhat analogous to the well-known effect of the zones of silence noticed in the neighbourhood of the prongs of a tuning-fork, but differs from it somewhat owing to the spherical shape of the balls, the non-periodic character of their motion, and their close contiguity at the instant of impact.

That the results shown in fig. 1 are quite reliable has been further tested by three methods. By measuring the deflexions of the spot of light for impacts at different distances from the mouth of the horn, the balls being made to impinge always with the same velocity and in the same direction, the deflexions are found practically to vary inversely as the square of the distances from the mouth of the horn. The

Fig. 2.



results are shown in fig. 2, where the squares of the reciprocal of the distances have been plotted against the deflexions. The curve shows that over a very wide range the deflexions practically vary inversely as the square of the

distances, and it is only when we come towards the origin that the curve shows a tendency to assume a parabolic shape. Further, measurement of the deflexions for different velocities of impact shows that within the range of the experiment the deflexions vary directly as the squares of the velocities. The results are shown in Table I. When the squares of the

TABLE I.

Nos.	Velocity of Impact.	Mean Deflexions.
1.	23.06	8.63
2.	21.32	7.53
3.	19.50	6.82
4.	17.75	5.20
5.	10.80	1.43

velocities are plotted against the deflexions they give practically a straight line passing through the origin. Experiments have also been made with pairs of balls of the same material but of different diameters. In this case it is found that the deflexions vary practically as the fourth power of the diameters of the balls. The results for the case of three pairs of wooden balls are given in Table II. All these results show, as we shall presently see, that the apparatus practically measures the intensity of the sound produced by impact.

TABLE II.

Nos.	Diameters of the Balls.	Mean Deflexions.
1.	3 inches.	21.92
2.	2½ inches.	7.02
3.	1½ inches.	1.35

3. *Nature of the Wave-Motion.*

A complete theoretical investigation of the nature of the wave-motion started by impact, assuming that no vibrations are excited in the balls, is beset with considerable mathematical difficulties. We shall confine ourselves to the case of two equal balls. As a result of the impact, the balls suffer

changes of velocity U during the short interval of time known as the duration of impact, in consequence of which an impulsive pressure is communicated to the surrounding medium and a train of sound-waves is started travelling forward with a definite wave-front. As a simplification we shall assume that the change in velocity is instantaneously acquired by the balls. As a matter of fact, if we examine the curve for the relative velocities of the centres of mass for the period the balls are in actual contact, we notice that the most rapid changes in velocity occur only at the epoch of greatest compression, and as the duration of impact itself is a very small quantity (usually less than the 2000th part of a second), we see that the effect of the duration of the impact on the sound-waves is generally not of very great importance. At any rate, we are not wide of the mark in taking the change in velocity as practically instantaneous.

(1) The case of a single sphere.

We shall first consider the case when a single sphere suffers an instantaneous change in velocity U .

If M be the mass of the sphere, its equation of motion can be written in the form

$$M \frac{d^2 x}{dt^2} = - \iint p \cos \theta \cdot a^2 d\omega, \quad . \quad . \quad . \quad (1)$$

where a is the radius of the sphere, p is the pressure at a point on the surface of the sphere, and $d\omega$ an elementary solid angle.

Also if ψ denote the velocity potential of the wave-motion started, the condition of continuity of normal motion on the surface of the sphere gives

$$\frac{\partial \psi}{\partial r} = - \frac{dx}{dt} \cos \theta, \quad \text{when } r=a. \quad . \quad . \quad . \quad (2)$$

The initial circumstances at time $t=0$ give

$$x=0 \quad \text{and} \quad \frac{dx}{dt} = U. \quad . \quad . \quad . \quad (3)$$

Further, the condition of discontinuity at the spherical boundary of the advancing wave gives

$$\frac{\partial \psi}{\partial t} = -c \frac{\partial \psi}{\partial r}, \quad . \quad . \quad . \quad . \quad (4)$$

to be satisfied for $r=ct+a$, c being the velocity of sound.

We can now assume for ψ the following expression

$$\psi = \frac{\partial}{\partial r} \cdot \frac{f(ct-r)}{r} \cdot \cos \theta, \quad (5)$$

and we can easily determine the arbitrary function involved in this expression by a method first given by Prof. Love* so as to satisfy all the conditions enumerated above.

The method consists in assuming

$$f(ct-r) = Ae^{\lambda(ct-r+a)} \quad \text{and} \quad x = Be^{\lambda ct},$$

and then on substitution in the boundary conditions (1) and (2), we notice that λ satisfies a biquadratic equation, two of whose roots are zero. The constants A's and B's are then determined with the help of the remaining conditions.

If we assume that the ratio of the mass of the air displaced by the sphere to its own mass is a very small quantity, we see that the expression for ψ can be written in the simple form

$$\psi = A \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \cos \theta - \frac{\sqrt{2} U a^3}{4} \frac{\partial}{\partial r} \left[\frac{e^{-\left(\frac{ct+a-r}{a}\right)}}{r} \cos \left(\frac{ct+a-r}{a} - \frac{1}{4} \pi \right) \right] \cos \theta, \quad (6)$$

where A is an indeterminate constant.

The first term in this expression is a degenerated function which does not satisfy the usual differential equation for wave-propagation, and consequently does not represent a wave-disturbance. This term arises from the subsequent motion of the sphere with a nearly constant velocity which involves only a local reciprocating motion of the neighbouring air.

The wave-motion produced is therefore given by the expression

$$\psi = - \frac{\sqrt{2} U a^3}{4} \frac{\partial}{\partial r} \left[\frac{e^{-\left(\frac{ct+a-r}{a}\right)}}{r} \cos \left(\frac{ct+a-r}{a} - \frac{1}{4} \pi \right) \right] \cos \theta. \quad (7)$$

Thus we see that the wave-motion generated by an instantaneous change in velocity of a single sphere is of the damped harmonic type which is practically confined to a small region near the front of the advancing wave.

* Love, "Some Illustrations of the Modes of Decay of Vibratory Motions," Proc. Lond. Math. Soc. (2) vol. ii. p. 88 (1904). [See also Lamb's 'Hydrodynamics,' Art. 295.]

(2) The case of two spheres.

The solution for the case when both the spheres undergo instantaneous change in velocity U cannot, however, be so easily obtained. In a recent paper* published in the 'Bulletin' of the Calcutta Mathematical Society, I have given a method by which a solution for this case can be obtained. For our present purpose, however, we see from symmetry that if we take as our origin the point of contact of two equal impinging spheres, the velocity potential of the wave-motion which satisfies the boundary conditions over the surfaces of both the spheres can be written in the form

[illegible]

due regard being paid to the dimensions of both the sides. $A_0, A_2, \&c.$; $\lambda_0, \lambda_2, \&c.$, are certain constants not depending on the radius of the spheres to be determined by the boundary conditions.

At a great distance from the source of sound we can neglect all powers of $\frac{1}{r}$ in this expression beyond the first. So that at a great distance we have approximately

$$\psi = \frac{Ua^2}{r} \left[A_0 e^{\lambda_0 \left(\frac{ct-r}{a} \right)} + A_2 \lambda_2^2 e^{\lambda_2 \left(\frac{ct-r}{a} \right)} P_2(\cos \theta) + \text{&c.} \right]. \quad (9)$$

Also, since the sound-pulse is practically confined to the wave-front, and also since in this region we have either r equal to ct or less than ct by a few diameters, we see that the expression within the bracket may be regarded as a simple function of the time and the inclination θ , independent of the radius of the spheres or the distance r .

The intensity of the sound for large values of r therefore varies as $\frac{U^2 a^4}{r^2}$.

* On "Sound-waves due to prescribed Vibrations on a Spherical Surface in the presence of a rigid and fixed Spherical Obstacle," Bulletin of the Calcutta Mathematical Society, vol. iv.

We thus arrive at the following results :—

(1) The intensity of the sound varies as the square of the change in velocity of the colliding spheres.

(2) It varies inversely as the square of the distance from the point of contact of the spheres.

(3) It varies as the fourth power of the radius of the spheres.

The truth of these results has already been verified experimentally, provided we assume that the apparatus measures the intensity, which we shall presently see it does.

We can easily study the forced and the free vibrations of the mica disk under the action of the sound-pulse. The forced and free vibrations of a membrane and those of a telephone plate have been studied by various writers. Without entering into mathematical details, we see that the disturbance produced by impact travels forward as a sound-pulse of the damped harmonic type which is sensible only within a few diameters from the inner side of the boundary of the advancing wave. Its action on the mica disk, which has usually a smaller natural frequency of vibration than that of the waves, is so very sudden and lasts for so short a time that the whole effect partakes of the character of an impulsive pressure in consequence of which the mica disk suddenly acquires a velocity and free vibrations of considerable amplitude are excited in it. Assuming that the mica disk is not displaced considerably, we can easily

see from elementary considerations that since $\rho \frac{\partial \psi}{\partial t}$ is the pressure per unit area on the mica disk, where ψ is the velocity potential of the sound-pulse and ρ the density of air, the initial velocity communicated to the mica disk would be practically proportional to the quantity

$$\int_{t_0}^t \rho \frac{\partial \psi}{\partial t} dt,$$

t_0 being the instant when the sound-pulse meets the mica disk. When this velocity attains the maximum value, the mirror leaves the pointer and moves with that maximum velocity. This velocity is therefore proportional to the quantity $[\rho \psi]_{t_0}^t$, the instant t being so chosen that this expression has the maximum value. If we denote this quantity by v and the deflexion of the mirror by θ , then we

must have v and θ connected by the relation

$$v^2 = a\theta + b\theta^2,$$

where a and b are two constants depending on the elasticity of the spring and its initial strained condition. In actual practice the mirror is initially in contact with the pointer with a sensible pressure, and as the deflexion is usually very small, the second term in the above expression is nearly negligible in comparison with the first. v^2 is, therefore, practically proportional to $a\theta$; in other words, the angular deflexion of the mirror is approximately proportional to the intensity of the sound incident on it.

On account of mathematical difficulties, it seems to be a hopeless task to attempt a numerical calculation of the distribution of intensities in different directions round the colliding spheres. But the analogous problem of two vibrating spheres whose distance apart varies periodically presents features similar to this problem, when the wave-length of the disturbance produced is sufficiently small. For this case, however, we can approximately calculate the distribution of intensities in different directions by the following method.

If we take as our origin the point symmetrically situated between the two spheres and the line joining the centres as our initial line, it is easy to see that the velocity potential of the wave-motion will be given by

$$\begin{aligned} \psi = & \left[A_0 f_0(kr) + A_2 \frac{(kr)^2 f_2(kr)}{\frac{d}{da} \{ (ka)^2 f_2(ka) \}} P_2(\cos \theta) \right. \\ & \left. + A_4 \frac{(kr)^4 f_4(kr)}{\frac{d}{da} \{ (ka)^4 f_4(ka) \}} P_4(\cos \theta) + \&c. \right] e^{ikot}, \end{aligned}$$

where

$$\begin{aligned} f_n(\zeta) = \frac{i^n e^{-i\zeta}}{\zeta^{n+1}} \left\{ 1 + \frac{n(n+1)}{2i\zeta} + \frac{(n-1)n(n+1)(n+2)}{2 \cdot 4 \cdot (i\zeta)^2} \right. \\ \left. + \dots + \frac{1 \cdot 2 \cdot 3 \dots 2n}{2 \cdot 4 \cdot 6 \dots 2n (i\zeta)^n} \right\}, \end{aligned}$$

and $\frac{2\pi}{k}$ is the wave-length.

The unknown constants A_0 , A_2 , &c. have to be determined

by means of the condition of continuity of normal motion on the surfaces of the spheres. We see from this expression that the disturbance produced at any point due to this system of two vibrating spheres will be the same as that due to a prescribed vibration given by

$$[A_0 + A_2 P_2(\cos \theta) + A_4 P_4(\cos \theta) + \dots] e^{ikct},$$

on the surface of a single sphere of radius equal to that of either sphere and having its centre at the origin. Now, from a consideration of the nature of the motion produced in the immediate neighbourhood of the two spheres, we can easily ascribe approximate values to the constants A_0 , A_2 , &c., which will conform as nearly as possible to the true state of affairs. As a first approximation, we can represent this disturbance by $U \cos 2\theta \cdot e^{ikct}$, which is the same as

$$U \left[\frac{4}{3} P_2(\cos \theta) - \frac{1}{3} P_0(\cos \theta) \right] e^{ikct},$$

so that $A_0 = -\frac{1}{3}$, $A_2 = \frac{4}{3}$ and the rest vanishes. This type of vibration shows that while the two caps bounded by the parallels of latitude of 45° and 135° are moving outwards the intermediate zone is moving inwards and *vice versa*.

Now, if we assume various values for the quantity ka which will determine the wave-length for a particular pair of balls, we can easily calculate the values of the intensity at a great distance from the source of sound. First suppose that $ka=1$, then since at a great distance

$$f_n(kr) = \frac{i^n e^{-ikr}}{(kr)^{n+1}},$$

we have

$$\begin{aligned} \psi = & - \left[\frac{A_0}{2} (1-i) - \frac{A_2}{89} (5-8i) P_2(\cos \theta) \right. \\ & \left. + \frac{A_4}{402337} (296-561i) P_4(\cos \theta) + \&c. \right] \frac{e^{ik(ct-r-a)}}{r}. \end{aligned}$$

Denoting the real and the imaginary parts of the expression within the bracket by F and G , we have

$$F = \frac{1}{2} A_0 - \frac{5A_2}{89} P_2(\cos \theta) + \frac{296A_4}{402337} P_4(\cos \theta) + \&c.,$$

$$G = -\frac{1}{2} A_0 + \frac{8A_2}{89} P_2(\cos \theta) - \frac{561A_4}{402337} P_4(\cos \theta) + \&c.$$

Since the intensity is proportional to $F^2 + G^2$, we see from the above expressions on substituting the values of A_0 , A_2 , &c., that the intensity is maximum in the direction of the line joining the centres, and that it gradually decreases and assumes the minimum value in the perpendicular direction.

Now, if we further diminish the wave-length—that is, if we assume $ka=2$, we get

$$\begin{aligned}\psi &= 100 \left[A_0 \frac{25-50i}{3125} + A_2 \frac{44+62.5i}{5842.25} P_2(\cos \theta) \right. \\ &\quad \left. + A_4 \frac{405+1170.6i}{1534329.36} P_4(\cos \theta) + \&c. \right] \frac{e^{ik(ct-r+a)}}{r}, \\ &= \frac{1}{30} \left[3A_0(8-16i) + 3A_2(7.53+10.7i) P_2(\cos \theta) \right. \\ &\quad \left. + \&c. \right] \frac{e^{ik(ct-r+a)}}{r}.\end{aligned}$$

Hence

$$F = -8 + 30.12 P_2(\cos \theta) + \&c.,$$

$$G = 16 + 42.8 P_2(\cos \theta) + \&c.$$

We thus see that in this case $F^2 + G^2$ is maximum at 0° where its value is 3941.44 nearly, and that its value gradually decreases and assumes the minimum value nearly 116 at an angle of 61° , and that it again increases and assumes a second maximum value at an angle of 90° , where its value is 558.16 nearly. If we further decrease the wave-length we get results analogous to the above case, namely, that the intensity is maximum in the line joining the centres, and that it assumes a minimum value at some angle intermediate between 0° and 90° and a second maximum value at 90° which is much less than the first maximum. We can easily proceed to a second approximation by determining the values of the coefficients in the expression for the prescribed vibration on the surface of the imaginary sphere so as to agree as closely as possible with the actual state of affairs.

4. *Experimental study of the character of the sound-wave.*

The experimental results described before would be very difficult to explain on any hypothesis other than that which

we have assumed for the origin of the sound. This hypothesis may therefore be regarded as confirmed by experiment to the practical exclusion of any others. It was, however, considered that further experimental study of the character of the sound-wave emitted in different directions by the colliding spheres would be of very great interest.

Attempts have been made to obtain photographic records of the motion of the mica disk under the action of the sound-wave by various methods. The first, due to Siegbahn*, was that of optically recording the motion of a pointer attached to the disk by the use of two microscopes focussed on it, one on either side. Another method which was used was simply to fix a small mirror to the mica disk at the place of greatest angular deflexions and to photograph on a falling plate the motions of the image of an illuminated slit formed by reflexions from the surface of the mirror. While both of these methods gave results confirming the broad indications of theory, the photographs obtained could not be regarded as satisfactory records of the character of the sound-wave owing to the free vibrations of the mica disk excited by the sound of the impact which continued for an appreciable period. Even the first two or three swings, which were much larger in amplitude than the others, showed the free vibrations somewhat prominently. This was evidently due to the highly impulsive character of the sound-wave. Some improvement was obtained by using a mica disk stiffened by attachment to a wire stretched in front of it under tension. (This was taken out from an old gramophone.) The motion of the wire was recorded photographically. Even with this arrangement, however, the free vibrations of the receiving apparatus were prominent. Unless a sufficiently sensitive and at the same time strongly damped recorder is found, there does not appear much hope of obtaining a satisfactory direct record of the character of the sound-pulse. The writer hopes shortly to try the use of an acetylene-gas manometric flame and will also make further experiments with receivers of various types for obtaining an accurate record of the character of the sound-wave.

Experiments have also been made with the ballistic apparatus described in the first part of the paper to compare the effects of the impacts of balls of the same size but of

* Phil. Mag., May 1914.

different materials. On trying the effect of a pair of wooden balls and those of a pair of billiard balls of equal size, the latter were found to give a deflexion about twice the deflexion due to the former. In this we may trace the effect not only of the larger coefficient of restitution of the billiard balls, but also probably of the shorter duration of impact which would be more effective in setting up an impulsive wave-motion in the fluid.

5. Summary and Conclusion.

The intensity of the sound generated by the collision of two solid spheres varies very greatly in different directions relative to the line of impact, and the character of the sound shows a similar pronounced variation. This observation was first made with the unaided ear and communicated to me by Prof. C. V. Raman, and the present work was undertaken at his suggestion to investigate this effect in detail, both theoretically and experimentally. A new type of apparatus in which the ballistic principle is utilized has been used to investigate the intensity of the sound in different directions. The results show a maximum intensity in the line of collision, practically zero intensity on the surface of the cone of semi-vertical angle 67° , and a second, but feeble maximum in the plane perpendicular to the line of impact. These results combined with the indications of theory and further observations on the character of the sound-wave show that, practically speaking, it is produced entirely by the accelerated motion of the spheres during the impact. The law of variation of the intensity of the sound with the velocity of impact and the radius of the balls has also been found and tested experimentally. The investigation was carried out in the Laboratory of the Indian Association for the Cultivation of Science. The writer hopes later on to carry out further work on the subject, particularly in the matter of getting direct records of the character of the sound-wave and observing the effects of oblique impact and the impact of spheres of unequal diameters.

Calcutta,
28th January, 1916.

VIII. *On the Dielectric Constant and Electrical Conductivity of Mica in Intense Fields.* By H. H. POOLE*.

Introduction.

IN view of the possible analogy between electric polarization and magnetization, it seems not unreasonable to suppose that some variation in the Dielectric Constant of a dielectric might be observed as the potential gradient approaches the sparking value. In the following paper an account is given of some experiments on mica (Muscovite), which was chosen on account of the ease of obtaining it in the form of uniform sheets of very small thickness. The method finally adopted consisted in charging a small mica condenser to a known voltage by means of a Wimshurst machine, and then discharging it through a suspended-coil ballistic galvanometer, suitable precautions being taken to avoid piercing the insulation of the latter. As a satisfactory electrostatic voltmeter for reading the pressures involved was not available, another condenser of nearly the same capacity and charged to the same pressure was simultaneously discharged through a second similar galvanometer. In many of the experiments a small leyden-jar was used for this purpose. As the potential gradient in the comparatively thick glass of the jar was only a very small fraction of that in the mica, it was assumed that the capacity of the jar remained constant. To remove any uncertainty on this point, an air-condenser was finally constructed and used in the latest experiments without, however, any appreciable effect on the results obtained. It was found that the ratio of the throws of the two galvanometers was almost independent of the time of charge if this was not less than 20 seconds: this condition was always fulfilled.

It soon became evident that, at high pressures, a very noticeable leak occurred through the mica, so the arrangement was modified so as to include a third galvanometer to measure this leakage current, the possibility of surface leak being eliminated by means of an earthed guard-ring.

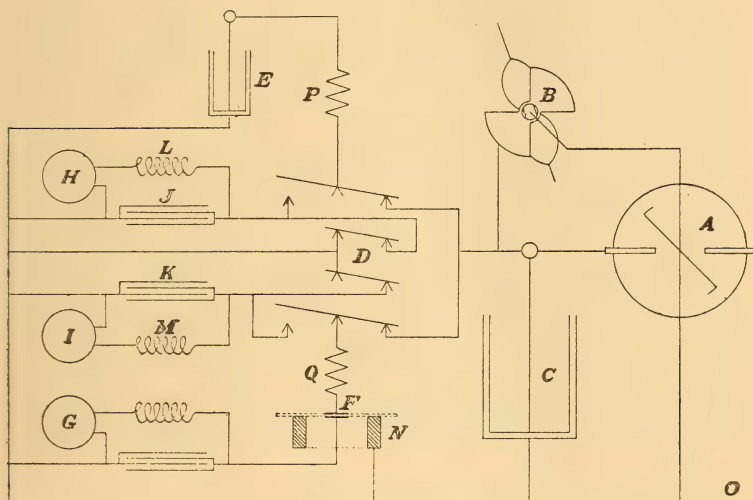
Apparatus.

The connexions are shown diagrammatically in fig. 1. A Wimshurst machine A is used to keep the system charged to an approximately constant potential, which is roughly measured by the electrostatic voltmeter B. A large leyden-jar C makes the pressure regulation easier by adding to the

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capacity of the system. The multiple key D, when in the position shown, connects the small leyden-jar E and the mica test-condenser F to the high pressure source. As already mentioned, E was subsequently replaced by an air-condenser. Any leakage through the mica is measured by

Fig. 1.



the galvanometer G, provided that the potential difference across the mica is constant. When D is in the position shown, the galvanometers H and I are short-circuited. On releasing a trigger the key discharges E and F through H and I respectively, the short-circuit connexions having first been broken. J and K are two condensers, each being about 5 microfarads, L and M are two self-inductances. This arrangement prevents large potential differences from occurring between the terminals of the galvanometers during discharge, while not affecting the quantities passing through them. A similar arrangement protects the galvanometer G in the event of the mica in F being pierced. N is an earthed guard-ring which prevents any surface leak over the mica from reaching G. The point O is connected to earth. P and Q are two high resistances to prevent surges, as explained later.

The first step in the construction of the test condenser was to obtain a uniform sheet of mica about 6 or 7 cm. square and from 0.01 to 0.02 mm. thick. The sheets, obtained by careful splitting, were examined under a microscope between

crossed nicols, and a suitable one selected, free from flaws and variations of thickness. Owing to the almost perfect cleavage these latter always occur abruptly, forming escarpments across the sheet, due to faulty splitting. These escarpments were at once visible between crossed nicols, owing to the variation in tint on either side. The thickness was at first measured with a spherometer reading to about 0.001 mm., but errors due to abrasion of the surface by the spherometer leg, and possibly also to slight buckling of the mica sheet, made it hard to be certain of the thickness to within 0.002 mm. Accordingly an optical method was adopted.

A Nernst filament was mounted a few centimetres in front of the sheet, and the image of its reflexion in the mica was thrown on the slit of a grating spectrometer by means of a lens, the arrangement being such that the reflexion was very nearly normal. Under these conditions the spectrum is crossed by a number of dark bands due to interference. The condition that a wave-length λ should be absent is $2\mu t = n\lambda$, where t is the thickness, μ is the index of refraction for the wave-length λ , and n is any integer. If λ' is the wave-length of the centre of the next dark band towards the red, then

$$2t\left(\frac{\mu}{\lambda} - \frac{\mu'}{\lambda'}\right) = 1,$$

or, if $\delta\lambda$ be the width of one complete band,

$$-2t \frac{d}{d\lambda} \left(\frac{\mu}{\lambda} \right) \cdot \delta\lambda = 1,$$

so

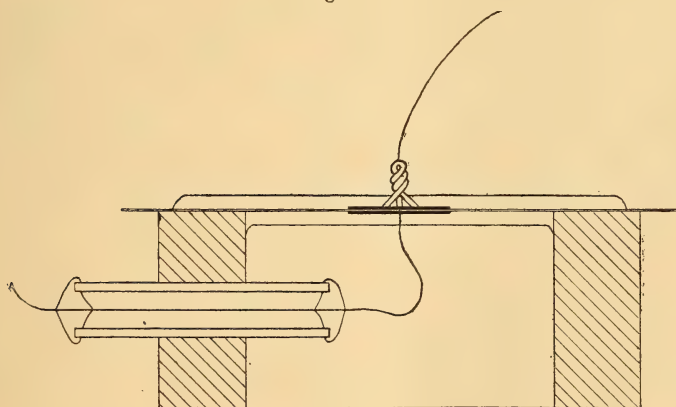
$$t = \frac{\lambda}{2\delta\lambda \left(\frac{\mu}{\lambda} - \frac{d\mu}{d\lambda} \right)}.$$

As the wave-lengths of a great number of dark bands can be accurately measured, we can find $\delta\lambda$ with very fair accuracy for any required part of the spectrum. If we knew μ and $-\frac{d\mu}{d\lambda}$ for the given specimen of mica and the given part of the spectrum we could find t , but, as there was some uncertainty as to the dispersive power of mica, the method was only used as a comparative one. A sheet of mica about 0.06 mm. thick was mounted in front of the spectrometer, and the value of $\delta\lambda$ found for various parts of the spectrum from red to green. The mica was then cut up into twelve pieces the total thickness of which was measured

with a micrometer-gauge and found to be 0.74 mm. So, the thickness of this sheet being known to be 0.0617 mm., we can find the thickness of any other sheet by comparison, since for a given part of the spectrum, $\delta\lambda$ varies inversely as t . The thickness of any sheet measured by this method always agreed with that obtained with the spherometer within the limits of error of the latter. The results were apparently reliable to within about 0.0001 mm.

The conducting sheets of the condenser were tinfoil disks about 1.15 cm. in diameter cut out with a sharp cork-borer. They were stuck on the opposite faces of the mica with seccotine, being pressed into close contact by means of a roller, care being taken to bring them exactly opposite to each other. When the seccotine was perfectly dry, the free surface of the mica was well washed under a tap to remove all traces of seccotine, and finally rinsed in distilled water and dried by gentle heat. The mica sheet was then stuck on the iron guard-ring with seccotine, as shown in fig. 2.

Fig. 2.



The insulated wire shown pressing lightly against the lower disk was eventually connected to the galvanometer G, while the small wire tripod standing on the upper disk was connected to the key D (fig. 1). In the earlier experiments the surfaces of the mica surrounding the disks were exposed to the air. In this case the capacity of the condenser was found to increase with pressure for voltages greater than 1000. This increase apparently depended on the hygrometric state of the air, as it was considerably reduced, though not entirely removed, by placing the condenser in a

desiccator. By carefully watching the condenser in a perfectly dark room, a faint flash was seen surrounding the tinfoil disk at the instant of discharge, indicating that the effective area was increased by a brush discharge over the surface. To prevent this the mica and the tinfoil disks were covered with a thick layer of shellac dissolved in absolute alcohol, and the whole gently warmed for several days. Great care was necessary not to overheat the condenser, as this softened the seccotine. Both sides of the mica were thus treated as indicated in fig. 2. The iron ring which formed the support was placed on an earthed metal stand in a desiccator. The base of the latter was formed of paraffin-wax, through which the various wires passed. As the diameters of the disks were from 600 to 1000 times the thickness of the mica sheet, the edge correction was very

small, so that $K = \frac{kd^2}{16t}$, where K is the capacity of the condenser in E.S.U., k the dielectric constant, d the diameter of the disks, and t the thickness of the sheet, both in cms.

The resistances P and Q were not used in the earlier experiments, but it was found that the mica was sometimes pierced by sudden connexion to the high-pressure source, even though the pressure of the latter was, if gradually applied, insufficient to cause breakdown. This was probably due to a surge, which might cause the pressure in the condenser to rise considerably above that of the source. To prevent oscillations in a condenser circuit, it is necessary

that $R^2 > \frac{4L}{K}$, where R is the resistance, L the self-inductance,

and K the capacity. It was not very easy to estimate the value of L in this case, but, as K was so small, it appeared that a considerable value of R was necessary to prevent risk. The resistances used were carbon lamps whose resistances when cold were about 1200 ohms. Wires were soldered to the contact pieces of the lamps, and paraffin-wax was then run into moulds surrounding the lamp caps to prevent the risk of sparks passing across from one terminal to the other. At first these were inserted between the high-pressure source and the key. This prevented surges at charge, but it was found that the act of discharge pierced a condenser which had stood the applied voltage for over half a minute. When this happened, the throw of the galvanometer I was much greater than it ought to have been, in fact off the scale. Apparently the condenser was pierced by being charged in the reverse direction to a somewhat lower voltage than it

had previously survived, an occurrence which is possibly of some interest in connexion with theories of polarization. To prevent this from happening again the resistances were ultimately connected as shown in fig. 1, thus preventing surges either at charge or discharge.

The discharge-key, shown diagrammatically at D in fig. 1, was formed of stout copper wires dipping into mercury cups in a block of paraffin-wax. The wires were mechanically connected by a rod of sealing-wax. A rubber spring held the key in the discharge position, but the key could be held in the charge position, as shown in the figure, by means of a trigger, on release of which the key very rapidly flew over to the discharge position. As the voltages employed exceeded 5000, it was found necessary to make the gap at the key at least a couple of centimetres long, as otherwise the mercury sometimes splashed up far enough to enable a spark to pass, thus discharging the large leyden-jar into one of the galvanometers. As it was necessary that as short a time as possible should elapse between the disconnexion of the condensers from the source and their connexion to the galvanometers, the mercury cups connected to the source were made about 5 cm. deep and filled with mercury to within about 2 cm. of their tops, which were only about 0.6 cm. wide. The wires of the key, when in the charge position, reached nearly to the bottom of the cups, so that, on releasing the trigger, the key had moved about 3 cm., and thus had acquired a considerable velocity, before it broke connexion with the source. This made it possible to combine rapidity of action with a sufficiently wide gap. The mouths of the mercury cups were made narrow to reduce splashing.

As the leakage through the mica is relatively large at the higher pressures, it is necessary to ascertain whether a perceptible loss of charge would occur during the time of operation of the key. To estimate this interval a standard $\frac{1}{3}$ microfarad condenser was charged to a suitable pressure and discharged by means of the key through a ballistic galvanometer. A resistance of 0.106 megohm was then connected as a shunt across the condenser and the reading repeated. The leak of the condenser through the given resistance in the interval between disconnexion from the source and discharge, is measured by the ratio of the two throws. Measurements made in this way gave values of this interval ranging from 0.0070 to 0.0093 sec. for the side of the key connected to the mica condenser. The figure 0.008 was taken as a mean, and a very small correction proportional to the leakage current was added to the observed

value of the charge on the condenser. This correction might have been neglected without serious error, as it was always very small.

The galvanometers H and I were similar and of the D'Arsonval type, their resistances being about 800 ohms each. As the throws to be observed were large the scales, which were 50 cm. long divided in mm., were set up so that the luminous images were at the ends of the scales when the galvanometers were short-circuited. The scales were set so that perpendiculars to them at their middle points bisected the mirrors. A careful standardization was necessary to test the relation between throw and quantity for each galvanometer. This was carried out with a standard $\frac{1}{3}$ microfarad condenser charged to various pressures up to about 10 volts, measured on a Weston voltmeter, and discharged through the galvanometer, the key D being used. It was also necessary to know the capacity of the condenser E (fig. 1), or, what comes to the same thing, to find what throw will be produced in the galvanometer H by charging E to some known potential. By combining a Tucker hygrometric battery giving about 600 volts with a set of small dry cells of the flash-lamp type and with the continuous current supply of the laboratory, pressures up to about 1000 volts could be obtained. These pressures were measured in sections by means of a Kelvin multicellular voltmeter reading up to 300 volts. Knowing the variation of sensitivity of the galvanometer H with the throw, a scale of voltage was constructed. The characteristics of the two galvanometers are shown in the following table. Here θ is the throw, V the factor by which the throw of H must be multiplied in order to find the voltage of the source, and Q the factor by which the throw of I must be multiplied in order to find the charge on the mica condenser in micro-coulombs.

θ .	V.	Q.
50	12.21	6.71×10^{-3}
100	12.29	6.78
150	12.37	6.84
200	12.44	6.90
250	12.50	6.96
300 and upwards.	12.56	6.99

The figures in the V column refer to the air-condenser

finally used; with the leyden-jar the figures were about 3 per cent. smaller.

The galvanometer G was of the Ayrton & Mather suspended-coil type, the coil being enclosed in a silver tube, so that the galvanometer was dead-beat even on open circuit. Its resistance was about 296 ohms, the constant was found to be 8.0×10^{-3} microampere per scale-division, the deflexion being very nearly proportional to the current over the range employed.

The Wimshurst machine, having vulcanite plates and insulation, generally excited itself without any special warming or other precautions, but it was impossible to say which way it would build up, so that the condensers were sometimes charged the reverse way accidentally. For determinations at voltages exceeding 2000 it was found best to run the machine at a nearly constant speed by means of a motor, and regulate the voltage by the pin-point method. This consists in fixing an earthed pin-point near the plate of the machine. The best position seemed to be just in advance of the collecting-comb connected to the condensers. By moving this pin in and out by means of a rack and pinion, the voltage could be adjusted over a wide range, and kept approximately constant for a considerable time. At voltages less than 4000 the electrostatic voltmeter was used for approximately indicating the pressure before discharge, above that the galvanometer G served the same purpose. The actual value of the pressure was obtained from the throw of H on discharge. The electrostatic voltmeter was of the vertical needle type, reading to the nearest 100 volts. As the divisions were very close together towards the upper part of the range, it could only be read rather roughly.

Experimental Results.

The first definite results were obtained with a mica sheet about 0.017 mm. thick. The value obtained for the dielectric constant k at voltages not exceeding 1000 was about 9. At higher voltages the value of k steadily increased. It was subsequently found that this increase was due to a brush discharge, as already mentioned. The conduction current through the mica was inappreciable for potential gradients less than 0.5 megavolt per cm. At higher gradients it increased rapidly, as shown by the following figures, which, with the exception of the last, are means taken from the curve obtained by plotting the results. The last figure is the highest obtained in this series.

X.	C.	C.
1.0	0.01	0.01
1.5	0.04	0.05
2.0	0.15	0.15
2.5	0.50	0.45
2.83	1.06	0.97

The figures in the first column are the potential gradients in megavolts per cm., those in the second column the current density in microamperes per sq. cm., while those in the third column are the current densities calculated from the formula

$$\log C = \log X + \bar{3}.286 + 0.794 \cdot X,$$

which was found to agree with the later results. It will be seen that the agreement is good at the lower gradients, but that the observed current is too large at the higher gradients. This is almost certainly due to the brush discharge increasing the effective area at high potentials.

This condenser stood a pressure of 4750 volts or 2.83 megavolts per cm.; it was eventually pierced by being inadvertently connected to the high-pressure source when the pressure of the latter had risen too high. The thickness of the mica sheet was subsequently measured optically and found to be 0.0168 mm. The figures given above are calculated on this value.

A new condenser was then made out of a sheet 0.012 mm. thick. The results obtained agreed fairly well with the previous ones, both as regards dielectric constant and conduction current. The latter seemed to be a function of the potential gradient only, as the voltage required to produce a given current was directly proportional to the thickness of the sheet. Further experiments on this point are, however, desirable, as the sheets employed did not differ very much in thickness. The sparking potential through an insulator is known not to be a linear function of the thickness, so possibly a similar effect might occur with the conduction current.

This condenser was in use when the brush discharge was discovered. The mica surfaces were then covered with a thick layer of shellac dissolved in absolute alcohol, and the condenser gently heated for several days in front of an electric radiator. The upper part of the jar E was also coated with shellac. The capacity and the leakage conductivity were both found to have increased by about 35 per cent., while the capacity was now almost independent of the voltage throughout the entire range. This increase in

capacity was probably due to overheating, which may have caused the seccotine to soften and creep along the surface, thus increasing the effective area. Only a few determinations were made, as the condenser was pierced by the discharge from a pressure of 2680 volts, or 2.23 megavolts per cm. The mica may possibly have been injured in some way during the heating as the condenser had previously survived 2800 volts. This sheet of mica was not measured optically, as it had been covered with shellac before the optical method was devised.

A third condenser was then made out of a sheet of mica 0.0173 mm. thick, measured optically. It was coated with shellac as before, but was only gently warmed for several days over a carbon-filament lamp. The resistances P and Q (fig. 1), which had previously been connected between the key and the high-pressure source, were transferred to the position shown in the figure, thus preventing surges at discharge.

Since the tinfoil disks were 1.15 cm. in diameter the capacity K of the condenser

$$= \frac{1.15^2 \times k}{16 \times 0.00173 \times 9 \times 10^5} = 5.315 \times 10^{-5} \times k \text{ microfarad,}$$

where k is the dielectric constant. Hence

$$k = 1.883 \times 10^4 \times \frac{Q}{V},$$

where Q is the charge on the condenser in microcoulombs, and V is the pressure in volts. Q and V are obtained by multiplying the throws of the galvanometers I and H by the appropriate factors, which have already been given.

Several sets of observations were taken using the leyden-jar E. Each day's readings agreed well amongst themselves, but small variations, amounting to a few per cent., occurred from day to day. The value found for k was about 9.0; it was apparently independent of the potential gradient within the limits of experimental error up to a gradient of 3.145 megavolts per cm., which must be very near the sparking value. At this gradient the leakage current amounted to 1.8 microampere per sq. cm. This current agreed well with the formula

$$C = aXe^{bX},$$

or

$$\log C = A + \log X + BX,$$

where C is the current density, X the potential gradient, a , b , A , and B are constants. From these determinations

the values found for A and B were $\bar{3}\cdot25$ and $0\cdot80$ respectively, the logs being to the base 10.

Experiments were also made on the value of k when the pressure was falling. The condenser was charged up to about 5000 volts, and allowed to leak down to the required potential and then discharged. A decided rise in the value of k was observed, evidently due to a hysteresis effect. This was not a permanent change caused by the high pressures, as, on again charging the condenser up to some voltage and then discharging, the same value was obtained for k as before the high voltages had been attained. The conduction current appeared to be about the same whether the pressure was stationary or falling; this could not be accurately determined, as the galvanometer G only measures this current correctly when the pressure is stationary, or nearly so.

As some doubt was felt as to the constancy of the leyden-jar E, an air-condenser was constructed. This consisted of three co-axial copper cylinders, of which the inner and the outer were earthed, and the intermediate one supported midway between them on vulcanite insulators. As the length of the insulated cylinder was about $26\cdot4$ cm., and its mean diameter $21\cdot3$ cm., the air-gap being about $0\cdot625$ cm., the capacity ought to be about 450 cm. or 5×10^{-4} microfarad. The value found by experiment was $5\cdot25 \times 10^{-4}$. This is about 3 per cent. less than the capacity of the leyden-jar.

A set of readings obtained with the air-condenser are shown below. Here V is the voltage, obtained by multiplying the throw of H by the proper constant, Q is the charge on the mica condenser, similarly obtained from that of I, k is the dielectric constant thus found, X is the potential

gradient in megavolts per cm. $= \frac{V}{1730}$, C is the conduction current in microamperes per sq. cm., and C' the current calculated from the formula $\log C = \bar{3}\cdot286 + 0\cdot794 \cdot X + \log X$; D is the difference between the last two columns.

The first two readings are rather low. The first one was obtained by charging with a battery in the course of the voltage calibration, and it is possible that the steadiness of the voltage may have caused k to have a lower value. It appeared, both from these and the previous experiments, that, when the voltage was falling, the value of k was increased, probably due to a hysteresis effect. With the machine it was hard to keep the voltage quite steady, and to this some of the variations observed in k must probably

be ascribed. It is possible that these fluctuations might, on the average, tend to increase the value obtained. The readings marked (a) were obtained after the condenser had been charged for 5 minutes; it will be seen that, at this pressure, there is no evidence of any increase in either the dielectric constant or the conduction current with time.

V.	Q.	k.	X.	C.	C'.	D.
847	0.374	8.33	0.49			
757	0.338	8.42	0.45			
1060	0.478	8.50	0.61			
1210	0.574	8.50	0.70			
1428	0.650	8.58	0.83			
1591	0.726	8.61	0.92	0.01	0.01	0.00
(a) 1576	0.712	8.51	0.91	0.01	0.01	0.00
1874	0.875	8.80	1.08	0.015	0.015	0.00
2184	0.996	8.59	1.26	0.025	0.025	0.00
2245	1.057	8.85	1.30	0.025	0.025	0.00
2450	1.118	8.59	1.42	0.03	0.035	-0.005
2512	1.151	8.65	1.45	0.04	0.04	0.00
2804	1.287	8.66	1.62	0.06	0.06	0.00
2996	1.366	8.58	1.73	0.075	0.08	-0.005
3176	1.468	8.70	1.84	0.10	0.105	-0.005
3369	1.559	8.72	1.95	0.14	0.135	+0.005
3532	1.624	8.66	2.04	0.17	0.165	+0.005
3752	1.713	8.60	2.17	0.215	0.22	-0.005
4272	1.952	8.63	2.47	0.43	0.435	-0.005
4513	2.060	8.60	2.61	0.58	0.595	-0.015
4736	2.152	8.56	2.74	0.725	0.79	-0.065
4849	2.217	8.63	2.81	0.92	0.92	0.00
5015	2.268	8.54	2.90	1.125	1.12	+0.005
5040	2.290	8.56	2.91	1.205	1.145	+0.06
5165	2.339	8.53	2.99	1.335	1.36	-0.025
5205	2.362	8.56	3.01	1.435	1.415	+0.02
5256	2.401	8.62	3.04	1.51	1.515	-0.005
5205	2.433	8.81	3.01	1.49	1.415	+0.075

The last readings were obtained after the condenser had been charged for two successive periods of 5 minutes each. The condensers were discharged at the end of the first period, but a spark at the key prevented a reading of V being obtained; they were immediately charged again and, at the end of a further period of 5 minutes, the figures given were obtained. It will be seen that a slight increase in both k and C was obtained, but the changes were almost within the limits of possible error and further measurements are desirable. They are, however, attended with considerable risk of piercing the mica. The hysteresis effect is clearly shown in the following figures, which were obtained by charging the condenser to about 5300 volts and allowing

it to leak down to the voltages given. The leakage-currents are not given, as, at the pressures at which they could otherwise be accurately read, the fall in potential was so rapid that the lag of the galvanometer prevented any reliance being placed on the readings.

V.	Q.	k.
3878	1.819	8.84
3138	1.483	8.90
2793	1.338	9.03
2463	1.185	9.04
2146	1.040	9.12
1704	0.815	8.98
1266	0.615	9.15
807	0.384	9.01

It would appear as the result of these observations that up to a gradient of 3 megavolts per cm. k is constant to within a few per cent., and also that a small hysteresis rise in k , amounting to about 4 or 5 per cent., is noticeable when the pressure is falling.

Turning to the leakage-current: all the evidence would seem to point to this being a true conduction current, as distinct from an effect due to increase of polarization. In the case of the high-voltage readings given above, the current through the condenser averaged 1.56 microampere for about ten minutes, during which time the quantity transferred through the condenser must have been about 935 microcoulombs. The actual charge on the plates available on discharge was only 2.4 microcoulombs, and may have increased by 0.05 microcoulomb in that time. Moreover, no appreciable reverse current could be observed through the condenser after discharge, as should be caused by a recovery from polarization.

The agreement with the formula given is within the limits of error over the entire range, that is for currents ranging from 0.01 to 1.5 microampere per sq. cm.; as the current is extremely sensitive to small changes of voltage near the upper limits, the actual errors are naturally greater in that part of the scale, though the relative errors are probably smaller.

This close agreement over so large a range is remarkable, and suggests that there may be some definite physical basis for the formula. However, it may be argued that the formula can only be an approximation, as, if it were a general one applicable to all fields, we would expect it only to contain odd powers of X , whereas it, or its equivalent $C = aX e^{bX}$, contains both odd and even powers. The formula $C = aX(e^{bX} + e^{-bX})$, which only contains odd powers, is indistinguishable from the preceding one over the range covered, as the second term is negligible. The first of these formulæ gives a value 5.3×10^{14} ohm cms. for the specific resistance of mica in weak fields, in which Ohm's law would be approximately obeyed, while the second gives 2.65×10^{14} ohm cms.: in this case the agreement with Ohm's law would be much closer, and would extend over a much greater range, as will be seen from a comparison of the curves in fig. 3. As the value given by Kaye and Laby for the specific resistance of mica is 9×10^{15} ohm cm., it seemed to be desirable to make some determinations of the conduction current in weaker fields, using an electrometer to measure the current.

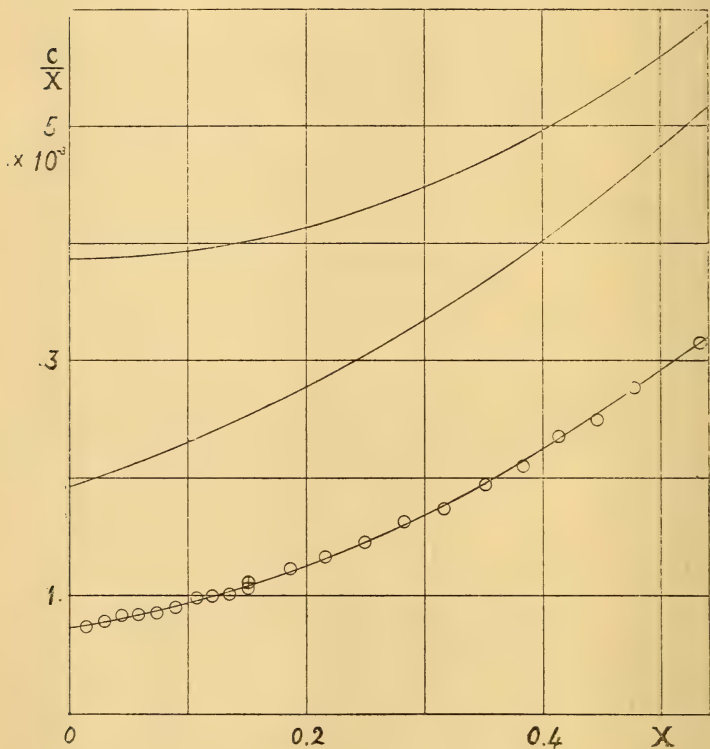
Accordingly some experiments were carried out in this way, the current being measured by the rate of rise of potential of a standard $\frac{1}{3}$ microfarad condenser, the potential being measured by means of a Dolezalek electrometer whose sensitivity was about 100 scale-divisions per volt. The

values of $\frac{C}{X}$ or the specific conductivity are plotted against

X in fig. 3. Here, as before, C is in microamperes per sq. cm., and X is in megavolts per cm. The upper curve has for its equation $C = aX(e^{bX} + e^{-bX})$, and the second curve the equation $C = aX e^{bX}$, where a and b are the values obtained from the previous results, so that either of these curves will fit the values obtained with large values of X . It will be seen that they both fail to represent the current correctly at small fields, though giving results of the right order of magnitude. The simpler formula differs less from the observed value than the other. The measurement of these small currents is rendered very uncertain by soakage and polarization effects, which make it hard to know how much of the observed current is to be ascribed to conduction. The reading marked \oplus was obtained after the voltage had been on for an hour; it lies slightly above the value previously obtained at this pressure, indicating a small rise of the

leakage-current with time. The lowest reading plotted in the figure represents a current density of about 10^{-5} microampere, or $\frac{1}{150,000}$ of the greatest current observed with the galvanometer. For these small gradients the value of the specific resistance found from the curve is about 1.36×10^{15} .

Fig. 3.



There was some evidence that, at still smaller gradients, a further increase in the specific resistance occurred, but the readings were rendered most uncertain by soakage currents.

On earthing the tinfoil disk which had been connected to the pressure source, a small reverse current was always noticeable. A reverse current about 3×10^{-4} microampere per sq. cm. was observed immediately after the mica had been subjected to a gradient of 0.536 megavolt per cm. (which caused a current 16.8×10^{-4}). This reverse current died away rapidly at first, but even after several days a current about 4×10^{-7} microampere per sq. cm. was observed.

This effect, though looked for, was never detected in the galvanometer experiments, showing that this reverse current does not increase with the field to at all the same extent as the direct current.

The low value found for the specific resistance in weak fields may be due to a permanent change in the mica caused by the intense fields to which it had been subjected. On the other hand, it seems equally probable that large variations in the resistivity may occur in different specimens. Some experiments on the insulation resistance of the standard condenser showed that the resistivity of the mica used in its construction was not less than 5×10^{16} . This value was obtained when the potential difference across the condenser was anything between 20 and 30 volts. With smaller pressures considerably larger values for the resistance were obtained, but in all cases the interpretation of the readings was rendered difficult by soakage effects.

Meaning of the Results obtained.

No evidence has been obtained in favour of a theory of electric polarization analogous to the molecular theory of magnetism, if we except the small hysteresis effect observed, as no certain variation of k with X has been established, even in the most intense fields. On the other hand, the results do not furnish any real evidence against such a theory. According to Sir J. J. Thomson's doublet-chain theory*, polarization and conduction are due to the formation of chains of electric doublets under the influence of the field. If there are N doublets per c.c. arranged in chains and the electric moment of each is M , the total electric moment per c.c. will be NM . Now, taking the highest of the figures given, the value found for this moment was about 6240 E.S.U. If we assume that each doublet consists of a + and a - charge each 4.8×10^{-10} E.S.U. at a distance 10^{-8} cm. apart, the value of M will be 4.8×10^{-18} , and therefore N will be 1.3×10^{21} . Taking the formula of Muscovite as $K_2O, 3Al_2O_3, 6SiO_2, 2H_2O$, there are about 2.3×10^{21} molecules per c.c., and each contains 42 atoms. It is therefore probable that, even in the most intense fields employed, the number of doublets arranged in chains formed only a small fraction of the whole, and so it is not surprising that no evidence of saturation was obtained.

* J. J. Thomson, 'The Corpuscular Theory of Matter,' p. 86. Also Phil. Mag. July 1915.

It seems possible that the exponential term in the expression for the current in the strong fields may be due to a distribution of electronic velocities in accordance with Maxwell's law *. According to this the number of electrons

possessing a given energy E should contain $e^{-\frac{3E}{2\bar{E}}}$ as a factor, where \bar{E} is the mean electronic energy. Now, in the case of insulators, few, if any, electrons normally possess enough energy to escape from the atoms to which they are attached, and so take part in the conduction current. In a very strong field, sufficient energy may be contributed by the field to enable some of the most energetic to escape. In order that this may be possible, it is necessary that the electron should possess energy not less than $E_1 - E_2$, where E_1 is the energy required to escape from the atom, and E_2 is the energy contributed by the field during the escape. We may write $E_2 = Xed$, where e is the electronic charge, and d some distance of atomic magnitude. Hence the number of available electrons, and so the current, might be expected to

contain $e^{+\frac{3Xed}{2\bar{E}}}$ as a factor. Comparing this with the results obtained, we find, after reducing X to E.S.U., that

$\frac{3ed}{2\bar{E}} = 5.5 \times 10^{-4}$. These results were obtained at about

9°C. , so, if we assume that the average electronic energy is the same as that for gaseous molecules †, we find that $\bar{E} = 2.8 \times 10^{-14}$, so as $e = 4.8 \times 10^{-10}$ we find that d is about 2×10^{-8} , which is of the right order of magnitude. In the case of the strongest fields, the energy contributed by the field would be nearly four times the mean electronic energy ‡.

If this explanation of the occurrence of the exponential factor is correct, and if, as assumed above, the mean electronic energy is proportional to the absolute temperature, the results obtained at different temperatures should vary considerably. It is proposed to modify the arrangement so as to enable determinations of the conduction current to be made at varying temperatures.

* See Richardson, *Phil. Mag.* August 1915.

† Thomson, *loc. cit.*

‡ The internal field due to the polarization is neglected here. As, however, we may assume that it is proportional to X it will not affect the form of the result.

Summary and Conclusion.

The results obtained may be summarized as follows :—

- (1) No variation in the dielectric constant could be detected up to gradients of 3×10^6 volts per cm.
- (2) The conduction current increases very rapidly with the gradient when the latter is large. The formula $\log C = A + \log X + BX$ represents the current within the limits of error over a wide range. A possible explanation of this formula is suggested on the basis of a distribution of electronic velocities in accordance with Maxwell's law.

In conclusion I wish to express my gratitude to Mr. H. Thrift, F.T.C.D., for the assistance he has so kindly given me in reading the galvanometers.

Physical Laboratory,
Trinity College, Dublin.
April 24, 1916.

IX. *The Tracks of the α Particles from Radium A in Sensitive Photographic Films.* By H. IKEUTI, Research Student Imperial University, Tokyo*.

[Plate IV.]

WHEN a point source of α rays is placed on a photographic plate, the paths of radial α rays can be traced in the film. If the rays are homogeneous, all the tracks will terminate on a spherical surface, whose radius is equal to their range in the substance. The consequence is that the radial tracks appear as a halo. Haloes obtained in this way with radium C have already been illustrated in a previous paper by Mr. Kinoshita and myself†. Although no particular difficulties were encountered in obtaining them with radium C, we were not able to get them with radium A.

Since the above paper was published, I made further experiments and obtained lately some photographs on which the α -ray tracks of radium A are clearly visible. The method employed consisted simply in knocking a plate at the film side with a small iron ball, which had just previously been exposed to radium emanation for a few minutes. The plate was developed usually half an hour later, and examined under a microscope on drying. In the present experiments, Ilford Process Plates only were used.

* Communicated by Prof. H. Nagaoka.

† S. Kinoshita & H. Ikeuti, *Phil. Mag.* March 1915.

A microphotograph of one of the plates is reproduced in Pl. IV. fig. 1; enlarged 120 diameters. Irregular dark areas seen in this figure are the spots at which the plate was struck in the process above mentioned. Around these areas there are to be seen a number of spots, each of which consists of a set of α -ray tracks radiating from a common centre. In these cases, fine dust particles which had adhered to the ball must have been set free by the shock and have settled down on the plate, thus forming the nuclei of the α -ray radiation. Rubbing the ball with a fine emery-paper before exposure to the emanation was found to be effective in obtaining a larger number of spots.

In order to show the spots more clearly, two of them, marked P and Q in fig. 1, are reproduced in figs. 2 and 3 respectively, magnified 345 diameters. The halo in fig. 2 has a mean radius of $\cdot 0507$ mm., and is, as we have shown in the previous paper, due to radium C. When closely examined, another concentric halo is to be seen inside this halo, evidently due to a set of homogeneous α rays from radium A, so that the outer one appears as a corona, resembling a pleochroic halo. The halo due to radium A is more clearly illustrated in fig. 3, in which, however, the halo due to radium C is feeble. In the above two cases, the radii of the outer and inner haloes are $\cdot 0507$ and $\cdot 0348$ mm. respectively, so that their ratio $1 : \cdot 686$ is very nearly the same as that of the ranges of the respective α rays in air, viz. 6.94 cm. : 4.75 cm. or $1 : \cdot 685$. This result shows that the radiant nuclei can be regarded practically as points.

It may be remarked that haloes produced simultaneously on the very same plate are somewhat different; the halo due to radium C is most conspicuous in P while it is very feeble in Q. This singularity seems to have been due to the manner of exposing the iron ball to the emanation. In the above experiments, the emanation was preserved in a glass vessel inverted over mercury, through which the ball was introduced. Under this circumstance, the active deposit found on the ball would consist of two different parts: one deposited directly from the emanation during the exposure, and the other, a portion of the deposit which had already been accumulated on the surface of the mercury in contact with the emanation, and which contained radium C in such a proportion that it was in equilibrium with radium A. It would be natural to suppose that the second part of the deposit covered the ball not uniformly all over the surface, but adhered to it rather irregularly. Consequently, which of the tracks of the α rays from radium A or radium C

predominated would largely depend on the portion of the surface of the ball whence the nucleus was detached from.

This interpretation seems to be confirmed by the following experiments. When the iron ball was first placed in the glass vessel and exposed for a short time to the emanation which was introduced subsequently, a similar process gave a normal result; in all the haloes produced on a plate, the relative numbers of the tracks of α rays from radium A and radium C were the same.

In conclusion, I wish to thank Professors Nagaoka and Kinoshita for their kind interest in this experiment. I have also to thank Professor Tawara of the Metallurgical Department for his kindness in allowing me to use the microphotographic apparatus.

March 1916.

X. The Decrease in the Paramagnetism of Palladium caused by Absorbed Hydrogen. By H. F. BIGGS, B.A., Assistant Lecturer in Mathematics in the University of Manchester.*

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- III. Particulars of the Experiment on Palladium.

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I. INTRODUCTORY.

THE importance of determining the magnetic properties of hydrogen, whether in the form of molecules, atoms, or bare nuclei†, suggested a measurement of the effect of

* Communicated by Sir Ernest Rutherford, F.R.S.

† Hoitsema (*Zeits. phys. Chem.* xvii. p. 1, 1895) has shown that the hydrogen in palladium is monatomic for small hydrogen-content.

E. Newbery and the author made a rough experiment to see whether the hydrogen given off from palladium carries a charge, but could observe no such effect at pressures of a few hundredths of a millimetre.

absorbed hydrogen on the susceptibility of palladium, an effect believed on the authority of Thomas Graham ⁽¹⁾ to be a strong increase to the paramagnetism of palladium. A preliminary experiment, however, contradicted Graham's result, showing that palladium when saturated with hydrogen becomes almost neutral. Since the method adopted is considerably modified from that used in previous absolute determinations of susceptibility, the experimental arrangements are described in detail.

II. METHOD.

1. *Principle of the Method.*

A disk of palladium foil is placed so that it fits within a small coil (coil A) carried on the arm of a torsion-balance, which swings in a non-uniform magnetic field. A current passed through the coil is then adjusted so that the force exerted by the field on the current balances the force on the palladium and on the matter of the arm itself. The process is repeated with the palladium removed. The difference of the balancing currents, multiplied by the total area of the coil, gives directly the magnetic moment, M , of the palladium. The intensity, H , of the field at the same spot is deduced from the throw of a galvanometer connected to the same coil when the circuit of the electromagnet is broken, this deduction being based on the comparison between the throws given by another coil (coil B),

- (a) when the circuit of the magnet is broken,
- (b) when the coil is snatched out of the field.

The mass-susceptibility (that is, the magnetic moment per gram per gauss) is then $K_m = M/Hm$, where m is the mass of the palladium.

2. *The Electromagnet and its Circuit.*

The magnet used was an "optical magnet" with cylindrical coaxial cores, 7 cm. in diameter, whose distance apart could be adjusted by screw collars. To the ends of these cores were screwed pole-pieces designed to give a field in which the ponderomotive force would be parallel to the axis of symmetry of the field, and roughly constant for all points on or near this axis. This form of field, essential in the preliminary experiment, is also useful in this method, as it renders unnecessary the exact replacement of the disk of foil in the same position in the coil—a difficult matter in the case of palladium-foil, which expands when charged with

hydrogen and becomes much crumpled on being charged and discharged. The design of the pole-pieces is given in the next section.

Several points of the circuit supplying the magnet could be connected at will with a voltmeter, and, by using currents which brought the voltmeter pointer exactly to a division of the scale, the current could be kept accurately constant over a range of 1 to 7 amperes.

The fine adjustment of the current was made by a weighted wire passing over a groove in a wide brass tube. The wire used for Bowden brakes was found suitable, and a run of about 10 feet was enough for the adjustment.

The voltmeter had to be placed at some distance to avoid the influence of the electromagnet on its field, and was read through a field-glass, an arrangement which had the advantage of automatically preventing parallax error in the reading.

As it was important for the ballistic measurements that the time taken to break the circuit should be constant, a mercury key actuated by a spring was included in the circuit. The core of the electromagnet was earthed to get rid of electrostatic effects.

3. *The Design of Pole-pieces to give a constant Ponderomotive Force.*

Suppose the field to be symmetrical about the axis of x . Let x, r be cylindrical coordinates. Then, if the ponderomotive force is constant,

$$H_x \partial H_x / \partial x = \text{const.}$$

$$\therefore H_x^2 = \text{const.} \times x.$$

Again, the flux across any section of a symmetrical tube of force is constant,

$$\therefore H_x r^2 = \text{const. for any line of force.}$$

Therefore the equation of a line of force will be

$$x r^4 = \text{const.}$$

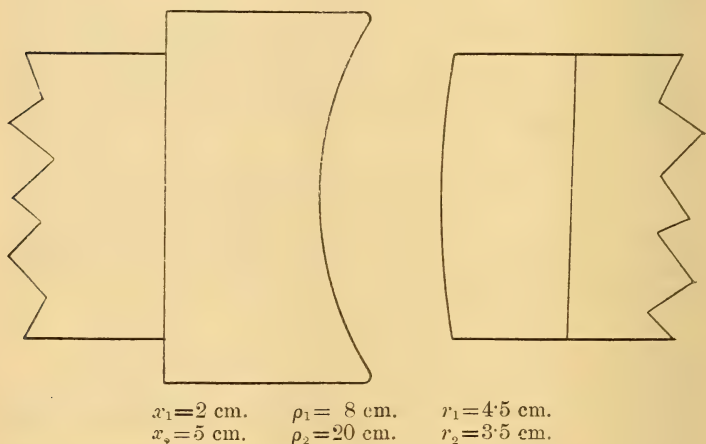
Let ϕ denote the inclination of a line of force to the axis,

$$\text{then } \tan \phi = \frac{dr}{dx} = -\frac{r}{4x}.$$

Then, since the lines of force are normals to the equipotential surfaces, the curvature of the latter at the axis will obviously be $\frac{1}{\rho} = \left(\frac{\partial \phi}{\partial r} \right)_{r=0} = -\frac{1}{4x}$, where ϕ is now considered as a function of x and r .

If, then, the faces of the pole-pieces are spherically curved with radii of curvature differing by four times the distance between the centres of the faces, these faces should coincide nearly enough with such equipotential surfaces, and should give a field with approximately constant ponderomotive force for points on or near the axis. Further, the radii r_1 , r_2 of the faces (measured perpendicular to the axis) should correspond to the radii of a single symmetrical tube of force. Thus we should make $r_1/r_2 = (x_2/x_1)^{1/4}$. In the preliminary experiment referred to, the palladium was balanced against weighed bits of bismuth in a field thus designed. The apparatus was very simple, and no knowledge of the field was necessary. The method gave results correct to about 10 per cent. In the final experiment larger pole-pieces were used, whose dimensions were those given in fig. 1.

Fig. 1.



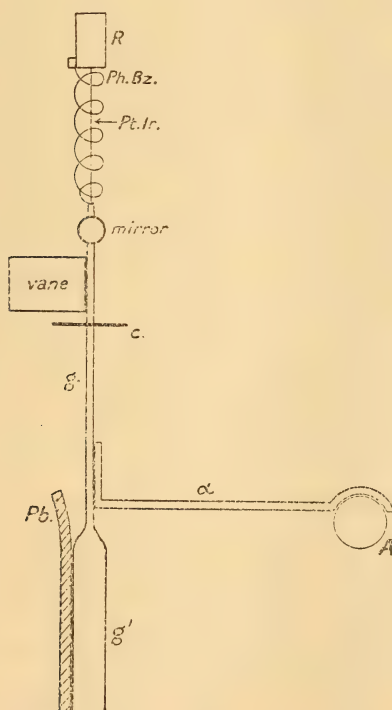
The edge of the concave pole-piece was rounded as shown to avoid disturbance of the field by sharp edges. To test the uniformity of the force a glass rider was made with a bit of bismuth at one end and a bit of aluminium at the other. The rider was placed on the coil in such a way that the ends lay in the axis of the field, and could be turned either with the bismuth in the strong field and the aluminium in the weak field or *vice versa*. By noting the difference of the balancing-current in the coil, when the rider was reversed, it was found that the ponderomotive force only varied by about 1 per cent. over 1.3 cm., the distance between the ends of the rider.

4. *The Torsion-system.*

The torsion-head was fixed to a bridge built on a slate bench, the space under the bridge being enclosed in a draught-tight chamber with a glass back. The space between the magnet-coils was also enclosed in a chamber, whose top came close under the bench. The torsion-stem passed through a hole in the bench, lined by a tube passing into the magnet-chamber close to its back wall.

R (fig. 2) represents the flat end-piece of the rod of the torsion-head. *Pt.Ir.* is the torsion-wire of platinum-iridium, a substance which has been found to give a very constant zero. Its diameter was $\cdot 05$ mm., its length about 5 cm.

Fig. 2.



The torsion-wire was connected to one end of the fine wire of the coil *A*. The other end of the coil-wire passed outside the tube *g*, making a couple of turns round the tube on the way, and was connected to a phosphor-bronze spiral, *Ph.Bz.*, enclosing the torsion-wire, and waxed at the top to *R*. Into

the lower end of the tube g was fixed the drawn-out end of a glass rod g' , whose weight was enough to keep the whole system nearly vertical when suspended. To g' was fixed an arm a of a narrow glass tube, through which passed the leads of coil A slightly twisted together. The object in having the leads twisted, as also in having the spiral *Ph.Bz.* at the top instead of at the bottom (as in a galvanometer), is, of course, that the linkages of the lines of the field with the circuit of coil A shall be confined to the coil itself. The end of the arm a was bent into the arc of a circle, fitting coil A , which was glued to it and thereby stiffened.

The image of the filament of a pocket flash-lamp in the mirror was projected on a scale at a distance of about 60 cm. The paper vane worked in a narrow metal chamber placed on the bench. This arrangement confined the motion to about 3° and provided the required damping. For the highly paramagnetic uncharged palladium in the stronger fields the equilibrium was found to be unstable; closer stops were therefore made by bending a tin wire so that its ends nearly met on either side of the vane which projected slightly above the top of the metal chamber.

Pb is a bit of lead wire whose free upper end was bent so as to bring the centre of gravity of the system on to the axis of g' and g . c is a small glass cross-arm on which a little lead rider was placed when the palladium was removed from the coil, thus keeping the centre of gravity in the same vertical. The arm c also served to clamp the system by being raised slightly on a small lever while the palladium was being inserted in, or removed from, the coil.

It was found necessary to earth the torsion-head as well as the core of the magnet to get rid of electrostatic effects.

5. *Winding and Measurement of the Coils.*

Coil A , the coil on the torsion-arm, and coil B , the exploring-coil, were made as nearly alike as possible, being wound as follows: 20 turns of No. 47 (.04 mm. diam.) silk-covered copper wire were wound on a layer of thin paper slightly waxed to a glass tube of diameter nearly 2 cm. The coil was wound in 3 layers (of 10 turns, 6 turns, and 4 turns, respectively), each layer when finished being thinly coated with seccotine. When dry it was slipped off the tube, and the paper cut away from each side of the coil.

To obtain the total area (mean area of a turn \times number of turns) of the coil, two points were marked on the parts of the wire not wound on, and the distance between these two points was accurately measured, (1) before the coil was

wound, and (2) after the coil was wound, the ends being held out in a straight line. From the length thus found of wire wound on, the total area of the coil can be deduced thus :—

If l be the length of the wire wound, n the number of turns, A the total area, we have, to a close approximation, if the normal to the wire always makes a small angle with the radius, the bar denoting mean value,

$$l = 2n\pi\bar{r},$$

$$A = n\pi\bar{r}^2 = n\pi\bar{r}^2 = \frac{l^2}{4n\pi}.$$

The materials of this coil (either the copper wire or the seccotine or both) proved to be badly chosen, the coil being almost as strongly paramagnetic as the uncharged palladium.

6. *Measurement of the Magnetic Force.*

As already mentioned, a similar coil B was wound, which was used as an exploring-coil to measure the value of the field at a certain spot in the usual way. The coil was waxed into a piece of ebonite fixed to the end of a light lever-arm, weighted just beyond the fulcrum, which, when released, raised the coil quickly out of the field. A change in the rate of raising the arm was found to make no difference to the galvanometer-throw, and it was therefore assumed that the movement was rapid enough to give correct readings. A piece of plate-glass was wedged flat against the lip of the concave pole-piece, and the ebonite piece containing the coil slid against this, so that the coil came accurately back to the same position each time. This position was so chosen that the coil was coaxial with the field and halfway between the pole-pieces, and was, as nearly as could be judged, the same as that afterwards occupied by coil A . Measurements of the field were then taken for four values of the current from 1.7 to 7 amperes. The current for each value, before measurements were taken, was reversed 50 times each way, after which the throw was constant, and the same for both directions of the current. The flux through the coil when raised out of the field was found, by breaking the exciting circuit, to be negligible.

In the galvanometer-circuit were included the coils of a Hibbert standard inductor (and for the stronger fields used, an extra resistance) and the galvanometer was thus standardized for each value of the resistance. The strength of the field at one particular spot is thus found for all the values of the current for the steady state of the magnet.

Further, for one particular value of the current the throw was observed when the circuit was broken by the spring break. The strength of the field at the spot actually occupied by the palladium and coil A is then determined by a comparison between the throws given by coils A and B respectively for the same exciting current when the circuit is broken, the galvanometer being standardized in each case by means of the standard inductor. In fact, if A, B be the total areas, H_A , H_B the values of the magnetic force for the same current, Θ_A , Θ_B the galvanometer-throws on breaking the circuit, S_A , S_B the throws given by the standard in series with coil A, coil B respectively, we may assume (since none of the quantities differ by more than a few per cent. as between A and B) that $\frac{AH_A}{BH_B} = \frac{\Theta_A/S_A}{\Theta_B/S_B}$.

While the magnet was excited, coil A was kept in position by turning the torsion-head.

7. *Adjustment and Measurement of Balancing Current.*

The current was taken from a storage-cell and adjusted by means of a pair of dial-resistances in series, and when the adjustment was reached the two resistances were varied, their sum being kept constant until the voltage-drop over one of them balanced against a standard cell.

III. PARTICULARS OF THE EXPERIMENT ON PALLADIUM.

The palladium used was prepared electrolytically by Messrs. Johnson and Matthey. Traces of iron were found in this, but they can hardly have affected the measurements of the effect of hydrogen, though the determinations of the susceptibility of palladium itself are probably too high on this account. The disk of foil weighed .17 gram.

To control the hydrogen-content of the palladium, the charging was done in a sodium hydrate electrolytic cell in series with a rough hydrogen voltameter. The actual measurement of the gas, however, was made by heating the palladium and pumping off the gas evolved into a tube calibrated for the purpose.

The observations were planned to answer two questions. Let M_1 , M_2 be the magnetic moment in the same field, H of the palladium with different hydrogen-contents v_1 , v_2 cm.³ at N.T.P., and let the (negative) quantity $(M_1 - M_2)/H(v_1 - v_2) = K_v$ be called the "susceptibility-gradient" due to hydrogen. Then the questions to which an answer was sought are:—(1) Is

the susceptibility-gradient independent of the field H ?
(2) Is it independent of the hydrogen-contents v_1 , v_2 ?
Both these questions seem to receive an affirmative answer, as far as the accuracy of the experiment goes. The Tables summarize the results.

H in gauss.	1040	1905	2807	3519
-10^3 K	7.46	7.48	7.70	7.67
10^6 K _m (Pd)	5.60	5.42	5.34	5.29

Susceptibility-gradient due to absorbed hydrogen for various
strengths of field. Hydrogen-content:—
550 × volume of palladium
to 4 × " "
Susceptibility of palladium. "

Vol. of hydrogen ÷ vol. of Pd.				
v_1/v	770	610	227	550
v_2/v	0	550	0	0
-10^3 K _v	7.61	7.01	7.88	7.92

Susceptibility-gradient due to absorbed hydrogen for various
hydrogen-contents.
 $H=1303$ gauss.

The volumes v_1 , v_2 are not accurately known, as is their difference, but the values given serve to indicate the ranges over which the observations were taken. Incidentally the susceptibility of uncharged palladium was found, the values agreeing with previous determinations. It may be noted that the irregular variations observed by Curie⁽²⁾ in the product KT (where T is the absolute temperature) might possibly be accounted for by supposing that the palladium, which was placed in a closed vessel, gave off and re-absorbed traces of hydrogen when the temperature was raised, but it seems more likely that they are due to slow reduction of iron impurity and subsequent re-oxidation, since similar irregularities were observed by Honda⁽³⁾ in the case of impure palladium, but not in the case of a very pure sample.

The temperature throughout the experiment varied within a few degrees of 15°C .

It will be seen from the Tables that the figures for what has been called the susceptibility-gradient show no systematic variation, in spite of the fact that palladium itself shows a marked decrease in susceptibility with increase of field. This result might lead us to suppose that the diamagnetism of the hydrogen absorbed is simply added algebraically to the paramagnetism of the palladium, if it were not that the value obtained for the susceptibility-gradient due to the absorbed hydrogen, 7×10^{-8} , is probably much greater than the diamagnetism of hydrogen gas. It is true that the latter has never been satisfactorily determined*, but it is probably beyond the range of the methods used. Again, on Langevin's⁽⁴⁾ theory, if the diamagnetism of hydrogen itself, whether in the form of molecules or atoms, were equal to the value here found, the radius of the orbit of the electrons in the hydrogen molecule or atom would be $2.4 \times 10^{-7}\text{ cm.}$, which is certainly too large in either case. The observed effect is therefore probably due to some interaction of the hydrogen and the palladium, whereby the latter's paramagnetism is reduced. It should be noticed that if the susceptibility-gradient due to the absorbed hydrogen remains constant till saturation is reached, since palladium will absorb 900 times its volume of hydrogen, the resulting combination should be diamagnetic. Unfortunately no pains were taken to saturate the palladium, and the apparatus was taken down to make way for an experiment on the susceptibility of hydrogen gas before this point was noticed.

I wish to express my best thanks to Sir E. Rutherford for allowing me the use of all the apparatus required for this research.

* Kammerlingh Onnes and Perrier (*Ak. Wet. Amst.* xx. p. 81, 1911) have made a rough determination of the susceptibility of liquid hydrogen; also Pascal (*Ann. Chim. Phys.* viii. sér. 19, p. 28, 1910) deduces a value for the susceptibility of hydrogen from that of its compounds. Both these determinations give a value of about 2.4×10^{-10} for the susceptibility per cm.^3 , which would mean that the radius of the electronic orbit in hydrogen is about $1.2 \times 10^{-8}\text{ cm.}$ Bohr's theory (*Phil. Mag.* xxvi. pp. 5, 863, 1913) gives $.5 \times 10^{-8}\text{ cm.}$

XI. *On the Change in the Resistance of a Sputtered Film after Deposition.* By Miss E. W. HOBBS, B.Sc. (Bristol)*.

A CONSIDERABLE amount of work has been carried out by various investigators on the electrical resistance of metallic films deposited by cathodic sputtering. In the main, the films dealt with have been previously aged by exposure to air and high temperature after their removal from the discharge-vessel, but in some instances arrangements have been made for measuring the resistance of a film in position during and after deposition. Thus in 1912† Kohlschütter and Noll published some results of experiments on thin films of silver deposited in hydrogen, nitrogen, and argon, in which measurements were made *in situ* of the changes in resistance with time which occur when a film is left *in vacuo* and in various gases after deposition. The following is an account of experiments on thin films of platinum and palladium carried out on somewhat similar lines, in which a fuller study of these changes was made. In particular, the effect of admitting gas into the discharge-vessel was investigated in detail; it is this that forms the chief subject of the paper.

APPARATUS.

The apparatus did not differ in essentials from that usually employed for the deposition of films. The films, 1.2 cm. long and 1 cm. wide, were deposited on polished strips of fused silica, which rested on the anode. In order to measure their resistance, leads were introduced into the discharge-vessel and contact was made with the film through thick end deposits, in some cases by soldering wires to them and in others by means of a mercury pellet. The latter method was found to be by far the more convenient, and was almost invariably employed. It was quite satisfactory if the mercury were covered by mica disks in order to protect it from electrical bombardment. An exploring point connected to a Braun electrometer was introduced for measuring the cathode fall of potential during deposition, and the current was controlled by a cadmium-iodide amyl alcohol adjustable resistance. While adjustments were being made, the silica could be screened by a mica shield, rotated by a ground-glass joint, but it was not necessary to use this in most of the work.

* Communicated by Dr. A. M. Tyndall.

† *Zeitschrift für Electrochemie*, vol. xviii. p. 419 (1912).

RESULTS.

The mechanism of cathodic sputtering at low pressures is probably similar to that of the production of colloidal solutions of metals in Bredig's method. The metallic particles are molecular aggregates possessing a negative charge, which they lose when, or shortly after, they strike the plate on which the film is deposited. Hence a semi-stable layer of particles is formed, having an electrical conductivity which depends on the conditions during deposition and on the thickness of the layer. In particular, Kohlschütter and Noll (*loc. cit.*) have shown that the nature of the residual gas during discharge has a great influence on the fineness of the particles deposited, and thus on the initial resistance of the film and on the subsequent changes which occur.

In the matter of adherence to the silica, films may vary from a soft film easily rubbed off to one of a hard metallic nature. An important factor in this connexion is the cathode fall of potential, which controls the speed with which the particles impinge on the plate, though the limits of adherence vary with the nature and pressure of the residual gas in the discharge-vessel.

The Ageing of Films.

It is well known that films deposited by cathodic sputtering undergo a process of ageing, entailing a change in the properties of the film. In 1898* Fawcett showed in this laboratory that the change may continue for months, but that it is hastened by rise of temperature.

The change in condition after deposition is made apparent by measuring the electrical resistance, which decreases with time. An investigation was made of the effect of ageing *in vacuo* on the electrical resistance of platinum and palladium films. The films were deposited at pressures varying from .14 mm. to .4 mm. in air, and with a modified form of apparatus from .5 mm. to .8 mm. in hydrogen; the pump was put on immediately discharge ceased, and resistance-time measurements were taken.

Deposition in Air.

A number of results obtained for platinum films deposited in air (the case most fully investigated) are given in fig. 1,

* Phil. Mag. xlv. p. 500 (1898).

Fig. 1.

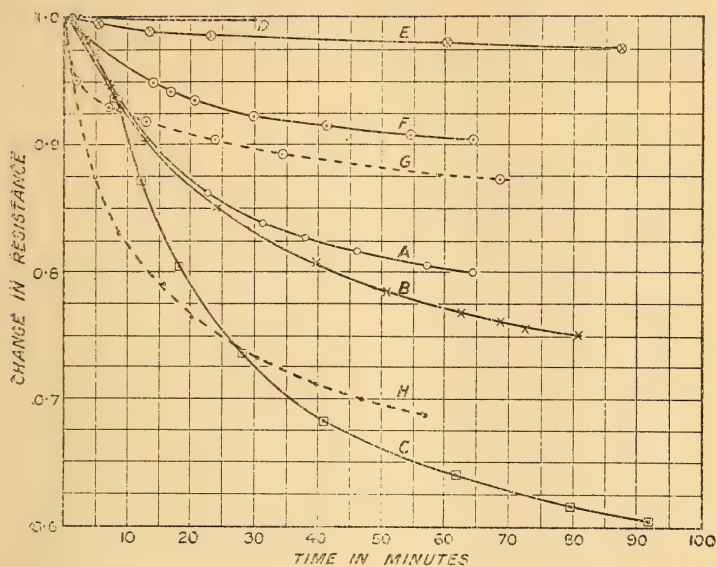
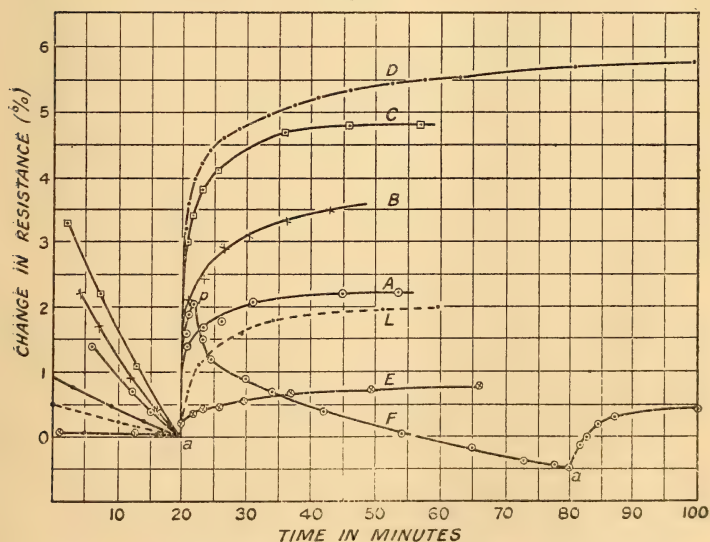


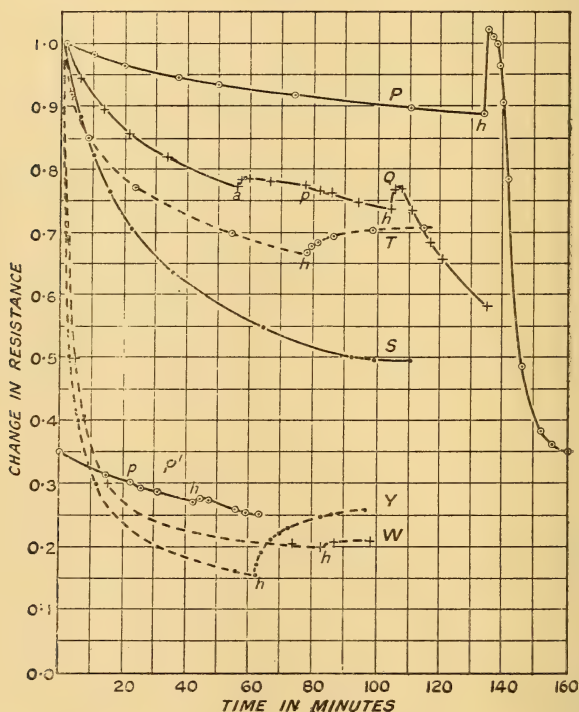
Fig. 2.



in which resistance-time measurements for films A to F are plotted as full curves. Similar results for palladium in air are shown in the early portions, as far as the point marked

(a) or (h), of the full curves in fig. 3. The ordinates express the resistance of the film as a fraction of the first value in each set of observations, which were always started about a minute after the discharge was cut off. Though the results

Fig. 3.



of the first 90 minutes of ageing are alone shown, the readings were in many cases extended over many hours; these confirm the view that the film is settling down to a steady state, which, however, may not be reached for days.

On the question whether the rate of ageing of a film depends on its thickness, there is no evidence one way or the other. This is made apparent on reference to Table I., in which all the results for platinum, including those plotted in fig. 1, are given. R_0 is the resistance of the film when the pump is put on after the discharge has ceased, and R its resistance one hour afterwards. It will be seen that $\frac{R}{R_0}$

is never greater than one, but varies in a haphazard manner from film to film. This is no doubt due to the impossibility of ensuring a constant texture of film throughout a given series.

TABLE I.

Film	R_0 , ohms.	$\frac{R}{R_0}$	Film	R_0 , ohms.	$\frac{R}{R_0}$	Film	R_0 , ohms.	$\frac{R}{R_0}$
C.	243	0.91		4000	0.85	M.	14,140	0.81
	278	0.98		4870	0.92		14,980	0.64
	407	0.90		5550	0.85		15,110	0.88
	1070	0.72		7500	0.89		15,930	0.74
B.	1545	0.98	L.	7600	0.95		16,270	0.51
D.	1790	0.80	E.	9200	0.77		20,500	0.60
	1970	0.87		11,800	0.89		29,000	0.53
A.	3100	1.00	F.	12,550	0.95		68,400	0.84
K.	3400	1.00						

The reason for the metastability of a newly deposited film cannot be definitely stated. The most feasible explanation is that of Kohlschütter and Noll (*loc. cit.*), who suggest that owing to surface-tension the finely divided particles in a freshly deposited film undergo a process of coalescence, by means of which the gaps between the particles are lessened in number, and so in general the resistance falls. Further evidence that such a process occurs was obtained from an examination of the optical properties of films. Kohlschütter and Noll point out that if the conditions are such that coalescence can only go on with increase of distance between the particles, the resistance will rise and may reach infinity. As evidence of this they cite the rise in resistance that they observed if silver films were deposited in hydrogen and left to age *in vacuo*, when it was found that the resistance decreased to a minimum and then rose to infinity. From a number of different observations they conclude that the metal is deposited in hydrogen in a coarse form, and hence that less material is required for a film of given resistance than if the deposition had occurred in nitrogen or argon; this increases the chance of a break in the film. In the author's experiments on platinum and palladium films, despite the extreme thinness of many of the films, this rise in resistance was never observed—the resistance always decreased with time when the film was left to age *in vacuo*, whether the deposition had taken place in air or in hydrogen.

The phenomenon may be viewed somewhat differently by
Phil. Mag. S. 6. Vol. 32. No. 187. July 1916. L

supposing that the material originally deposited in an amorphous state undergoes a slow process of crystallization; in other words, that this is another example of the metastability of metals observed by Cohen and his co-workers. Unfortunately, no light is thrown upon the true nature of the change by an analysis of the ageing curves, as they are not expressible in terms of any simple formula.

On both views, the effect of heat, unless it caused a break in the film, would be to bring about a marked fall in resistance, in the former case owing to the decrease in viscosity and the consequent acceleration of the process of coalescence, and in the latter because of the increased velocity of the allotropic change at higher temperatures. Fawcett (*loc. cit.*) showed that the decrease in resistance which occurred in his experiments when a film was heated was accompanied by a marked hardening of the surface of the film. This effect was observed in the present work.

It is important to note that the ageing must be proceeding *during* deposition, and consequently it is another factor influencing the nature of the film obtained at the cessation of discharge. Again, while the discharge is taking place, films being deposited near the cathode will be affected to some extent by its temperature, which Hodgson and Mainstone* have shown is a function of the current through the discharge-vessel and the cathode-fall of potential. All attempts, therefore, to obtain quantitative results of the rate of laying down of the film by measurements of resistance during deposition are doomed to failure.

Deposition in Hydrogen.

An investigation of the change in resistance with time was made for films deposited in hydrogen, and the ageing curves in this case were found to be similar to those obtained with air as the residual gas during discharge. In fig. 1 the dotted curves G and H represent two ageing curves for platinum films ($R_0 = 8400$ ohms and 9300 ohms respectively) deposited in hydrogen, and the early portions of the dotted curves in fig. 3 give similar results for palladium. It will be noted in the case of palladium that rates of ageing are shown which are far greater than those obtained under any other conditions (*vide* curves W and Y, fig. 3). There is strong evidence that in many instances this change is a continuation of a change which has been proceeding at a very rapid rate during the discharge itself, and this probably applies to platinum films as well, though their case has not received such full investigation.

* Phil. Mag. vol. x. vi. p. 411 (1913).

Thus a number of films have been observed in which the resistance *during* deposition has fallen at a very rapid rate, far greater than in air, although it is known that hydrogen is not a gas specially favourable to cathodic sputtering. Moreover, the resulting film had a transparency which suggested a much higher resistance. It is probable that this rapid change is set in action by the sorption* of hydrogen during deposition; and this view receives support from results given below on the effect of admitting hydrogen to the discharge-vessel; it will be discussed again in that connexion.

Effect of Addition of Gas.

It is known that films of sputtered metal take up gas during deposition. Owing to the relatively large surface, some sorption of gas in a film will occur when gas is admitted into the discharge-vessel after deposition; an investigation of its effect on the electrical resistance was made.

In the first experiments, dry air was admitted to the vessel containing a film which had been ageing for some time. The pressure reached a constant value (atmospheric) within a few seconds after the stopcock, through which the air entered, had been opened. Typical results for a number of films are shown as full curves in fig. 2. The ordinates are changes in resistance expressed as a percentage of the resistance of the film at the moment that the gas is admitted. The abscissæ are values of time in minutes. The initial parts of the curves give the later stages of ageing, the air being admitted at the point (*a*) in every case. The films to which the full curves shown here refer are those for which the complete ageing curves are given in fig. 1; they are designated by the same symbols.

Kohlschütter and Noll (*loc. cit.*) record some results of admitting nitrogen to silver films, deposited in nitrogen, immediately after their deposition. In one case only do they appear to have observed a rise in resistance, but state that in general the ageing is apparently stopped, though a measurement of the resistance of one film 15 hours after the admission of the gas shows that in reality the process is delayed and not brought to a complete standstill. On the removal of the nitrogen, the ageing proceeded.

In the author's experiments a rise in resistance with time

* The term sorption has been suggested by McBain to include under one title the phenomena of adsorption (surface condensation) and absorption (solid solution).

is shown in *every* case on the admission of air*. Except in the curve F, which is separately discussed below, the rise continued for not less than 25 minutes, and for most films still longer at a slow rate. There is no fall in resistance in any of the cases given to indicate that the process of ageing, which is shown to a marked extent in curves A, B, and C, continues after the admission of gas; but in some of the films investigated the initial rise was followed by a very gradual decrease in resistance. In such cases, the ageing was evidently still continuing though at a decreased rate. It may therefore be present in curves A, B, and C though swamped by the gas effect, and consequently the effect of the admission of gas on these films cannot be examined with any certainty, since the effect of ageing is quantitatively unknown. Fortunately investigations have been carried out on films in which the ageing was negligibly small. Curves D and E are examples of these. Since they no doubt represent the true effect of the air and, moreover, show extremes of magnitude, the experimental numbers are given in Table II.

TABLE II.

Film D.		Film E.	
Time in mins.	Resist. in ohms.	Time in mins.	Resist. in ohms.
0.0	3100	0.0	1545
15.0	3090	5.5	1537
30.0	3090	7.75	1534
40.0	3090	13.5	1526
41.0	Air admitted.	23.2	1520
41.75		44.7	1516
42.3	3200	60.2	1514
44.4	3218	82.5	1512
45.3	3223	105.5	1511
47.5	3229	106.5	Air admitted.
49.4	3234	107.0	
52.75	3229	108.5	1516
58.25	3245	110.0	1517
62.0	3248	116.0	1519
73.0	3254	123.8	1520
84.7	3258	136.0	1521
102.5	3262	154.3	1522
138.0	3267	4 hours later.	1524
16 hours later.	3296	Next morning.	1526
21 " "	3297		

* It is possible that this might be missed if the experimental method were such that the ageing was proceeding at a rapid rate at the moment when the gas was admitted.

The main point to notice in these results is the rapid initial rise in resistance followed by a more gradual rise which may be prolonged for a long period of time. Again, no relationship between the change in resistance and the thickness was obtained, the effect of the gas depending on the nature of the film.

In film F, the pump was put on 2 minutes after the air had been admitted. The result was to produce a rapid fall in resistance changing to a more gradual fall. It is seen that the effect of the air is not immediately counteracted by putting on the pump, and that, as in the experiments of Kohlschütter and Noll with nitrogen, the ageing continues in a film placed *in vacuo* after a treatment with air.

It would have been interesting to have obtained sorption curves by measuring the change in pressure with time in order to compare them with the resistance-time curves; but owing to the smallness of the amount of platinum in a film, it was quite impossible to do this. Indirect evidence that there is an intimate connexion between the two phenomena may, however, be given. Of the various experiments on the sorption of gases by solids, those of Bergher* on charcoal are of particular interest, as they were carried out under conditions of temperature and pressure comparable with the above. The curves that he obtained by plotting volumes of gas taken up with time are very similar to the resistance-time curves shown here.

This connexion between sorption and change of resistance if real is interesting. A mere cessation of ageing could be attributed to a decrease in surface-tension accompanying the adsorption of gas. But the fact that a rise in resistance occurs, and moreover continues for a long time, suggests that the result of admitting gas into the discharge-vessel is not solely a surface-tension effect. It is more possible that some change in orientation of the individual units taking part in conduction is brought about by the air adsorbed on the film and absorbed in the molecular clusters themselves. In the present state of the theory of conduction in metals, further discussion of this point would be unprofitable.

Admission of Hydrogen.

The effect of adding hydrogen to platinum and palladium films is not in the general case a simple one. There may be (1) the normal rise in resistance following the admission of gas; (2) a fall in resistance due to the evolution of heat

* *Ann. d. Phys.* xxxvii, p. 472 (1912).

accompanying the sorption of hydrogen which involves a marked change in the nature of the film; and (3) a fall in resistance due to the combination with evolution of heat between the hydrogen and any traces of oxygen present in the film under the catalytic action of the finely divided metal. The relative values of these three effects may vary greatly with the condition of the film and with its previous history. Evidence of (2) is given in the ageing curves of films deposited in hydrogen, shown as dotted curves in figs. 1 and 3. It will be noticed that in the hydrogen curves, both for platinum and palladium, the initial portions are much steeper than those of the air curves, which often show a lag in the ageing rate at the beginning (*e. g.* curve C, fig. 1).

Still greater changes have been observed in the short interval of time that elapsed before the pump was put on after the cessation of the discharge. For instance, in curve H (fig. 1) the film was deposited in hydrogen at a pressure of $\cdot 5$ to $\cdot 6$ mm., and was left for about 1 minute at this pressure. During that time its resistance fell from something greater than 30,000 to 9300 ohms.

Striking examples of the effect wrought in a film by hydrogen are given in the curves of fig. 3. They refer to palladium films, aged *in vacuo*, to which dry hydrogen was admitted, the point of admission being marked (*h*). In the case of curve Q, the film was previously supplied with air at the point (*a*) and the pump was put on at the point (*p*). Films P and Q were deposited in air; films T, W, and Y in hydrogen. In the films P and Q the initial rise in resistance at (*h*) may be taken to represent the direct effect of the gas. The subsequent rapid fall, which sometimes almost swamps the initial rise, is attributable to the changes produced in the film by the heat evolution which accompanies the large sorption of hydrogen in palladium. Further complications may be introduced by the combination between hydrogen and traces of oxygen present in the film. This effect in the case of the curves T, W, and Y, deposited in hydrogen, will be absent. Some hydrogen will have already been taken up by the films, and the consequent evolution of heat will have taken effect; so that on admission of hydrogen at the point (*h*) the ordinary rise in resistance due to the sorption of gas alone is shown.

In most instances, when hydrogen is admitted to a palladium film, the percentage rise in resistance due to the *direct* effect of the gas is much greater than that produced by letting air into the discharge-vessel. This might be

expected from what is known of the relative sorption of hydrogen and air in palladium.

The latter stages in the examination of film P deserve notice. After exposure to hydrogen for 48 minutes, the film was placed *in vacuo* for 21 minutes and hydrogen was again added. The results are shown as P' at a lower level. The second dose of hydrogen, it will be seen, only caused a small rise in resistance and no succeeding marked fall. It may be assumed that at this stage the film has been transformed into a state in which the sorption of hydrogen is only slight. The rate at which palladium takes up hydrogen is known to vary not only with the form of palladium employed, but also with different samples of the same kind.

The result of adding hydrogen to platinum films deposited in hydrogen is similar to that with palladium, and needs no further comment. The addition of hydrogen to platinum films deposited in air gave changes in resistance which varied in magnitude and in sign for different films. The following examples may be cited of films, deposited in air, to which hydrogen and air were added *alternately*, the pump being applied between the additions.

- (1) Film K. (Resistance 3400 ohms.) The first addition of hydrogen gave a small rise of 0.8 per cent. in 14 minutes; subsequent additions gave small falls from 2 per cent. to 0.5 per cent. in 10 minutes. Air always gave a rise in resistance.
- (2) Film L. (Resistance 7660 ohms.) Small rises, from 2 per cent. to 0.2 per cent. in 25 minutes, followed every addition of hydrogen. The rise given by the first addition of hydrogen is shown in curve L (fig. 2). It will be seen that the change of resistance-time curve is similar to those for the addition of air. As before, a rise in resistance always followed the admission of air.
- (3) Film M. (Resistance 4900 ohms after having been subjected to several treatments of air but none of hydrogen.) The resistance fell to 1900 ohms in 80 minutes on the addition of hydrogen. Subsequent additions of hydrogen and air both gave slight rises, in the former case followed by a fall.

If it be assumed, as seems probable, that the relative values of the direct effect of hydrogen on the resistance and the indirect effect due to evolution of heat in sorption

and in chemical action may vary widely with different films, the results given above are not necessarily conflicting.

In no case examined, whether the film was deposited in air or hydrogen, has air produced a fall in resistance, though in many instances the film had been previously treated with hydrogen which had probably not been completely removed by the pump. It is clear, therefore, that in such cases, namely when air is in excess, the effect—if it be present at all—of the combination of hydrogen and oxygen in the film under the catalytic action of the platinum must be negligibly small.

Kohlschütter and Noll state that the catalytic power of a platinum deposit varies with the nature of the residual gas in the discharge-vessel during deposition. It is great if the gas be argon, when the deposit consists of very fine particles, and less if the gas be hydrogen, when a coarser film is obtained.

It is noteworthy that the electrical properties of a film deposited by cathodic disintegration are not the same as those of a powder. Thus Goddard* has shown that the resistance of the powders he used decreased when gas was admitted, and increased when the gas was pumped out. Some rough experiments of the author on palladium-black gave qualitatively similar results. The above results show that with sputtered films the converse is true.

McClelland and Dowling† have observed some curious effects by applying a transverse electrical field to a thin layer of conducting powder. On removal of the field it was found that the powder had in general a greatly increased conductivity. There is no evidence that this occurs in the case of sputtered films. Thus in a few cases the author tried the effect of placing a plate parallel to and at a distance of 3 or 4 mm. from the film and charging it to 1600 volts at atmospheric pressure. On removal of the field no change in conductivity was observed to have taken place.

Summary.

1. A study of the physical changes occurring in a metallic film after deposition by cathodic sputtering has been carried out by measurements of the changes with time in the electrical resistance of the film when retained *in vacuo*.

2. The *direct* effect of admitting gas into the vessel containing the film under observation has been shown to be a

* Phys. Review, xxviii. p. 405 (1909).

† Proc. Royal Irish Acad. vol. xxxii. A. No. 5 (1915).

rise in resistance which may continue for a considerable time. Evidence is given that this phenomenon is intimately connected with the sorption of gas by the film.

The above experiments were carried out in the Physical Laboratory of the University of Bristol, under the direction of Dr. A.M. Tyndall. I am greatly indebted to him for the help and encouragement that he has given me throughout the work. My thanks are also due to Mr. J. D. Fry for his help in the earlier stages of the work, particularly in the construction of the apparatus.

The expenses of the research were defrayed by a grant from the University of Bristol Colston Society.

April 20th, 1916.

XII. *A Semi-automatic High-Pressure Installation.*

By The Earl of BERKELEY, *F.R.S.*, and C. V. BURTON, *D.Sc.**

AS the primary object of the installation was to determine the compressibilities of solutions, we add to the description of the pressure-apparatus itself details of two devices found useful in this connexion.

Previous work† on the subject has shown us that when altering the pressure, the alteration must take place slowly; otherwise, the mercury rising in the capillary of the piezometer will leave behind it an appreciable quantity of the solution clinging to the walls, and thus vitiate the estimation of the change of volume. The method by which this is overcome is given in some detail on p. 159. Again, when the pressure has been changed, it is essential that the thermodynamic heating or cooling brought about thereby should have time to dissipate before the level of the mercury meniscus is read. In other words, the new pressure must be maintained constant for a period of one to two hours; an automatic device for doing this is shown in fig. 6.

The general arrangement.—The working fluid is castor oil‡, and the pressure is applied by means of a pump made by Messrs. Schaffer and Budenberg. As this pump is of a pattern regularly made and catalogued by the makers, its

* Communicated by the Authors.

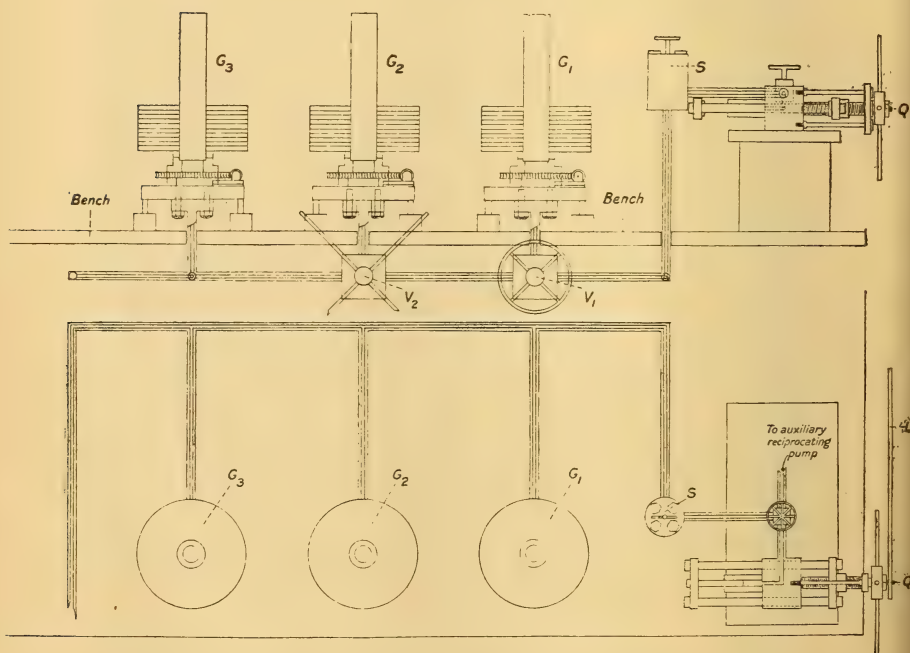
† This research is not yet published.

‡ Ordinary "cold pressed" castor oil becomes somewhat gelatinous under high pressures. If, however, it is cooled to 0° C. until no further precipitate comes out and then filtered, the oil will remain unchanged for quite a long time.

construction and working need not be referred to in detail. For raising the pressure rapidly to (say) 600 atmospheres, there is a low-pressure reciprocating plunger operated by a lever and working in conjunction with automatic suction and discharge-valves in the usual way. For the attainment of higher pressures, the low-pressure pump is then shut off by screwing down a needle-valve, and a high-pressure plunger of small section is gradually forced inwards by means of a screw provided with a star-handle.

The arrangement of pump, tubing, gauges, and valves is shown diagrammatically in plan and elevation in fig. 1.

Fig. 1.
Elevation.



Plan.

The gauges (G_1 , G_2 , G_3) are of the "dead-weight" type, the essential feature being an accurately cylindrical piston of hardened steel fitting as closely as possible without actual rubbing contact in a mild steel cylinder, so that, whilst the combination forms an almost oil-tight joint, the mechanical friction opposing a lengthwise motion of the piston is

in any case small, and under working conditions is reduced to a negligible amount by keeping the piston in rotation.

The pressure-ranges for which the three gauges are designed are—

0	to	200 atmospheres	for G_1 ,
200 atmos	to	800	„ G_2 ,
800 „	to	1500	„ G_3 .

The corresponding piston-areas being approximately—

0.10 sq. in. for G_1 , 0.02 sq. in. for G_2 ,

and 0.01 sq. in. for G_3 .

The needle-valves V_1 , V_2 (fig. 1) are used for shutting off the gauges G_1 , G_2 when these are not in use. No such valve is required for the high-pressure gauge G_3 , which carries a sufficient load to keep it from lifting when the pressure is low enough to be measured by G_1 or G_2 .

Details of gauges.—A piston intended for high-pressure measurements must have a small sectional area if the weights are to be kept within manageable limits, and it must be very closely fitted. The oil which supports the rotating piston must consequently be under a considerable pressure before the lubrication is free enough to render the arrangement sufficiently frictionless. It is for this reason that any single gauge is only serviceable over a restricted range of pressure.

The gauge G_2 , used for intermediate pressures, is shown in more detail in figs. 2 and 3. The upper end of the piston* has the form of a spherical cap, upon which rests the hardened steel pin P spherically hollowed to a somewhat greater radius. This pin is inserted in, and thus supports, the tubular weight-carrier C. To the lower end of C an annular plate Q (fig. 3) is pivottally connected, with freedom to tilt about a horizontal axis A A, while two pins B project downwards from Q and engage in radial slots in the worm-wheel V. The worm-wheel is kept in rotation, and the weight-carrier and weights must necessarily partake of this rotation, the transmission of the motion being such that the forces applied to the carrier are sensibly equivalent to a

* The pistons and cylinders were made for us by Messrs. Elliott, and are copies of those used at the National Physical Laboratory. The designs were given us by Mr. Jakeman of that laboratory, to whom our best thanks are due.

couple. The rotation of the weight-carrier is communicated to the piston in much the same way, except that the equality between the constituents of the driving couple depends, in

Fig. 2.

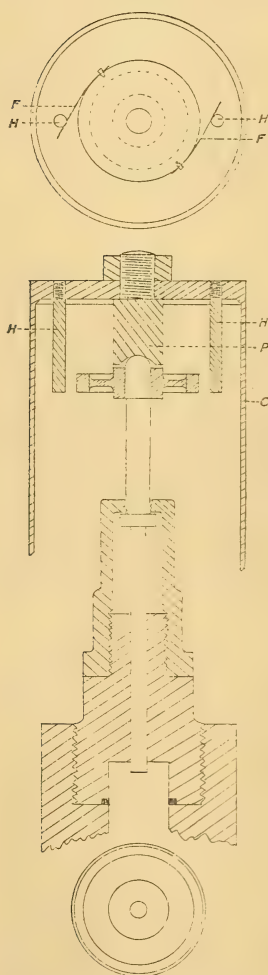
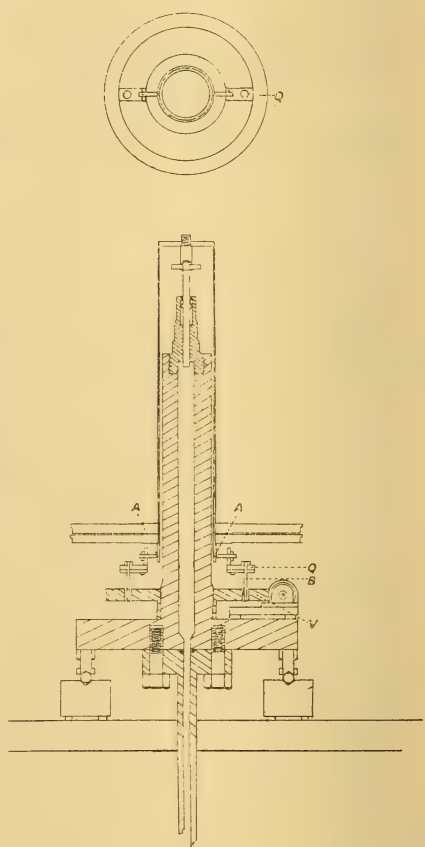


Fig. 3.

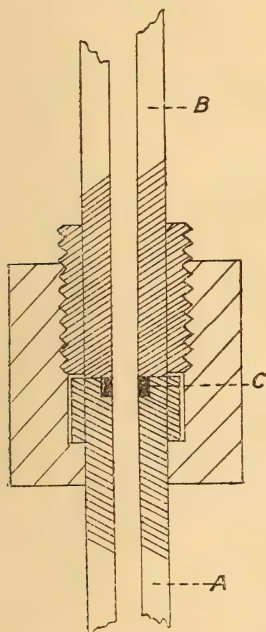


this case, on the approximately equal flexibility of two flat springs *FF* attached to the piston-head, and engaging with pins *HH* as indicated in the upper part of fig. 2.

Description of joints.—The various tees, elbows, and extensions (there are some 40 joints in all) used in this

apparatus are made tight in a simple way which we have used extensively for some years, and have found to be entirely satisfactory. For example, A and B (fig. 4) are

Fig. 4.



two tube-ends which have to be joined together. One of them, A, is recessed, and the recess is filled by a ring of dermatine* C, which before the joint is assembled stands a little proud of the tube. It is now only necessary to bring the tubes firmly together, and a fluid-tight joint results.

In the figure the tube-ends are held together by an ordinary union, but the same form of joint is equally serviceable when, instead of this, the tubes are provided with flanges which are bolted together; and it is equally applicable to large and to small bores. It can, of course, be used where one of the pieces to be coupled has a bore smaller than the other or is a blank. "Red" dermatine is rather harder than "black," and is correspondingly easier to machine; it stands oil or water.

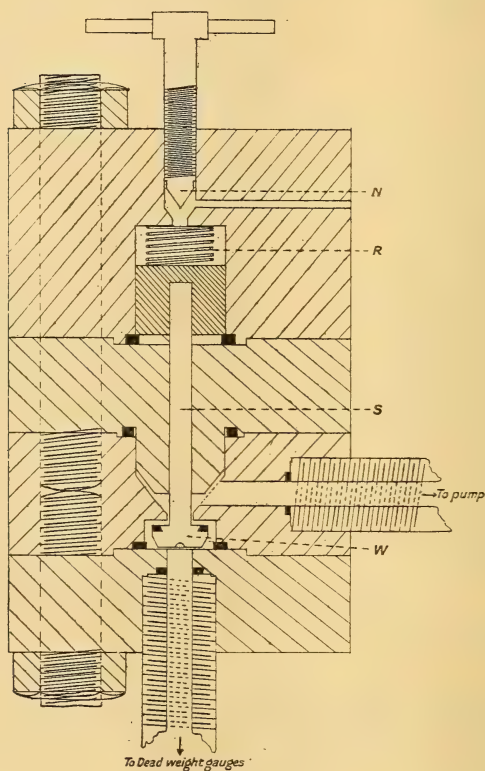
The advantages of this joint are:—It is fluid-tight from the outset, before any pressure has been applied, and remains tight up to any pressure which the couplings are strong enough to stand. It can be taken apart and remade† a great number of times without renewing the dermatine ring, and is extremely simple to make, requiring only ordinary workshop accuracy. On the other hand, where two pieces have to be coupled fluid-tight and in accurate relation to one another, this result is readily secured, as the pieces are in direct contact. If under working conditions the tubes or couplings are subjected to mechanical strain, the joint itself is unaffected.

* Gutta-percha also makes an excellent *temporary* joint, and is to be preferred to dermatine should it be necessary to use very thin (in the radial direction) rings. It is not acted upon by castor oil.

† This is essential, as the apparatus must be cleaned out at intervals, because the castor oil gets contaminated by small metallic particles (possibly scale), and also appears to have a slow action on the steel.

The semi-automatic valve — This is shown in fig. 1 at S, and in fig. 5 in section. It is used to shut off the pump from the

Fig. 5.



rest of the pressure-system when the plunger has been pushed in to its limit, and has therefore to be run out again so as to draw a fresh lot of oil into the pump. This valve must accordingly be efficient up to the highest pressures used (1500 atmospheres), and we have not found needle-valves of the ordinary type satisfactory. The connexions to the pump and to the pressure-tubes are as indicated; and in the view shown there is free communication between them because the valve W is withdrawn from its seat. The valve-face is formed by a ring of dermatine, sectioned in black. The slow leakage of oil which would otherwise take place around the unpacked valve-stem S is prevented by the auxiliary needle-valve N. It is only while the valve is being altered

from the open to the closed position that any such leakage is possible. To close the valve, the needle N is opened and the valve-stem S is thus free to rise and seat itself securely, the light spring R becoming compressed at the same time. It will be noticed that the dermatine ring is so arranged that it becomes automatically tighter under the pressure which it has to hold. The screw plunger-pump can now be refilled with oil, and, after closing N, the valve will automatically reopen when the pump-pressure has been brought to a value a little in excess of that in the remainder of the pressure-system.

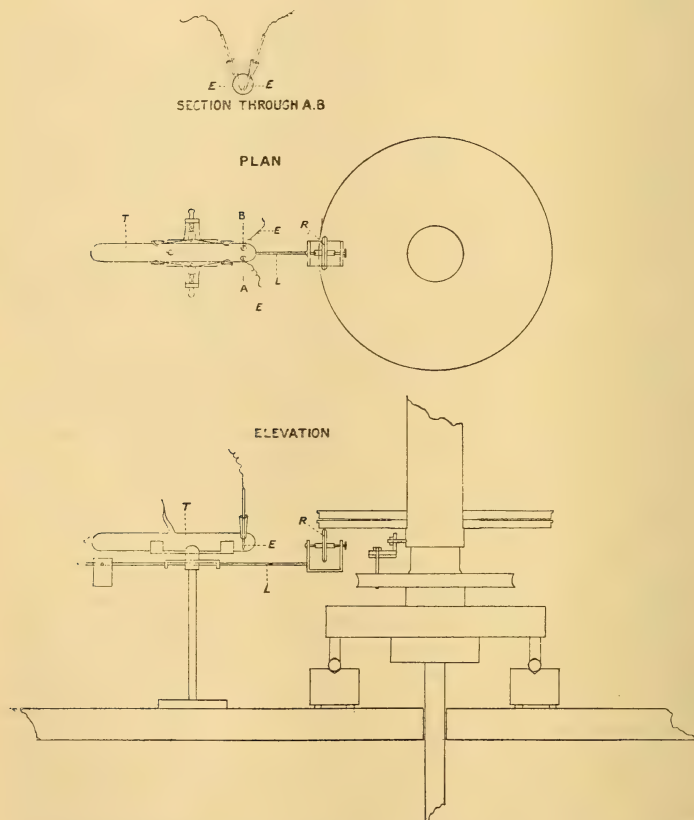
Method of altering the pressure.—As already stated, in much of the work for which this installation is used it is necessary to avoid rapid changes, and especially rapid increases of pressure. For this reason, when a weight has to be added to the load carried by the rotating piston of the gauge (G_1 , G_2 , G_3), means must be provided for allowing the extra weight to come on gradually. The plan adopted is to suspend the weight from three triggers on a cage carried upon a ball-bearing at the end of a fairly extensible spring. The suspended system is lowered by the slow unwinding of the cord to which it is attached; and as the unwinding proceeds, the extra load is placed gently in position and its weight gradually transferred from the cage to the pile of weights below it, the spring shortening meanwhile. As soon as the cage has been relieved of the whole of the weight, the three triggers withdraw automatically, leaving the cage clear for lifting. The gradual removal of weight is effected by the same mechanism; but in this case the triggers are differently set, so that they are adapted for picking up instead of for releasing.

To obtain an increment of pressure less than that due to a weight unit, a light container is placed on the top of the weights, and above it is suspended a tin funnel containing the requisite quantity of shot. On opening the orifice of the funnel the shot runs into the container at a rate which depends (within limits) only on the area of the orifice and not on the head of shot.

Automatic compensation for leak.—It is sometimes necessary to maintain a pressure sensibly constant for a considerable time. To this end the head of the pump-screw is fitted with a ratchet-clutch, Q, fig. 1 (the "free-wheel" clutch of a bicycle), to one member of which is attached a horizontal lever L. It happened to be most convenient to arrange this lever to the right hand of the pump, which meant that the down-stroke of the lever would be the operative one. As it was desired

to reciprocate the lever by means of a wire rope, a sufficient weight was hung on to it, and the rope was carried up to a ball-bearing crank-pin, set in the face of a worm-wheel. The worm is driven through a magnetic clutch, by means of which the automatic control of the pumping is effected whichever of the three gauges G_1 , G_2 , G_3 is in use; the fall of the gauge-piston below a certain level closes the circuit of the electromagnet and causes the pump to work; the rise of the piston above a slightly higher level breaks the circuit and stops the pumping.

Fig. 6.



The (continuously-driven) pulley on the worm-spindle provides a choice of speeds, the aim being to arrange matters so that even with the greatest expected leak only intermittent pumping will be needed to keep up the pressure. A detail of the contact-maker is shown in fig. 6.

The roller R is journaled in one end of the lever L, which is so loaded that R always rests in light contact with the undermost of the pile of weights. When the weights fall the lever L is tilted clockwise and a few cubic centimetres of mercury contained in the glass tube T attached to L run down to the right-hand end of the tube and make contact between two iron electrodes E. The tube T is exhausted to a Fleuss vacuum and hermetically sealed, the electrodes being closely fitted through ground conical tubulures, and made tight with sealing-wax.

It may be worth while noting that this type of vacuum contact-maker is capable of carrying currents up to some 5 to 7 amperes at 100 volts D.C. without sustaining injury.

Sensitiveness of the gauges.—It is well known that in this form of gauge the sensitiveness depends largely on the rate at which the pistons are rotated. Several experiments were made to find out the most suitable rate, *i. e.* the slowest rate compatible with sufficient sensitiveness. The outcome of these being that G_3 should revolve once in 6 seconds, G_2 once in 26 seconds, and G_1 once in 45 seconds. At these rates it was found that with the pistons of both G_2 and G_3 floating under a pressure of 800 atmospheres, but with the latter slowly falling while the former is slowly rising, an addition of 20 grams to G_2 (equivalent to a change of pressure of 0.17 atmosphere) the movements would reverse, and G_2 would fall while G_3 rose.

Pressure release-valve.—Whilst working with this installation it was found an advantage to be able to release the pressure somewhat rapidly, but at the same time to do so without jerks. An ordinary needle-valve is not suitable because the small opening soon gets clogged by any solid particles there may be in the oil (or even any material of greater viscosity?), and on enlarging the opening the obstruction flows away, and a rapid variation ensues.

The following device successfully avoided the trouble, especially at the higher pressures, where it is most needed.

A cylinder, closed at one end by a needle-valve, was bored out to $\frac{3}{8}$ inch and tapped 26 threads to the inch. A plug carrying the same thread was loosely fitted into the cylinder and the whole joined on to the pressure-system. On opening the needle-valve, the plug travels forward until there is a firm contact between the adjacent thread surfaces, thus forming a narrow channel through which the oil can flow. The length of the channel (some 50 inches) acts as a resistance to the flow, but its depth is such as to avoid clogging.

XIII. *On the Mechanical Relations of Dielectric and Magnetic Polarization.* By G. H. LIVENs*.

1. **T**HERE still appears to be some uncertainty as to the exact expression for the mechanical forcive of electric or magnetic origin acting on the elements of a polarizable medium in an exciting field of force, and as to the consequent reduction of this forcive to its representation by means of an applied stress system.

The original procedure of Maxwell†, based, for the case of magnetic media, on the most elementary physical ideas respecting the nature of the polarization involved, leads to a definite expression for the forcive on any medium which is independent of the law, or even of the existence of a law, for the induction of the polarization by the exciting field, and which should therefore hold in the case of all substances in which the polarization is of the nature of that described, even if this polarization involves hysteretic qualities. This, the original, procedure has been elaborated and extended in the more recent developments of the theory of electrons, and the general validity of the result obtained is thereby fully confirmed and substantiated both for dielectric and magnetic media.

A more general mode of discussion based on the method of energy, but avoiding molecular theory, has been originated by Korteweg‡, formulated in general terms by von Helmholtz§ and further developed by Lorberg, Kirchhoff||, Hertz¶, Cohn**, and others. The expression for the forcive on the polarized medium obtained by this method is, however, inconsistent with that obtained by Maxwell on his simpler form of the theory. The discrepancy is regarded by some authors as due to incompleteness in Maxwell's formulation, but Larmor†† has shown that, at least in the simplest case of isotropic media, it is due mainly to fundamental errors both in the physical assumptions on which the energy method is based and also in the analytical processes by which that method is developed. This criticism appears, however, to have been entirely overlooked and the energy method is still

* Communicated by the Author.

† Treatise, vol. ii. §§ 641, 642.

‡ Wied. Ann. ix. (1880).

§ Wied. Ann. xiii. (1882); *Abhandlungen*, i. p. 798.

|| Wied. Ann. xxiv., xxv. (1885); *Abhandlungen*, 'Nachtrag,' p. 91.

¶ Wied. Ann. xli. (1890); 'Electric Waves' (English Edition), pp. 259-268.

** 'Das electromagnetische Feld,' Ch. viii.

†† Phil. Trans. A. cxc. p. 280 (1897).

very generally reproduced and it has subsequently been further developed and extended by Cohn *, Gans †, Sano ‡, and others.

It seems therefore desirable to examine in closer detail the criticism of Larmor, applying it to the more general case when the law of induction is no longer linear or isotropic. but with the view rather to emphasising the underlying physical principles than of adding anything original to the criticism itself. To aid in the examination a short account of Maxwell's theory based on a correct interpretation of the energy method is added, and it is thus hoped to establish on a firmer footing the only consistent specification of the problem which has so far been put forward.

The discussion is confined entirely to the case of dielectric media, and the methods and notations of the vectorial calculus will be employed throughout.

2. According to the theory of Maxwell as elaborated and extended in the theory of electrons, a body polarized to intensity P in a field of force of intensity E requires on account of the polarization induced in it an amount of energy per unit volume at any place expressed by

$$-(PE),$$

which together with the energy of the free charges makes up a total equivalent to a distribution throughout the whole field of density $\frac{1}{8\pi}E^2$ at each place.

This energy does not, however, arise as a result simply of the action of mechanical forces applied to the media as a whole, and is not therefore compensated by a reduction of the available mechanical energy associated with these forces; for part of it has arisen from the store of internal elastic or thermal energy in the medium, which may for the purposes of the present argument be regarded as of effectively non-electric nature. The separation of these two fundamentally different parts is effected by the usual method, which consists in imparting a small virtual variation to the general configuration, both geometrical and electrical, of the medium and calculating the work in each part separately. In this way it is easily seen that the part

$$W_m = - \int_0^E (PdE)$$

of the total electric energy per unit volume is associated

* *L. c.* p. 512.

† *Ann. Phys.* xiii. p. 631.

‡ *Phys. Zeitschr.* iii. p. 401 (1902). See also the articles by Pockels and Gans in the *Encyclopädie der mathematischen Wissenschaften*, Bd. v.

mainly with the mechanical forces on the medium as a whole, for it is the part of the total that remains when the internal configuration determined by the polarization is maintained constant during the establishment of the system.

The function W_m may now be regarded as the potential function of the required mechanical forces. The expression for the forcive on the element on account of the polarization in it then follows as a matter of course, and its linear constituents are represented per unit volume by the components of the vector

$$-\text{grad } W_m = \text{grad } (PE),$$

the differential operations not, however, affecting P ; the angular components are similarly determined and are represented by the vector

$$[PE].$$

When a potential of force exists the former vector is equivalent to

$$(P\nabla)E,$$

wherein ∇ is the usual Hamiltonian vector operator. Thus in the case of isotropic media, when the induction of the polarization follows a linear law so that

$$P = \frac{K-1}{4\pi} E,$$

the forcive per unit volume is completely specified by the vector

$$\frac{K-1}{8\pi} \text{grad } E^2.$$

The additional forcive which acts on the medium on account of the distribution of free charge of density ρ throughout it is represented by a force on the element of volume at any point whose intensity per unit volume is

$$\rho E,$$

and this, combined with the above forcive arising on account of the polarization, is completely represented in the most general possible case by an applied stress system whose nine components are of the types

$$T_{xx} = E_x D_x - \frac{1}{8\pi} E^2, \quad T_{xy} = E_x D_y, \quad T_{yx} = E_y D_x,$$

wherein

$$D = \frac{1}{4\pi} E + P$$

is the total displacement vector of Maxwell's theory.

The remaining part of the total electric energy of the polarizations, viz.

$$-\int_0^P (E dP),$$

represents the energy supplied from the total store of internal energy of other than electric type in the medium during the process of setting up the polarization. This expression with sign changed thus represents the increase in the energy of non-electric nature in the dielectric medium consequent upon the induction of its polarized condition.

3. In the theory of von Helmholtz the whole argument is placed on a different footing. No distinction is now drawn between the fundamental constituents, æther and polarizable matter, of the dielectric field, which is regarded as consisting of a single uniform medium capable of transmitting the electric actions in the same manner as an ordinary elastic solid transmits mechanical forces; the electric force at any point of the field is the straining force and the total electric displacement in Maxwell's sense represents the strain produced. The displacement D is subject to the usual characteristic equation of the theory, viz.

$$\text{div } D = \rho,$$

and the electric force exciting the displacement is derived from a potential.

The first step in the formulation of the theory is to express the energy per unit volume in terms of the field vectors and the potential in such a form that the variation of the integral which represents the energy for the whole volume leads on integration by parts to the characteristic equation for the displacement as one of the conditions of internal equilibrium; the integral is then asserted to be in its normal form, which means that it represents the actual distribution of the energy of the medium as well as its total amount. Its variation with sign changed, owing to change of material configuration, should then give the extraneous force that must be applied in order to maintain mechanical equilibrium; the variation with respect to the electrical configuration being null, so that electric equilibrium is provided for by the characteristic equation already satisfied. The variation without change of sign should thus give the mechanical force of electric origin that acts on the medium.

The organized energy in the medium is known to be

$$W = \int dv \int_0^D (E dD),$$

where

$$\text{div } D = \rho \quad \text{and} \quad E = -\text{grad } \phi,$$

so that the forces acting will be derived from the variation of W ; the variation with respect to ϕ leads to the electric forces, and that with respect to the material configuration leads to the mechanical ones. The problem is to determine the mechanical forces when there is electric equilibrium, that is, when the variation with respect to ϕ yields a null result. The form of W above expressed does not lead to this null result. We can, however, by integration by parts derive the form

$$W = \int dv \int_0^{\rho} \phi d\rho,$$

the essence of this transformation being that in the new integral the distribution of the energy among the elements of volume dv of the medium has been altered. This form does not satisfy the above requirements either, but by combining the two forms we obtain

$$W = \int dv \left[\rho\phi - \int_0^E (DdE) \right],$$

whose variation with respect to ϕ is null as required; although, as integration by parts is employed, the variation is *not* null for each single element of mass. This integral is then taken to represent the actual distribution of the organized energy in the medium when in electric equilibrium and not merely its total amount, and variation of it with respect to the material configuration should thus give the actual bodily distribution of mechanical force, not merely its statical resultant on the hypothesis that the system is absolutely rigid. Now, in finding the variation of W arising from a virtual displacement δs of the polarized material, we have to respect the condition that the free charge ρdv is merely displaced, so that by the equation of continuity

$$\delta\rho + (\nabla, \rho\delta s) = 0,$$

and also that each element of the material is moved on with its own elastic constants, so that if ∇' denotes the vector of space variation taken without reference to the variation of E ,

$$\delta D + (\delta s, \nabla')D = 0,$$

while things have been arranged so that a variation of ϕ produces no result,—but only, however, no aggregate result on integration by parts. The transitions at interfaces are supposed to be gradual, so that the volume integrals can all be extended over the entire field, without the necessity for

the introduction of surface integrals. We have therefore

$$\begin{aligned}\delta W &= \int dv \left[\phi \delta \rho - \int_0^E (\delta D \cdot dE) \right] \\ &= - \int \phi (\nabla, \rho \delta s) dv + \int dv \int_0^E ((\delta s, \nabla') D, dE).\end{aligned}$$

Now

$$\int_0^E ((\delta s, \nabla') D, dE) = - (D, (\delta s, \nabla) E) + (\delta s \nabla) \int_0^E (D dE),$$

so that on integrating the first integral by parts, we get

$$\delta W = \int dv \left[\rho (\delta s \nabla) \phi - (D (\delta s \nabla) E) + (\delta s \nabla) \int_0^E (D dE) \right].$$

The vector coefficient of δs with sign changed is now taken to determine the linear components of the mechanical forcive on the dielectric medium; its x -component for example is

$$-\rho \frac{\partial \phi}{\partial x} + \left(D \frac{\partial E}{\partial x} \right) - \frac{\partial}{\partial x} \int_0^E (D dE).$$

This expression is far more complex than that given by Maxwell's theory, and it leads to a stress whose specification differs from that of Maxwell in its leading terms, which are now of the type

$$T_{xx} = E_x D_x - \int_0^E (D dE).$$

These general formulæ were first given by Cohn for the particular case of isotropic magnetic media for which a general law of induction is valid; they are here seen to be in no way limited by these assumptions. In the most general case, however, their form alone suggests an obvious difficulty. The type of forces which are represented in them are much more general than is the case in all known mechanical systems, in so far as the force on any element of the medium is a function not only of the conditions in that element at the instant when it is under examination, but also of the whole of its past history. In any ordinary physical theory the existence of such forces is entirely excluded as it would, for instance, result in a force on an unpolarized element of matter formed by the combination of two opposite but equal polarized elements with different past histories.

We need not, however, dwell upon the difficulties of this kind inherent in the present form of the theory, as it will soon be seen that the deduction given is entirely fallacious.

4. On close inspection it will be seen that the analysis of the preceding paragraph determines in reality something quite different from that which is the ultimate object of search. In calculating the mechanical force on the element of the dielectric medium by the variational method, care must be exercised to determine the change in the energy of the moving element of the medium, because it is this change which is brought about by the action of the mechanical forces when the constitution is maintained constant. But in the argument of the preceding paragraph, the work done by the bodily forces acting on the material element during its displacement is equated to the change in the energy in the element of volume originally occupied by the element before its displacement, which is quite a different thing from the change in the energy of the moving element itself. The analysis is therefore fundamentally unsound and will require considerable modification. To obtain its proper legitimate form we must confine our attention to a finite portion of the dielectric medium and follow it in its motion. We enclose the portion by a surface f on its outer boundary and then notice, with Larmor, that in the displacement a space is left unoccupied on the one side of the surface and a new space is occupied on the other; the result is that the above expression for variation of W , or at least that part of it referring to the portion of the dielectric medium inside f , is now represented by

$$\delta W = \int dv \left[-\phi(\nabla, \rho \delta s) - (D, (\delta s \nabla) E) + (\delta s \nabla) \int_0^E (D dE) \right] \\ + \int df \left[\rho \phi - \int_0^E (D dE) \right] \delta s_n,$$

wherein the volume integral is taken throughout the portion of the medium enclosed by the surface f and the surface integral over its outer boundary or f itself; δs_n represents the outward normal component on f of the virtual displacement given to the medium. The volume integral of the first term in the first integral now transforms by integration by parts into the integral

$$\int \rho (\delta s \nabla) \phi,$$

together with a surface integral over f equal and opposite in sign to the first part of the surface integral already present in δW ; and the integral of the last term in the first integral is equal but opposite in sign to the remaining part of this

same surface integral: thus finally we get

$$\delta W = \int dv [\rho(\delta s \nabla) \phi - (D, (\delta s \nabla) E)]$$

as the expression for the variation of the energy in the specified portion of the medium followed in its motion. As the result now applies for any arbitrary surface f , the integrand represents quite properly the distribution of the variational energy throughout the material dielectric; the coefficient of δs in it therefore gives properly the linear constituent of the forcive on the element; its x component per unit volume is

$$-\rho \frac{\partial \phi}{\partial x} + \left(D \frac{\partial E}{\partial x} \right) = \rho E + (D \nabla) E,$$

provided a potential of force exists. This type of forcive is, however, quite an impossible one, as it points to the existence of a bodily forcive of amount

$$\text{grad} \left(\frac{1}{8\pi} E^2 \right)$$

on the elements of free æther, which could not therefore be in equilibrium.

The final form of this result deduced by an elaboration of Larmor's argument can be verified by a simple direct calculation if it is noticed that it is in reality the variation of W with respect to the electrical potential, when the variation of this is brought about by a virtual displacement of the matter, that is really required; for it is the work of the mechanical forces balancing the electrical attractions that is to be determined; and if the elastic conditions of the dielectric medium specified completely by the vector D remain unaltered in any displacement, we may be sure that all the work done is used up as purely electrical energy of configuration, none of it having then been absorbed by the medium into its store of internal energy of effectively non-electric nature. In fact, if we perform the variation in this way with D , and consequently also ρ , constant we get

$$\delta W = \int dv [\rho(\delta s \nabla) \phi - (D(\delta s \nabla) E)],$$

exactly as above.

This simple mode of treatment also emphasises what it is that really determines the internal electrical coordinates of the medium. Helmholtz asserts that they, or at least the conditions in them, are determined by the potential distribution, and for the simple case examined by himself and

Larmor, where the medium is isotropic and the law of induction linear, this is probably true; but the present argument shows that it is the electrical displacement in Maxwell's sense that more properly defines degrees of electrical freedom in the dielectric medium.

The fact that in the aggregate for the whole field the above expression for δW vanishes, which Helmholtz is at such pains to secure and which results from the characteristic equation for D , viz.

$$\text{div } D = \rho,$$

merely verifies that there is on the whole no resultant force on the infinitely extended system.

The second term in the expression for the forcive just deduced is exactly analogous to the corresponding term depending on the polarization of the elementary theory of Maxwell. It is only to be noticed that the whole circumstances summed up in the vector D are subject to the virtual displacement of the matter, whereas in Maxwell's theory the

part $\frac{1}{4\pi}E$ of this vector defining conditions in the æther does not partake of motions of the matter, the remainder $P = D - \frac{E}{4\pi}$ being the only portion that so moves. Herein

lies the second cause of discrepancy between the two theories, and it would appear that the method described by von Helmholtz must be radically unsound: it would be valid if there were only one medium under consideration, of which W is the energy function; but there is here in the same space the æther with its stress and the polarized matter with its reacting mechanical forces, and there is no means of disentangling from a single energy function in this way the portions of energy which are associated with these different effects.

To sum up, we may say that the theory of electric stress formulated by Helmholtz, Hertz, Cohn, and others is fundamentally at fault. In its correct analytical form it involves physical assumptions which have been long regarded as wholly untenable, besides leading to an impossible type of dielectric body forcive. We are therefore virtually thrown back to Maxwell's original and simple theory, which is the only one that has really proved to be consistent with all the facts, among which it must not be forgotten to include those fundamental ones concerning the electrodynamic properties

of moving dielectric media, which necessitated in the first instance the introduction of the electron theory in its main aspects.

5. Before concluding this discussion reference must perhaps be made to a small discrepancy of a different kind, which occurs in the form of the theory given by Pockels in his article on "Magneto- and Elektro-striction" in the *Encyklopädie der mathematischen Wissenschaften*. It will be observed that the stress system obtained on Maxwell's theory is not in general self-conjugate, as must necessarily be the case with all stresses in material media without polar molecules, in order that the energy principle may be verified. Now Pockels obtains by von Helmholtz's method a self-conjugate stress in which the corresponding cross-terms are each half the sum of the corresponding terms in Maxwell's stress; he secures this by including the effect of a general virtual displacement possessing rotational as well as translational qualities. He attempts to assign the discrepancy in Maxwell's determination to a neglect of the rotational part of the displacement, and supports his contention by a reference to Hertz's interpretation of the energy method of von Helmholtz, as given in Lorentz's article on "Maxwell's Theory" in the same work. It appears, however, that Pockels criticism really redounds on himself, inasmuch as he has neglected to include in the hypothetical expression of the virtual work the effect of any possible applied couples such as are necessitated by Maxwell's theory. His reference to Hertz's work in no way helps the matter for, as is clearly recognized by both Hertz and Lorentz, this method only determines the sum of corresponding cross-terms of the stress matrix and not each of them independently. Besides, the difficulties encountered above and involved in the method of application of the integration by parts also occurs in Pockels' work, and in such a way as to obscure what in reality comes to a mutual cancelling of the two additional parts of the variation, one of which alone is included by Pockels, arising on account of the rotational terms.

The University, Sheffield,
April 15th, 1916.

XIV. *Strength of the Thin-plate Beam, held at its ends and subject to a uniformly distributed Load (Special Case).*
By B. C. LAWS, B.Sc., A.R.C.Sc., A.M.I.C.E.*

IN a recent communication† the author gave the solution of a special case of the thin-plate beam, in which the uniformly distributed load was taken as acting downwards, *i. e.* so as to press the beam against the broad supports.

As therein explained the case is met with in those structures as *e. g.* ships, and floating docks, built to withstand hydrostatic pressure, where the shell plating is riveted to and supported at frames or girders forming the foundation of the structure; and if we take a section of the plating in the direction perpendicular to the frames we obtain the profile of the beam forming the subject of discussion.

In the present case the problem is considered wherein the load acts in the reverse direction; the beam is not now supported directly by the frames but by the rivets which secure it to the latter.

The appearance of the beam between consecutive frames is of the nature indicated in fig. 1, in which F , F_1 denote the frames, and r , r_1 the lines of rivets.

Fig. 1.



Plate beam with a uniformly distributed load w per square inch.

Such cases are commonly met with in practice, as *e. g.* in the plating of the inner bottom and bulkheads of floating structures, and give rise to stresses in the material different from those obtained in the previous case.

As before, consider a strip of plating of unit width and thickness t , constituting a beam of length $rr_1 = 2a$, and

* Communicated by the Author.

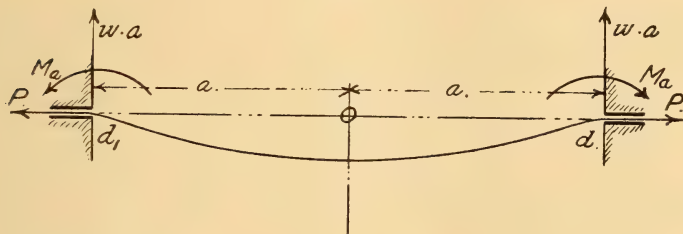
† Phil. Mag. vol. xxxi. April 1916.

subject to a uniformly distributed load w per unit length due to a head of water h .

Take the axis of X at the mid-surface of the plate, *i. e.* in a plane distant $t/2$ from the faces of the supporting frames, the origin O midway between the supports, and the axis of Y downwards.

The diagrammatic representation of the beam is shown in fig. 2.

Fig. 2.



Distribution of forces acting on beam subject to load w per square inch.

The solution is worked out on lines similar to those indicated in the previous paper, and we are led to the fundamental equation :—

$$E . I . \frac{d^2 y}{dx^2} = M_a + P . y - \frac{w(a^2 - x^2)}{2},$$

where P denotes the pull along the beam.

Put $P = m^2 E . I .$

The solution to this equation is :—

$$E . I . \frac{d^2 y}{dx^2} = \frac{w}{m^2} \left\{ \frac{m . a (e_0^{m,x} + e_0^{-m,x})}{e_0^{m,a} - e_0^{-m,a}} - 1 \right\}, \quad . \quad . \quad (a)$$

giving the bending moment at any point of the beam, where e_0 is the base of the natural or Napierian logarithms.

The deflexion y at any point of the beam is given by :—

$$m^2 E . I . y = - \frac{w . a}{m} \left\{ \frac{(e_0^{m,a} + e_0^{-m,a}) - (e_0^{m,x} + e_0^{-m,x})}{e_0^{m,a} - e_0^{-m,a}} \right\} + \frac{w(a^2 - x^2)}{2}. \quad . \quad . \quad (b)$$

The greatest bending moment occurs at d and is obtained by putting $x = a$ in equation (a); and the maximum deflexion y_0 is obtained by putting $x = 0$ in equation (b).

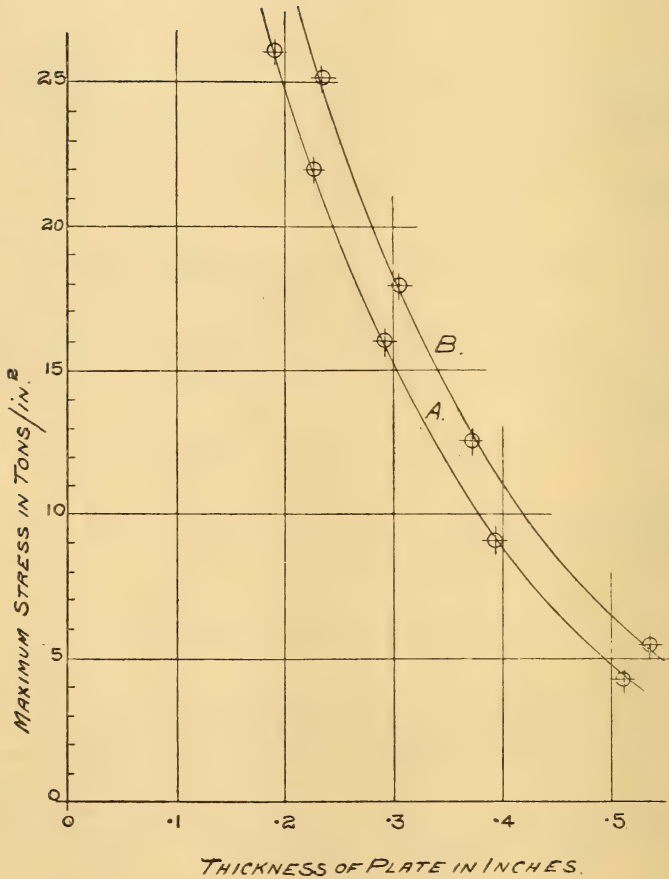
Now the elongation of the beam

$$= \frac{2a \cdot P}{E \cdot t} = \frac{m^2 \cdot t^2 \cdot a}{6} = \frac{4 \cdot y_0^2}{3a},$$

or

$$y_0 = \frac{\sqrt{m \cdot a \cdot t}}{2\sqrt{2}} \dots \dots \dots (c)$$

Fig. 3.



By assuming a value of m , t may be obtained by combining equations (b) and (c), thence by the aid of (a) and

the relation $P = m^2 E \cdot I$ the stresses due to bending and stretching.

Again, taking the case where $a = 11$ ins., $w = \cdot 007$ ton per inch run, $E = 13500$, and $m = \cdot 1$, we obtain :—

$$t = \cdot 305 \text{ in.}$$

$$\text{Max. stress due to bending} = 16\cdot86 \text{ tons/in.}^2$$

$$\text{,, ,, stretching} = 1\cdot05 \text{ ,,}$$

$$\text{Maximum tensile stress} = 17\cdot91 \text{ ,,}$$

By varying the value of m other values of t and stress may be obtained, from which data a curve showing the "variation of stress with plate thickness" may be constructed as indicated by B in the diagram, fig. 3.

The curve A—drawn for comparison—in the same diagram is reproduced from the previous paper.

Taking an allowed stress of 10 tons/in.², the corresponding thickness of plating is $\cdot 418$ in. as compared with $\cdot 376$ in. in the previous case.

XV. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xxxi. p. 572.]

April 5th, 1916.—Dr. Alfred Harker, F.R.S., President,
in the Chair.

THE following communication was read :—

'The Picrite-Teschenite Sill of Lugar (Ayrshire) and its Differentiation.' By George Walter Tyrrell, A.R.C.Sc., F.G.S.

This sill occurs near the village of Lugar in East Central Ayrshire, and is magnificently exposed in the gorges of the Bellow and Glenmuir Waters, just above the confluence of these streams to form the Lugar Water. It has a thickness estimated at 140 feet, and is intrusive into sandstones of the 'Millstone Grit.' The contacts consist of a curiously-streaked and contorted basaltic rock, passing at both margins into teschenite. The upper teschenite, however, becomes richer in analcite downwards, and ends abruptly at a sharp junction with fine-grained theralite. The lower teschenite becomes somewhat richer in olivine upwards, but passes rapidly into hornblende-peridotite. The central unit of the sill is a graded mass beginning with theralite at the top and passing gradually into

picrite, and finally peridotite, by gradual enrichment in olivine and elimination of felspar, nepheline, and analcite.

The field detail of the Bellow, Glenmuir, and other sections is given in Part 2 of the paper; and the petrographic detail, with several chemical analyses, in Part 3. A unique rock, named *lugarite* in 1912, with 50 per cent. of analcite and nepheline, occurs as an intrusion into the heart of the ultrabasic mass of the sill. Part 4 deals with the special significance of this sill in petrogenetic theory. The mineral and chemical variations are described and illustrated by diagrams. It is shown that the average rock of the sill, obtained by weighing the analyses of the various components according to their bulk, is much more basic than the rock now forming the contacts. Hence, assuming that the sill is a unit and represents a single act of intrusion, the main differentiation cannot have occurred *in situ*. Other special features of the sill are the identity and banding of the contact-rocks, its asymmetry, the density-stratification of the central ultrabasic mass, and the sharp junction between the upper teschenite and the underlying theralite.

The theory is advanced that the differentiation units were produced by the process of liquation, but that their arrangement within the sill took place under the influence of gravity. There are sharp interior junctions between a unit consisting mainly of calcic ferromagnesian silicates, and a unit consisting mainly of alkali-alumina silicates with water, the former giving rise to the central ultrabasic stratum, and the latter to the teschenites. These partly immiscible fractions arranged themselves according to density. Then within the central ultrabasic stratum there was a subsidiary gravity-stratification—due to the subsidence of olivine-crystals, giving rise to the graded mass described above. If differentiation had occurred subsequent to the arrival of the sill in the position that it now occupies, the contact-rocks should have the same composition as the average rock of the sill. This, however, is not the case, as the average rock has the composition of an almost ultrabasic theralite, entirely different from the teschenites of the contacts. Hence it is believed that, after forming contact-sheaths of theralite, and undergoing gravity-stratification subsequent to liquation, the intrusive activity was renewed, and the sill was moved on along bedding-planes into cold rocks, leaving its contact-sheath behind adhering to the old contacts, and establishing new contacts with its upper and lower teschenite-layers. Here crystallization began, and, by the subsidence of olivine, the subsidiary gravity-stratification of the central ultrabasic layer was effected. The extraordinary flow-banding shown by the contact-rocks affords confirmation of the renewed movement thus postulated.

In conclusion, the sill is compared with five other teschenite-picrite sills in Scotland, those of Ardrossan, Saltcoats, Blackburn, Barnton, and Inchcolm.

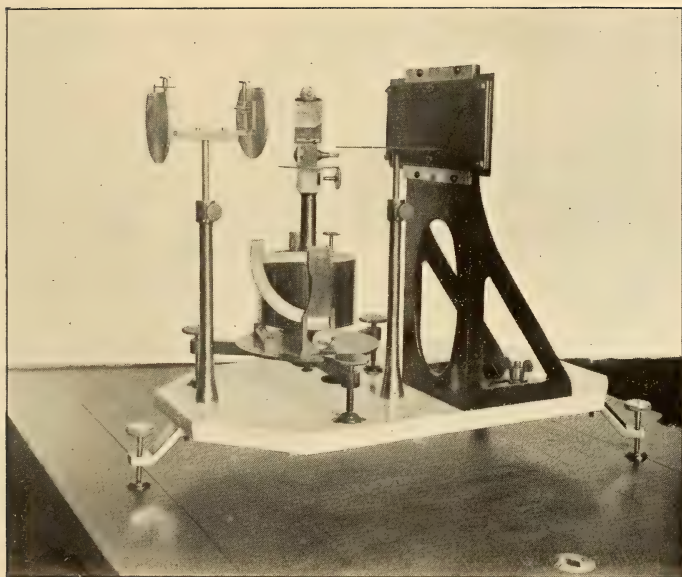






FIG. 1.



FIG. 2.



FIG. 3.



FIG. 4.



FIG. 5.



FIG. 6.

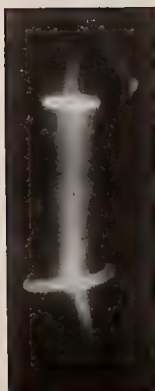


FIG. 7.



FIG. 8.



FIG. 9.



FIG. 10.

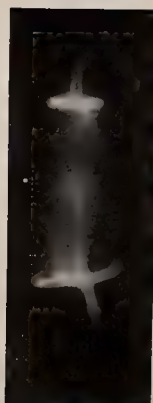


FIG. 11.



FIG. 13.



FIG. 14.



FIG. 15.

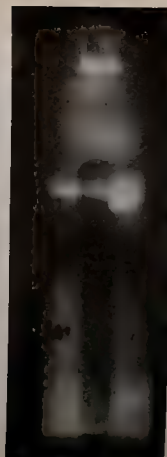
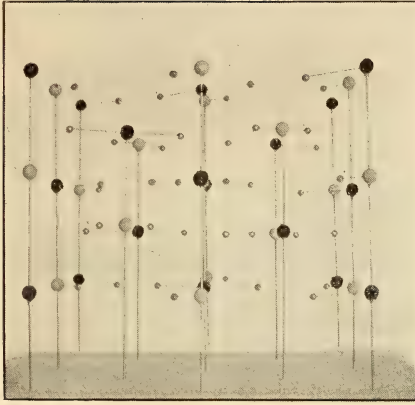


FIG. 16.

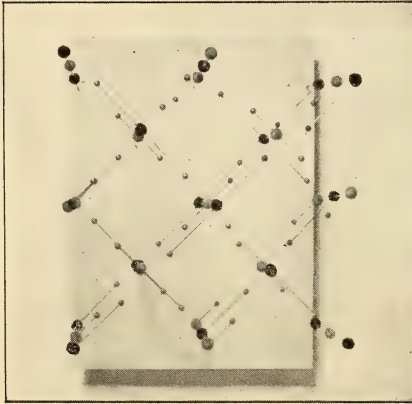


FIG. 17.





a.

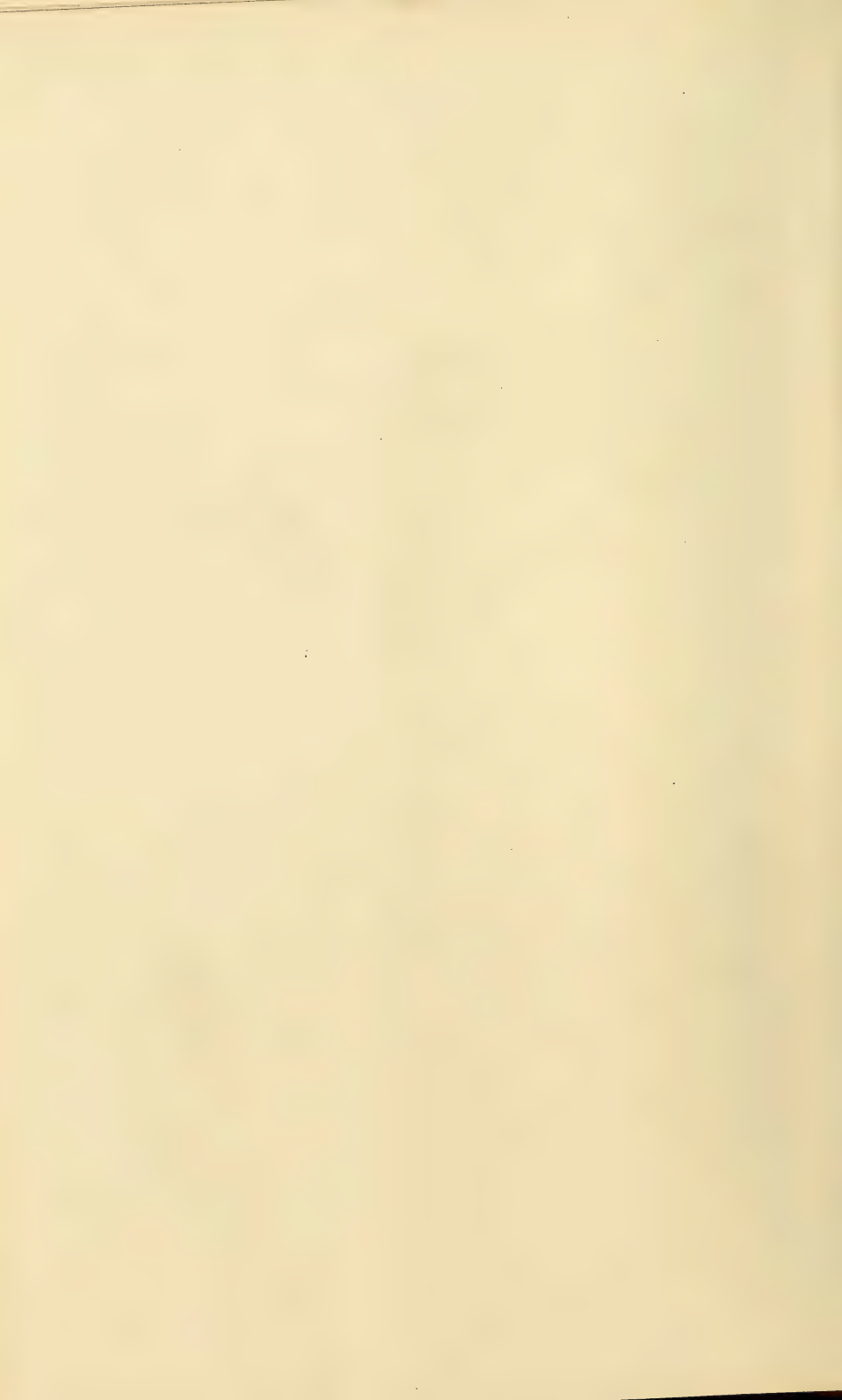


b.

Zircon lattice.

a in the direction of the *a*-axis.

b " " " *c*-axis.



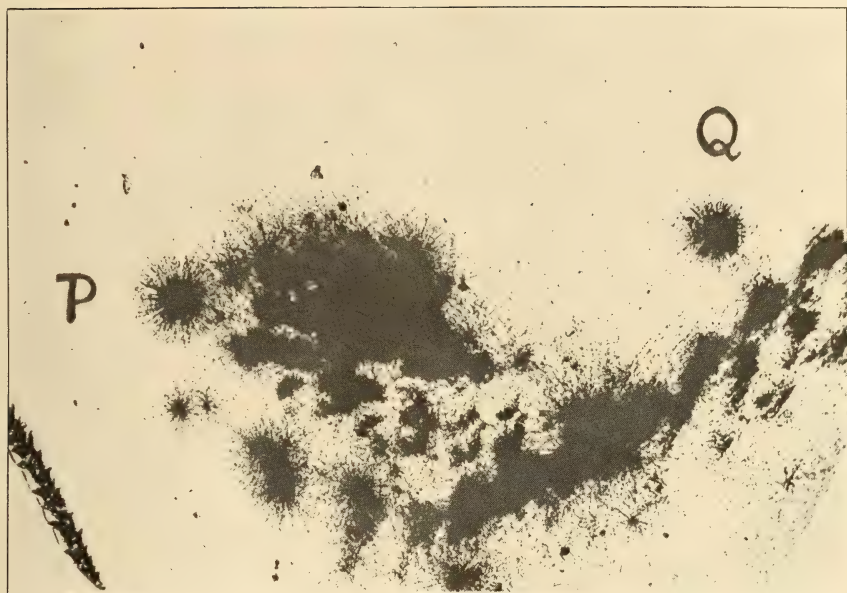


FIG. 1.
(Magnified 120 diameters.)

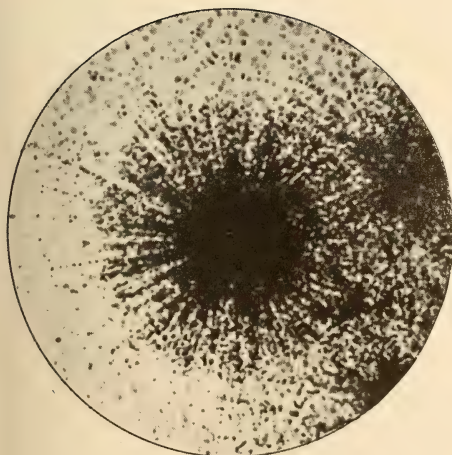


FIG. 2.
(Magnified 345 diameters.)

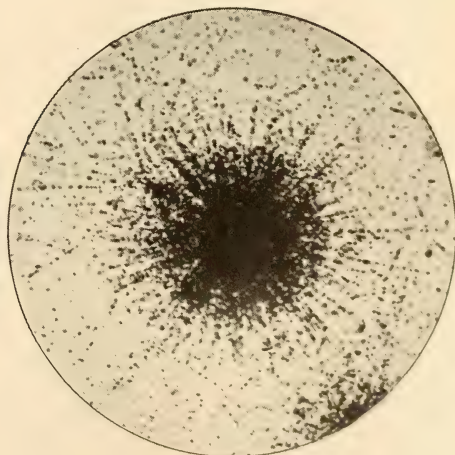


FIG. 3.
(Magnified 345 diameters.)



THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

AUGUST 1916.

XVI. *On the Discharge of Gases under High Pressures.*
By Lord RAYLEIGH, O.M., F.R.S.*

THE problem of the passage of gas through a small aperture or nozzle from one vessel to another in which there is a much lower pressure has had a curious history. It was treated theoretically and experimentally a long while ago by Saint-Venant and Wantzel † in a remarkable memoir, where they point out the absurd result which follows from the usual formula, when we introduce the supposition that the pressure in the escaping jet is the same as that which prevails generally in the recipient vessel. In Lamb's notation ‡, if the gas be subject to the adiabatic law ($p \propto \rho^\gamma$),

$$q^2 = 2 \int_p^{p_0} \frac{dp}{\rho} = \frac{2\gamma}{\gamma-1} \frac{p_0}{\rho_0} \left\{ 1 - \left(\frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right\} = \frac{2}{\gamma-1} (c_0^2 - c^2), \quad (1)$$

where q is the velocity corresponding to pressure p ; p_0 , ρ_0 the pressure and density in the discharging vessel where $q=0$; c the velocity of sound in the gas when at pressure p and density ρ ; c_0 that corresponding to p_0 , ρ_0 . According to (1) the velocity increases as p diminishes, but only up to a maximum, equal to $c_0 \sqrt{\{2/(\gamma-1)\}}$, when $p=0$.

* Communicated by the Author.

† "Mémoire et expériences sur l'écoulement de l'air, déterminé par des différences de pressions considérables," *Journ. de l'École Polyt.* t. xvi. p. 85 (1839).

‡ 'Hydrodynamics,' §§ 28, 25 (1916).

Phil. Mag. S. 6. Vol. 32. No. 188. Aug. 1916.

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If $\gamma=1.408$, this limiting velocity is $2.214c_0$. It is to be observed, however, that in considering the rate of discharge we are concerned with what the authors cited call the "reduced velocity," that is the result of multiplying q by the corresponding density ρ . Now ρ diminishes indefinitely with p , so that the reduced velocity corresponding to an evanescent p is zero. Hence if we identify p with the pressure p_1 in the recipient vessel, we arrive at the impossible conclusion that the rate of discharge into a vacuum is zero. From this our authors infer that the *identification cannot be made*; and their experiments showed that from $p_1=0$ upwards to $p_1=.4p_0$ the rate of discharge is sensibly constant. As p_1 still further increases, the discharge falls off, slowly at first, afterwards with greater rapidity, until it vanishes when the pressures become equal.

The work of Saint-Venant and Wantzel was fully discussed by Stokes in his Report on Hydrodynamics*. He remarks "These experiments show that when the difference of pressure in the first and second spaces is considerable, we can by no means suppose that the mean pressure at the orifice is equal to the pressure at a distance in the second space, nor even that there exists a contracted vein, at which we may suppose the pressure to be the same as at a distance." But notwithstanding this the work of the French writers seems to have remained very little known. It must have been unknown to O. Reynolds when in 1885 he traversed much the same ground†, adding, however, the important observation that the maximum reduced velocity occurs when the actual velocity coincides with that of sound under the conditions then prevailing. When the actual velocity at the orifice reaches this value, a further reduction of pressure in the recipient vessel does not influence the rate of discharge, as its effect cannot be propagated backwards against the stream. If $\gamma=1.408$, this argument suggests that the discharge reaches a maximum when the pressure in the recipient vessel falls to $.527p_0$, and then remains constant. In the somewhat later work of Hugoniot‡ on the same subject there is indeed a complimentary reference to Saint-Venant and Wantzel, but the reader would hardly gather that they had insisted upon the difference between the pressure in the jet at the orifice and in the recipient vessel as the explanation of the impossible conclusion deducible from the contrary supposition.

In the writings thus far alluded to there seems to be an

* B. A. Report for 1846; Math. and Phys. Papers, vol. i. p. 176.

† Phil. Mag. vol. xxi. p. 185 (1886).

‡ Ann. de Chim. t. ix. p. 383 (1886).

omission to consider what becomes of the jet after full penetration into the receiver. The idea appears to have been that the jet gradually widens in section as it leaves the orifice and that in the absence of friction it would ultimately attain the velocity corresponding to the entire fall of pressure. The first to deal with this question seem to have been Mach and Salcher*, but the most elaborate examination is that of R. Emden†, who reproduces interesting pictures of the effluent jet obtained by the simple shadow method of Dvorák‡. Light from the sun or from an electric spark, diverging from a small aperture as source, falls perpendicularly upon the jet and in virtue of differences of refraction depicts various features upon a screen held at some distance behind. A permanent record can be obtained by photography. Emden thus describes some of his results. When a jet of air, or better of carbonic acid or coal-gas, issues from the nozzle into the open under a pressure of a few millimetres, it is seen to rise as a slender column of the same diameter to a height of perhaps 30 or 40 cm. Sometimes the column disappears without visible disturbance of the air; more often it ends in a small vortex column. When the pressure is raised, the column shortens until finally the funnel-shaped vortex attaches itself to the nozzle. At a pressure of about one-fifth of an atmosphere there appears again a jet 2 or 3 cm. long. As the pressure rises still further, the jet becomes longer and more distinct and suddenly exhibits thin, bright, and fairly equidistant disks to the number of perhaps 10 or 12, crossing the jet perpendicularly. The first disks have exactly the diameter of the nozzle, but they diminish as the jet attenuates. Under still higher pressures the interval between the disks increases, and at the same time the jet is seen to swell out between them. These swellings further increase and oblique markings develop which hardly admit of merely verbal description.

Attributing these periodic features to stationary sound waves in the jet, Emden set himself to determine the wavelength (λ), that is the distance between consecutive disks, and especially the pressure at which the waves begin to develop. He employed a variety of nozzles, and thus sums up his principal results:—

1. When air, carbonic acid, and hydrogen escape from equal sufficiently high pressures, the length of the sound

* Wied. *Ann.* Bd. xli. p. 144 (1890).

† Wied. *Ann.* Bd. lxi. pp. 264, 426 (1899).

‡ Wied. *Ann.* Bd. ix. p. 502 (1879).

waves in the jet is the same for the same nozzle and the same pressure.

2. The pressure at which the stationary sound waves begin to develop is the same in air, carbonic acid, and hydrogen, and is equal to .9 atmosphere.

This is the pressure-*excess* behind the nozzle, so that the whole pressure there is 1.9 atmosphere. The environment of the jet is at one atmosphere pressure.

Emden, comparing his observations with the theory of Saint-Venant and Wantzel, then enunciates the following conclusion:—The critical pressure, in escaping from which into the atmosphere the gas at the nozzle's mouth moves with the velocity of sound, is equal to the pressure at which stationary sound waves begin to form in the jet. So far, I think, Emden makes out his case; but he appears to over-shoot the mark when he goes on to maintain that *after* the critical pressure-ratio is exceeded, the escaping jet moves everywhere with the same velocity, viz. the sound-velocity; and that everywhere within it the free atmospheric pressure prevails. He argues from what happens when the motion is strictly in one dimension. It is true that then a wave can be stationary in space only when the stream moves with the velocity of sound; but here the motion is not limited to one dimension, as is shown by the swellings between the disks. Indeed the propagation of any wave at all is inconsistent with uniformity of pressure within the jet. At the *surface* of the jet, but not within it, the condition is imposed that the pressure must be that of the surrounding atmosphere.

The problem of a jet in which the motion is completely *steady* in the hydrodynamical sense and approximately uniform was taken up by Prandtl*, both for the case of symmetry round the axis (of z) and in two dimensions. In the former, which is the more practical, the velocity component w is supposed to be nearly constant, say W , while u and v are small. We may employ the usual Eulerian equations. Of these the third,

$$\frac{dw}{dt} + u \frac{dw}{dx} + v \frac{dw}{dy} + w \frac{dw}{dz} = -\frac{1}{\rho} \frac{dp}{dz},$$

reduces to

$$W \frac{dw}{dz} = -\frac{1}{\rho} \frac{dp}{dz}, \quad (2)$$

when we introduce the supposition of steady motion and

* *Phys. Zeitschrift*, 5 Jahrgang, p. 599 (1904).

neglect the terms of the second order. In like manner the other equations become

$$W \frac{du}{dz} = -\frac{1}{\rho} \frac{dp}{dx}, \quad W \frac{dv}{dz} = -\frac{1}{\rho} \frac{dp}{dy}. \quad (3)$$

Further, the usual equation of continuity, viz.

$$\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} = 0, \quad (4)$$

here reduces to

$$\rho \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) + W \frac{d\rho}{dz} = 0. \quad (5)$$

If we introduce a velocity-potential ϕ , we have with use of (2)

$$\nabla^2 \phi = -\frac{W}{\rho} \frac{d\rho}{dz} = \frac{W^2}{a^2} \frac{dw}{dz} = \frac{W^2}{a^2} \frac{d^2 \phi}{dz^2}, \quad (6)$$

where $a = \sqrt{(dp/d\rho)}$, is the velocity of sound in the jet. In the case we are now considering, where there is symmetry round the axis, this becomes ($r^2 = x^2 + y^2$)

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + \left(1 - \frac{W^2}{a^2} \right) \frac{d^2 \phi}{dz^2} = 0, \quad (7)$$

and a similar equation holds for w , since $w = d\phi/dz$.

If the periodic part of w is proportional to $\cos \beta z$, we have for this part

$$\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} + \left(\frac{W^2}{a^2} - 1 \right) \beta^2 w = 0, \quad (8)$$

and we may take as the solution

$$w = W + H \cos \beta z \cdot J_0 \{ \sqrt{(W^2 - a^2)} \cdot \beta r / a \}, \quad (9)$$

since the Bessel's function of the second kind, infinite when $r=0$, cannot here appear. The condition to be satisfied at the boundary ($r=R$) is that the pressure be constant, equal to that of the surrounding quiescent air, and this requires that the variable part of w vanish, since the pressure varies with the total velocity. Accordingly

$$J_0 \{ \sqrt{(W^2 - a^2)} \cdot \beta R / a \} = 0, \quad (10)$$

which can be satisfied only when $W > a$, that is when the mean velocity of the jet exceeds that of sound. The wave-

length (λ) of the periodic features along the jet is given by $\lambda = 2\pi/\beta$.

The most important solution corresponds to the first root of (10), viz. 2.405. In this case

$$\lambda = \frac{2\pi R \sqrt{(W^2/a^2 - 1)}}{2.405} \quad . \quad . \quad . \quad (11)$$

The problem for the two-dimensional jet is even simpler. If b be the width of the jet, the principal wave-length is given by

$$\lambda = 2b \sqrt{(W^2/a^2 - 1)} \quad . \quad . \quad . \quad (12)$$

The above is substantially the investigation of Prandtl, who finds a sufficient agreement between (11) and Emden's measurements*.

It may be observed that the problem can equally well be treated as one of the small vibrations of a stationary column of gas as developed in 'Theory of Sound,' §§ 268, 340 (1878). If the velocity-potential, symmetrical about the axis of z , be also proportional to $e^{i(kat + \beta z)}$, where k is such that the wave-length of plane waves of the same period is $2\pi/\kappa$, the equation is § 340 (3)

$$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + (k^2 - \beta^2)\phi = 0, \quad . \quad . \quad . \quad (13)$$

and if $k > \beta$

$$\phi = e^{i(kat + \beta z)} J_0\{\sqrt{(k^2 - \beta^2)} \cdot r\} \quad . \quad . \quad . \quad (14)$$

The condition of constant pressure when $r = R$ gives as before for the principal vibration

$$\sqrt{(k^2 - \beta^2)} \cdot R = 2.405 \quad . \quad . \quad . \quad (15)$$

The velocity of propagation of the waves is ka/β . If we equate this to W and suppose that a velocity W is superposed upon the vibrations, the motion becomes steady. When we substitute in (15) the value of k , viz. $W\beta/a$, we recover (11). It should perhaps be noticed that it is only after the vibrations have been made stationary that the effect of the surrounding air can be properly represented by the condition of uniformity of pressure. To assume it generally would be tantamount to neglecting the inertia of the outside air.

* When $W < a$, β must be imaginary. The jet no longer oscillates, but settles rapidly down into complete uniformity. This is of course the usual case of gas escaping from small pressures.

The above calculation of λ takes account only of the principal vibration. Other vibrations are possible corresponding to higher roots of (10), and if these occur appreciably, strict periodicity is lost. Further, if we abandon the restriction to symmetry, a new term, $r^{-2}d^2\phi/d\theta^2$, enters in (13) and the solution involves a new factor $\cos(n\theta + \epsilon)$ in conjunction with the Bessel's function J_n in place of J_0 .

The particular form of the differential equation exhibited in (13) is appropriate only when the section of the stream is circular. In general we have

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + (k^2 - \beta^2)\phi = 0, \quad . \quad . \quad . \quad (16)$$

the same equation as governs the vibrations of a stretched membrane ('Theory of Sound,' § 194). For example, in the case of a square section of side b , we have

$$\phi = \cos \frac{\pi x}{b} \cdot \cos \frac{\pi y}{b} \cdot e^{i(kat + \beta z)}, \quad . \quad . \quad . \quad (17)$$

vanishing when $x = \pm \frac{1}{2}b$ and when $y = \pm \frac{1}{2}b$. This represents the principal vibration, corresponding to the gravest tone of a membrane. The differential equation is satisfied provided

$$k^2 - \beta^2 = 2\pi^2/b^2, \quad . \quad . \quad . \quad (18)$$

the equation which replaces (15). It is shown in 'Theory of Sound' that provided the deviation from the circular form is not great the question is mainly one of the *area* of the section. Thus the difference between (15) and (18) is but moderate when we suppose πR^2 equal to b^2 .

It may be worth remarking that when V the wave-velocity exceeds a , the group-velocity U falls short of a . Thus in (17), (18)

$$V = \frac{ka}{\beta}, \quad U = \frac{d(\beta V)}{d\beta} = a \frac{dk}{d\beta} = \frac{\beta a}{k};$$

so that

$$UV = a^2. \quad . \quad . \quad . \quad (19)$$

Returning to the jet of circular section, we may establish the connexion between the variable pressure along the axis and the amount of the swellings observed to take place

between the disks. From (9)

$$\phi = \int w dz = Wz + H\beta^{-1} \sin \beta z \cdot J_0 \{ \sqrt{(W^2/a^2 - 1)} \cdot \beta r \},$$

and

$$\left(\frac{d\phi}{dr} \right)_R = H \sqrt{(W^2/a^2 - 1)} \cdot \sin \beta z \cdot J_0'(2.405). \quad (20)$$

The latter equation gives the radial velocity at the boundary. If δR denote the variable part of the radius of the jet,

$$\delta R = \int \frac{1}{W} \left(\frac{d\phi}{dr} \right)_R dz = - \frac{H \cos \beta z}{\beta W} \sqrt{\left(\frac{W^2}{a^2} - 1 \right)} \cdot J_0'(2.405) \quad (21)$$

Again, if δp be the variable part of the pressure at the axis ($r=0$),

$$\frac{\delta p}{\rho} = C - \frac{1}{2} q^2 = C' - \frac{1}{2} w^2 = -W \delta w,$$

where ρ is the average density in the jet and δw the variable part of the component velocity parallel to z . Accordingly

$$\frac{\delta p}{\rho} = -WH \cos \beta z; \quad (22)$$

and

$$\frac{\delta R}{\delta p/\rho} = \frac{J_0'(2.405) \sqrt{(W^2/a^2 - 1)}}{\beta W^2}. \quad (23)$$

In (23) we may substitute for β its value, viz.

$$\frac{2.405 a}{R \sqrt{(W^2 - a^2)}},$$

and for $J_0'(2.405)$ we have from the tables of Bessel's functions -0.5191 , so that

$$\frac{\delta R}{\delta p/\rho} = -0.2158 R (a^{-2} - W^{-2}). \quad (24)$$

As was to be expected, the greatest swelling is to be found where the pressure at the axis is least.

A complete theory of the effects observed by Mach and Emden would involve a calculation of the optical retardation along every ray which traverses the jet. For the jet of circular section this seems scarcely practicable; but for the jet in two dimensions the conditions are simpler and it may be worth while briefly to consider this case. As before, we

may denote the general thickness of the two-dimensional jet by b , and take $b + \eta$ to represent the actual thickness at the place (z) where the retardation is to be determined. The retardation is then sufficiently represented by Δ , where

$$\Delta = \int_0^{\frac{1}{2}(b+\eta)} (\rho - \rho_1) dy = \int_0^{\frac{1}{2}(b+\eta)} \rho dy - \frac{1}{2} \rho_1 (b + \eta), \quad (25)$$

ρ being the density in the jet and ρ_1 that of the surrounding gas. The total stream

$$= \int_0^{\frac{1}{2}(b+\eta)} \rho (W + \delta w) dy = W \int_0^{\frac{1}{2}(b+\eta)} \rho dy + \rho \int_0^{\frac{1}{2}b} \delta w dy;$$

and this is constant along the jet. Thus

$$\Delta = C - \frac{1}{2} \rho_1 \eta - \frac{\rho}{W} \int_0^{\frac{1}{2}b} \delta w dy, \quad (26)$$

C being a constant, and squares of small quantities being omitted.

In analogy with (9), we may here take

$$\delta w = H \cos \beta z \cdot \cos \{ \beta y \sqrt{(W^2/a^2 - 1)} \}, \quad (27)$$

and for the principal vibration the argument of the cosine is to become $\frac{1}{2}\pi$ when $y = \frac{1}{2}b$. Hence

$$\int_0^{\frac{1}{2}b} \delta w dy = \frac{H \cos \beta z}{\beta \sqrt{\{W^2/a^2 - 1\}}} \cdot \quad (28)$$

Also

$$\begin{aligned} \phi &= \int w dz = Wz + \beta^{-1} H \sin \beta z \cdot \cos \{ \beta y \sqrt{(W^2/a^2 - 1)} \}, \\ \left(\frac{d\phi}{dy} \right)_{\frac{1}{2}b} &= -H \sqrt{\{W^2/a^2 - 1\}} \cdot \sin \beta z. \end{aligned}$$

Thus

$$\frac{1}{2}\eta = \frac{1}{W} \int \left(\frac{d\phi}{dy} \right)_{\frac{1}{2}b} dz = \frac{H \cos \beta z \cdot \sqrt{\{W^2/a^2 - 1\}}}{\beta W}$$

Accordingly

$$\Delta = C - \frac{H \cos \beta z}{\beta W} \left[\rho_1 \sqrt{\{W^2/a^2 - 1\}} + \frac{\rho}{\sqrt{\{W^2/a^2 - 1\}}} \right]; \quad (29)$$

so that the retardation is greatest at the places where η is least, that is where the jet is narrowest. This is in agreement

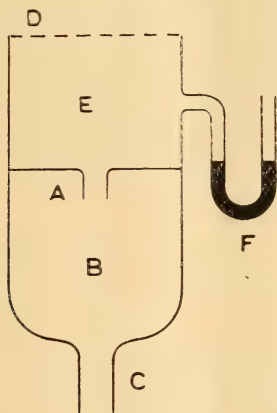
with observation, since the places of maximum retardation act after the manner of a convex lens. Although a complete theory of the optical effects in the case of a symmetrical jet is lacking, there seems no reason to question Emden's opinion that they are natural consequences of the constitution of the jet.

But although many features are more or less perfectly explained, we are far from anything like a complete mathematical theory of the jet escaping from high pressure, even in the simplest case. A preliminary question is—are we justified at all in assuming the adiabatic law as approximately governing the expansions throughout? Is there anything like the “bore” which forms in front of a bullet advancing with a velocity exceeding that of sound?* It seems that the latter question may be answered in the negative, since here the passage of air is always from a greater to a less pressure, so that the application of the adiabatic law is justified. The conditions appear to be simplest if we suppose the nozzle to end in a parallel part within which the motion may be uniform and the velocity that of sound. But even then there seems to be no reason to suppose that this state of things terminates exactly at the plane of the mouth. As the issuing gas becomes free from the constraining influence of the nozzle walls, it must begin to expand, the pressure at the boundary suddenly falling to that of the environment. Subsequently vibrations must set in; but the circumstances are not precisely those of Prandtl's calculation, inasmuch as the variable part of the velocity is not small in comparison with the difference between the mean velocity and that of sound. It is scarcely necessary to call attention to the violence of the assumption that viscosity may be neglected when a jet moves with high velocity through quiescent air.

On the experimental side it would be of importance to examine, with more accuracy than has hitherto been attained, whether the asserted independence of the discharge of the pressure in the receiving vessel (supposed to be less than a certain fraction of that in the discharging vessel) is absolute, and if not to ascertain the precise law of departure. To this end it would seem necessary to abandon the method followed by more recent workers in which compressed gas discharges into the open, and to fall back upon the method of Saint-Venant and Wantzel where the discharge is from atmospheric pressure to a lower pressure. The question is

* Proc. Roy. Soc. A. vol. lxxxiv. p. 247 (1910); Scientific Papers, vol. v. p. 346.

whether any alteration of discharge is caused by a reduction of this lower pressure beyond a certain point. To carry out the investigation on a sufficient scale would need a powerful air-pump capable of absorbing the discharge, but otherwise the necessary apparatus is simple. In order to measure the discharge, or at any rate to determine whether it varies or not, the passage of atmospheric air to the nozzle might be somewhat choked. The accompanying diagram will explain the idea. A is the nozzle, which would be varied in different series of experiments; B the recipient, partially exhausted, vessel; C the passage to the air-pump. Above



the nozzle is provided a closed chamber E into which the external air has access through a metal gauze D, and where consequently the pressure is a little below atmospheric. F represents (diagrammatically) a pressure-gauge, or micro-manometer, whose reading would be constant as long as the discharge remains so. Possibly an aneroid barometer would suffice; in any case there is no difficulty in securing the necessary delicacy*. Another manometer of longer range, but only ordinary sensitiveness, would register the low pressure in B. In this way there should be no difficulty in attaining satisfactory results. If F remains unaffected, notwithstanding large alterations of pressure in B, there are no complications to confuse the interpretation.

Terling Place, Witham.
June 10, 1916.

* See for example Phil. Trans. cxvii. A. p. 205 (1901); Scientific Papers, vol. iv. p. 510.

XVII. *On the Energy acquired by small Resonators from incident Waves of like Period.* By Lord RAYLEIGH, O.M., F.R.S.*

IN discussions on photo-electricity it is often assumed that a resonator can operate only upon so much of the radiation incident upon it as corresponds to its own cross-section†. As a general proposition this is certainly not true and may indeed differ from the truth very widely. Since 1878‡ it has been known that an ideal *infinitely small* acoustical resonator may disperse energy corresponding to an area of wave-front of the primary waves equal to λ^2/π , an efficiency exceeding to any extent the limit fixed by the above mentioned rule. The questions of how much energy can be absorbed into the resonator itself and how long the absorption may take are a little different, but they can be treated without difficulty by the method explained in a recent paper§. The equation (49) there found for the free vibration of a small symmetrical resonator was

$$M \frac{d^2\rho}{dt^2} + \mu\rho + 4\pi\sigma r^3(1 - ikr) \frac{d^2\rho}{dt^2} = 0, \quad . \quad . \quad . \quad (1)$$

in which ρ denotes the radial displacement of the spherical surface from its equilibrium value r , M the mass, μ the coefficient of restitution, σ the density of the surrounding gas, and $k=2\pi\div$ wave-length (λ) of vibrations in the gas. The first of the two terms containing σ operates merely as an addition to M . If we write

$$M' = M + 4\pi\sigma r^3, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(1) becomes

$$M' \frac{d^2\rho}{dt^2} + \mu\rho - i.4\pi\sigma kr^4 \frac{d^2\rho}{dt^2} = 0. \quad . \quad . \quad . \quad (3)$$

Thus, if in free vibration ρ is proportional to e^{int} , where n is complex, the equation for n is

$$n^2(-M' + i.4\pi\sigma kr^4) + \mu = 0. \quad . \quad . \quad . \quad . \quad (4)$$

* Communicated by the Author.

† See for example Millikan's important paper on a direct determination of Planck's constant " h "; Physical Review, vol. vii. March 1916, p. 385.

‡ 'Theory of Sound,' § 319; λ =wave-length.

§ Phil. Mag. vol. xxix. Feb. 1915, p. 210.

The free vibrations are assumed to have considerable persistence, and the coefficient of decay is e^{-qt} , where

$$q = 2\pi\sigma kr^4 \sqrt{(\mu/M')} = 2\pi\sigma pkr^4/M', \quad . \quad . \quad . \quad (5)$$

if $p^2 = \mu/M'$.

We now suppose that the resonator is exposed to primary waves whose velocity-potential is there

$$\phi = \alpha e^{ipt}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The effect is to introduce on the right hand of (3) the term $4\pi r^2 \sigma \alpha . ip e^{ipt}$; and since the resonance is supposed to be accurately adjusted, $p^2 = \mu/M'$. Under the same conditions $i d^2\rho/dt^2$ in the third term on the left of (3) may be replaced by $-p d\rho/dt$, whether we are dealing with the permanent forced vibration or with free vibrations of nearly the same period which gradually die away. Thus our equation becomes on rejection of the imaginary part

$$M' \frac{d^2\rho}{dt^2} + 4\pi\sigma pkr^4 \frac{d\rho}{dt} + \mu\rho = -4\pi r^2 \sigma \alpha p \sin pt, \quad . \quad (7)$$

which is of the usual form for vibrations of systems of one degree of freedom. For the permanent forced vibration $M' d^2\rho/dt^2 + \mu\rho = 0$ absolutely, and

$$\frac{d\rho}{dt} = -\frac{\alpha \sin pt}{kr^2}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The energy located in the resonator is then

$$\frac{M\alpha^2}{2k^2r^4}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and it may become very great when M is large and r small.

But when M is large, it may take a considerable *time* to establish the permanent regime after the resonator starts from rest. The approximate solution of (7), applicable in that case, is

$$\rho = \frac{\alpha \cos pt}{pkr^2} (1 - e^{-qt}), \quad . \quad . \quad . \quad . \quad (10)$$

q being regarded as small in comparison with p ; and the energy located in the resonator at time t

$$= \frac{1}{2} M \left(\frac{d\rho}{dt} \right)_{\max.}^2 = \frac{M\alpha^2}{2k^2r^4} (1 - e^{-qt})^2. \quad . \quad . \quad . \quad (11)$$

We may now inquire what time is required for the accumulation of energy equal (say) to one quarter of the limiting value. This occurs when $e^{-qt} = \frac{1}{2}$, or by (5) when

$$t = \frac{\log 2}{q} = \frac{\log 2 \cdot M'}{p \cdot kr \cdot 2\pi\sigma r^3} \quad \dots \quad (12)$$

The energy propagated in time t across the area S of primary wave-front is ('Theory of Sound,' § 245)

$$\frac{1}{2} S \sigma a k^2 \alpha^2 t, \quad \dots \quad (13)$$

where a is the velocity of propagation, so that $p = ak$. If we equate (13) to one quarter of (9) and identify t with the value given by (12), neglecting the distinction between M and M' , we get

$$S = \frac{\pi}{2 \log 2 \cdot k^2} = \frac{\lambda^2}{8\pi \log 2} \quad \dots \quad (14)^*$$

The resonator is thus able to capture an amount of energy equal to that passing in the same time through an area of primary wave-front comparable with λ^2/π , an area which may exceed any number of times the cross-section of the resonator itself.

XVIII. On the Mutual Magnetic Energy of two Moving Point Charges. By Prof. A. ANDERSON †.

THE method here given of finding the mutual magnetic energy of two moving point charges of electricity is elementary. It does not claim to have the elegance of Heaviside's method. It is, however, a confirmation of his result by simple mathematics, in which there are no vector potentials, no curls of any kind, and no trouble of adjusting the vector potential to make the space integral of the scalar product of it and the magnetic force vanish for the polar, or displacement, currents in the medium.

Consider, first, the case of two charges, e_1 and e_2 , at A and B moving along parallel lines AL and BM with velocities w_1 and w_2 , AB being perpendicular to AL and BM (fig. 1).

Let P be any point in space. Draw the plane PLM perpendicular to AL and BM and join PA, PB.

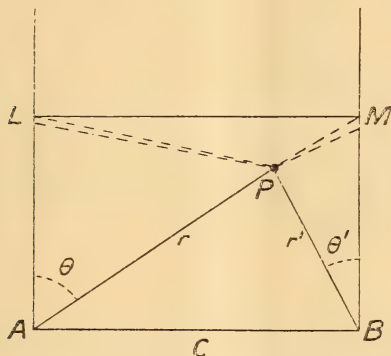
* $\log 2 = 0.693$.

† Communicated by the Author.

The mutual magnetic energy at P per unit volume is

$$\frac{\mu e_1 e_2 w_1 w_2 \sin \theta \sin \theta' \cos \angle \hat{LPM}}{4\pi r'^2 r'^2}.$$

Fig. 1.



Let angle $APB = \alpha$, $PAB = A$, and $PBA = B$; also, let $AB = c$.

By means of a spherical triangle it is easily seen that

$$\cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \angle \hat{LPM}.$$

Also, $\cos \theta = \sin A \cos \phi$ and $\cos \theta' = \sin B \cos \phi$,

where ϕ is the angle between the planes APB and $LABM$.

The mutual energy per unit volume is consequently

$$\mu e_1 e_2 w_1 w_2 (\cos \alpha - \cos^2 \phi \sin A \sin B) / 4\pi r^2 r'^2.$$

Take an element of area dS in the plane APB at P ; then the element of volume at P may be taken

$p d\phi dS$, where p is the perpendicular from P on AB .

If the triangle APB be rotated about AB through a complete revolution, the mutual energy in the ring traced out by dS is

$$\begin{aligned} \frac{\mu e_1 e_2 w_1 w_2}{4\pi r'^2 r'^2} p dS \left(2\pi \cos \alpha - \sin A \sin B \int_0^{2\pi} \cos^2 \phi d\phi \right) \\ = \frac{\mu e_1 e_2 w_1 w_2}{4\pi r'^2 r'^2} dS (2p \cos \alpha - p \sin A \sin B). \end{aligned}$$

Since $rr' \sin \alpha = pc$ and $\sin A \sin B = p^2 / rr'$, this is equal to

$$\frac{\mu e_1 e_2 w_1 w_2}{4c^2} dS \left[\frac{2 \cos \alpha \sin^2 \alpha}{p} - \frac{\sin^3 \alpha}{c} \right].$$

This must be integrated over the whole half plane on one side of the base AB.

There are two integrals to be found,

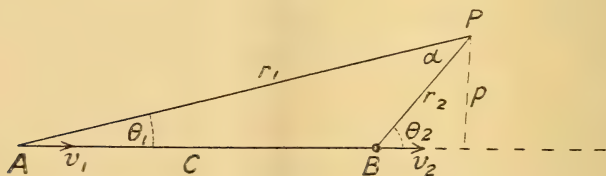
$$\int \frac{2 \cos \alpha \sin^2 \alpha}{p} dS \quad \text{and} \quad \int \frac{\sin^3 \alpha}{c} dS.$$

They present no difficulty, the first being obtained by reducing to polar coordinates, A being pole and AP radius vector. The second is very easily got by taking as element dS the difference between the areas of two segments of circles on AB, one containing an angle α and the other an angle $\alpha + d\alpha$. Their values are respectively $4c$ and $2c$. The total mutual energy is, therefore,

$$\frac{\mu e_1 e_2 v_1 v_2}{2c}.$$

Consider now the case of two charges, e_1 and e_2 , at A and

Fig. 2.



B both moving in the direction AB (fig. 2) with velocities v_1 and v_2 . The mutual energy per unit volume at P is

$$\frac{\mu e_1 e_2 v_1 v_2}{4\pi} \sin \theta_1 \sin \theta_2 / r_1^2 r_2^2.$$

And the mutual energy in the ring traced out by dS at P is

$$\begin{aligned} dS \cdot \frac{\mu e_1 e_2 v_1 v_2}{2} \cdot p \sin \theta_1 \sin \theta_2 / r_1^2 r_2^2 \\ = dS \cdot \frac{\mu e_1 e_2 v_1 v_2}{2} \cdot \frac{p^3}{r_1^3 r_2^3} = dS \cdot \frac{\mu e_1 e_2 v_1 v_2}{2} \cdot \frac{\sin^3 \alpha}{c^3}. \end{aligned}$$

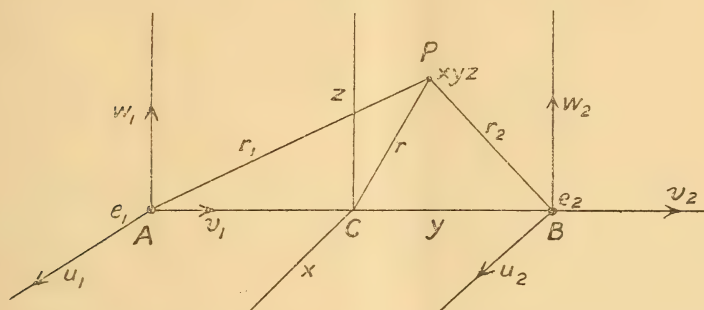
But $\int \sin^3 \alpha dS = 2c^2$.

Hence the total mutual energy $= \frac{\mu e_1 e_2 v_1 v_2}{c}$, that is, double what it is in the former case for the same values of charges and velocities.

We are now in a position to solve the general case.

The component velocities of e_1 at A (fig. 3) are u_1, v_1, w_1 ,

Fig. 3.



and those of e_2 at B u_2, v_2, w_2 . The origin is taken at C the middle point of AB, the axis of y being along CB.

$$r_1^2 = r^2 + yc + \frac{c^2}{4}, \quad \text{and}$$

$$r_2^2 = r^2 - yc + \frac{c^2}{4}, \quad \text{where } AB = c, \text{ as before.}$$

The components of magnetic force due to e_1 are

$$\alpha_1 = e_1 \frac{v_1 z - w_1 \left(y + \frac{c}{2} \right)}{r_1^3},$$

$$\beta_1 = e_1 \frac{w_1 x - u_1 z}{r_1^3},$$

$$\gamma_1 = e_1 \frac{u_1 \left(y + \frac{c}{2} \right) - v_1 x}{r_1^3};$$

and those due to e_2 are

$$\alpha_2 = e_2 \frac{v_2 z - w_2 \left(y - \frac{c}{2} \right)}{r_2^3},$$

$$\beta_2 = e_2 \frac{w_2 x - u_2 z}{r_2^3},$$

$$\gamma_2 = e_2 \frac{u_2 \left(y - \frac{c}{2} \right) - v_2 x}{r_2^3}.$$

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$$\begin{aligned} \therefore r_1^3 r_2^3 (\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2) \\ = x^2 (v_1 v_2 + w_1 w_2) + y^2 (u_1 u_2 + w_1 w_2) + z^2 (v_1 v_2 + u_1 u_2) \\ - zy (v_1 w_2 + w_1 v_2) - xz (u_1 w_2 + u_1 w_2) - xy (u_1 v_2 + u_2 v_1) \\ + \frac{c}{2} z (v_1 w_2 - w_1 v_2) + \frac{c}{2} x (v_1 u_2 - u_1 v_2) \\ - \frac{c^2}{4} (u_1 u_2 + w_1 w_2). \end{aligned}$$

The total mutual magnetic energy may be found by evaluating the integrals $\int \frac{x^2 dx dy dz}{r_1^3 r_2^3}$, &c...., but it is not necessary to do this. Rejecting those which evidently vanish, and observing that

$$\int \frac{x^2 dx dy dz}{r_1^3 r_2^3} = \int \frac{z^2 dx dy dz}{r_1^3 r_2^3},$$

it is clear that the total mutual magnetic energy is

$$A(2v_1 v_2 + u_1 u_2 + w_1 w_2) + B(u_1 u_2 + w_1 w_2),$$

where A and B are constants to be determined.

Let u_1, u_2, w_1, w_2 all vanish, the energy is $2Av_1 v_2$. Hence, by the former result for this case,

$$A = \frac{\mu e_1 e_2}{2c}.$$

Again, let u_1, u_2, v_1, v_2 all vanish, the energy is

$$(A + B)w_1 w_2,$$

and, therefore, by the former result,

$$A + B = \frac{\mu e_1 e_2}{2c},$$

thus $B=0$, and the energy is given by

$$\frac{\mu e_1 e_2}{2c} (2v_1 v_2 + u_1 u_2 + w_1 w_2),$$

or, if V_1 and V_2 are velocities of e_1 and e_2 , ϵ the angle between their directions, and θ_1, θ_2 the angles between these directions and AB, by

$$\frac{\mu e_1 e_2 V_1 V_2}{2c} (\cos \epsilon + \cos \theta_1 \cos \theta_2).$$

XIX. *On the Principle of Least Action in the Theory of Electrodynamics.*—I. By G. H. LIVEN^{*}.

IT is a fundamental problem in the general theory of electrodynamics to formulate a scheme for the description of the general phenomena of the electromagnetic field on an analogy with the behaviour of some hypothetical system obeying the ordinary laws of generalized mechanics. Now, any dynamical problem can be enunciated in a single formula as a variation problem, and thus if the laws of electrodynamics can be derived in a minimum or variational theorem, the formulation is virtually complete; there remain only such interpretations, explanations, and developments as will correlate the corresponding integrals relating to known dynamical systems.

In this form the problem has been solved in slightly different ways by Larmor[†], Lorentz[‡], Macdonald[§], and others. The integral which is the subject of variation in each case is virtually the same as

$$\int_{t_1}^{t_2} \left[L + \int \left\{ \frac{1}{2c} (AC) - \frac{1}{8\pi} E^2 \right\} dv \right] dt,$$

wherein L is the part of the Lagrangian function of the system not depending on the conditions of the field; A is the vector potential of the magnetic field; C is the total current of Maxwell's theory and includes both the true electric and fictitious æthereal fluxes, and E is the electric force intensity at the typical field-point. It is assumed that the system consists entirely of free æther and electrons, and the integration with respect to v is taken over the whole field.

On analogy with the known properties of electrostatic fields it is assumed that the term in E^2 represents the true potential energy of elastic strain in the æther, both as regards its distribution and total amount; it then follows that the integral

$$\frac{1}{2c} \int (AC) dv$$

represents the totality and distribution of the kinetic energy of electric origin in the æthereal field.

* Communicated by the Author.

† 'Æther and Matter' (Cambridge, 1900), ch. iv.

‡ 'La théorie électromagnétique de Maxwell, &c.' (Leiden, 1892),

§§ 55-61.

§ 'Electric Waves' (Cambridge, 1902), App. C.

But in all other applications of the general theory of electromagnetism it is found more convenient to assume that the magnetic energy, which for other reasons is regarded as of kinetic type, is distributed throughout the field with a density at any place expressed by

$$\frac{1}{8\pi} B^2,$$

where B is the vector of magnetic induction.

Now, these two expressions for the magnetic energy do not in the most general case agree, even in total amount, for, as Macdonald points out *, in the derivation of the one from the other by the method of integration by parts, an integral over the infinite boundary is brought in which is not generally negligible. It thus becomes a question whether the results of the dynamical theory can be used in the more usual formulations and developments of the subject. An attempt to justify such usage so far as concerns the expression for the force on a moving charge has been made by Larmor † by the examination of a special problem with restricted conditions, but some doubt may still exist as to the general validity of the argument thus employed.

The question is settled in the theory of relativity ‡, where it is verified that the usual expression for the force is consistent with the second expression for the magnetic energy; but insofar as this verification is based on the differential invariant theory associated with Minkowski's four-dimensional analysis of the general theory, it can hardly be said to throw much light on the physical bearing of the problem.

It seems therefore desirable to attempt a direct formulation of the general dynamical theory on the basis of the second and more usual expression for the kinetic energy of electric origin in order to confirm, if possible, the result obtained by Larmor under special circumstances, which is generally used without hesitation or restriction in either form of the theory. The object of the present note is to show that such a formulation can easily be effected and the results derived from it, insofar as they are identical with those deduced on the older basis, fully substantiate Larmor's conclusions from his special problem.

The principle of least action is applied, in the manner already elaborated in full detail by Larmor, to determine the

* *L. c.* p. 33.

† *L.M.S. Proc.* 1915.

‡ *Cf.* Cunningham, 'The Principle of Relativity' (Cambridge, 1914), p. 158.

sequence of changes in a system consisting entirely of free electrons and æther, but for which the Lagrangian function is

$$L + \int \frac{1}{8\pi} (B^2 - E^2) dv.$$

The main difficulty experienced in using this form of function is that it is not on either account explicitly expressed in terms of the independent coordinates of the system, which in the present case are usually taken to be the æthereal displacement $\frac{1}{4\pi}E$ and the position coordinates (x, y, z) of the various electrons. The main claim for the alternative expression for the magnetic energy is that it is so expressed, but even then the difficulty still persists as regards the electric energy. It appears, however, that any such explicit interpretation of the functions can be avoided by the use of Lagrangian undetermined multipliers, as, in fact, is done by Larmor for the potential electric energy. In this way the variations of E and B can be temporarily rendered independent of each other and also of the coordinates of the electrons, the form of the undetermined functions thus introduced being, however, finally chosen to secure the correct relations of dependence between the different variables.

Now the functions E and B are connected with one another and with the coordinates of the electrons by the relations

$$\begin{aligned} \int \operatorname{div} E dv - 4\pi \sum e &= 0, \\ \int \left(\operatorname{Curl} B - \frac{1}{c} \frac{dE}{dt} \right) dv - \frac{4\pi}{c} \sum ev &= 0, \end{aligned}$$

wherein Σ represents a sum taken over all the electrons, the typical one having a charge e and velocity v ($\dot{x}, \dot{y}, \dot{z}$). The second relation being a vector one is in reality equivalent to three independent equations. We now introduce four Lagrangian undetermined multipliers ϕ, A_x, A_y, A_z , all of which are functions of position in the field. The last three may be taken to be the rectangular components of a vector A . It is thus the variation of

$$\begin{aligned} \int_{t_1}^{t_2} dt \left[L + \frac{1}{8\pi} \int dv \left\{ B^2 - E^2 + 2\phi \operatorname{div} E \right. \right. \\ \left. \left. - 2 \left(A, \operatorname{Curl} B - \frac{1}{c} \frac{dE}{dt} \right) \right\} - \sum \phi e + \sum \frac{e}{c} (Av) \right], \end{aligned}$$

that is to be made null, afterwards determining the forms of

ϕ and A to satisfy the restrictions which necessitated their introduction. The complete variation is easily effected and is, in fact,

$$\begin{aligned} \int_{t_1}^{t_2} dt \Sigma \left[\left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} - e \frac{\partial \phi}{\partial x} + \frac{e}{c} \left(v \frac{\partial A}{\partial x} \right) \right\} \delta x \right. \\ \left. + \{ \dots \} \delta y + \{ \dots \} \delta z + \frac{e}{c} (A_x \delta \dot{x} + A_y \delta \dot{y} + A_z \delta \dot{z}) \right] \\ + \frac{1}{4\pi} \int_{t_1}^{t_2} dt \int dv \left[(B \delta B) - (E \delta E) + \phi \operatorname{div} \delta E \right. \\ \left. - (A \operatorname{Curl} \delta B) + \frac{1}{c} \left(A \frac{d \delta E}{dt} \right) \right]. \end{aligned}$$

On removal by integration by parts of the various differential operations affecting the independent variations and noticing that the time differentiations affecting the coordinates of an electron are total differentiations following the motion of that electron, we get finally for the total variation

$$\begin{aligned} \int_{t_1}^{t_2} dt \Sigma \left[\left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} + e \frac{\partial \phi}{\partial x} - \frac{e}{c} [v \operatorname{Curl} A]_x \right. \right. \\ \left. \left. + \frac{e}{c} \frac{\partial A_x}{\partial t} \right\} \delta x + \{ \dots \} \delta y + \dots \right] \\ + \frac{1}{4\pi} \left(\int_{t_1}^{t_2} dt \int dv \left[(B - \operatorname{Curl} A, \delta B) - \left(E + \operatorname{grad} \phi + \frac{1}{c} \frac{\partial A}{\partial t}, \delta E \right) \right] \right. \\ \left. - \frac{1}{4\pi} \int_{t_1}^{t_2} dt \int dS \{ \phi \delta E_n - [A \delta B]_n \} \right. \\ \left. + \left| \Sigma \frac{e}{c} (A_x \delta x + A_y \delta y + A_z \delta z) - \frac{1}{c} (A \delta E) \right|_{t_1}^{t_2} \right). \end{aligned}$$

The integral with respect to S is taken over the infinitely extended surface bounding the field.

The variations δx , δy , δz , which give the virtual displacement of an electron e , and the variations δE , δB , which specify the condition variations in the æther, can now be considered as all independent and perfectly arbitrary; hence the coefficients of each must vanish separately in the dynamical variational equation. We conclude that ϕ and A must tend to zero regularly at infinity and then at each point of the field

$$E + \frac{1}{c} \frac{\partial A}{\partial t} + \operatorname{grad} \phi = 0,$$

$$B - \operatorname{Curl} A = 0,$$

whilst for each electron there are three equations of type

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} - \frac{e}{c} [v \text{Curl } A]_x + \frac{e}{c} \frac{\partial A_x}{\partial t} + e \frac{\partial \phi}{\partial x} = 0.$$

With the exception of the second equation, these are identical with Larmor's equations deduced from the older alternative basis. It thus appears that the dynamical equations determining the strain conditions of the æther and the motion of the various electrons are identical on either form of the theory.

It also appears that the scalar and vector multipliers ϕ and A , which were originally introduced into the theory as undetermined functions, are, in fact, the scalar and vector potentials of Maxwell's theory; but they are here relegated to their proper position as auxiliary functions introduced to procure analytical simplicity in the relations of the theory, and are so far without direct physical significance.

The present discussion does not, of course, help in the elucidation of any difficulties experienced in considering a definite choice of one of the specified forms for the magnetic energy distribution and the consequent rejection of the other, except in so far as it has removed one of the difficulties in the way of a general acceptance of the more usual expression in terms of the magnetic induction. It does, however, show that, dynamically interpreted, the two different forms of the expression are identical, and this covers the whole field of observable physical activities, with the possible exception of those associated with the phenomenon of radiation. The exclusion of radiation is important, and emphasises the essential difference between the two forms of the theory, as well as the reason for the similarity in their dynamical aspects. It appears, on general grounds, that the purely radiation portions of the field, representing as they do detached portions of the general dynamical system which have become isolated from the remaining parts with a definite quota of the total energy, will be completely inoperative as regards the dynamical relations of the electrons and of the æther in their immediate neighbourhood; and this view is supported by the known properties of such fields, inasmuch as the energy in them, being equally divided between the potential and kinetic types, is not represented at all in the generalized Lagrangian function of the system. Thus, from the point of view of the dynamics of the electron, and this is all we are ultimately concerned with, we may omit altogether the purely radiation portions of the field;

and this procedure would secure the analytically necessary localization of the fields, which is essential to the mathematical developments of either form of the theory, and it removes the origin of the discrepancy between the two estimates of the magnetic energy in the field. It would thus justify a further simplification of the problem by restricting it to stationary or quasi-stationary motions, and secure full confirmation for the generality of the procedure adopted by Larmor.

The University, Sheffield,
April 15th, 1916.

XX. *On the Hall Effect and Allied Phenomena.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN the April number of your Magazine, in a short review of my paper on the "Electron Theory of the Hall Effect and Allied Phenomena," Mr. A. W. Smith directs attention to two points in my work which may have given rise to some misunderstanding. In reviewing the apparent discrepancies in the sign of the four effects I stated, what appeared to be a general rule, that the relative signs of the four phenomena concerned are always the same. This statement was based on the only evidence I had access to when compiling the paper in question; but I am not yet inclined to modify it to any extent. On page 110 of Baedeker's "Die elektrischen Erscheinungen in metallischen Leitern," a table of the four coefficients for twelve metals is compiled from the work of Zahn. Of these twelve all but silver, copper, and antimony conform to the rule. To this list Mr. Smith adds four more substances of which three are irregular. There is therefore, on the whole, still a balance of 10 to 6 in favour of my original statement.

However, apart from statistics, I quite agree with Mr. Smith that the rule in all its simplicity cannot possibly be infallible; but this is only what we might expect. The necessity for simplicity of the physical hypotheses required in the analysis of a mathematical theory puts that theory quite out of keeping with the actual facts, which are concerned in the present instance with substances whose composition is known to be extremely complex and irregular. The constitutional irregularities of matter are in no way susceptible of exact mathematical specification, and have therefore always to be

excluded in a strictly analytical theory, except in so far as in such cases where the average aspect of the irregularities may be conveniently determined and expressed. Nevertheless, the general compatibility or otherwise of the results of an ideal theory based on the hypothesis of a suitable average may provide some indication of the validity of the general theory; and in this way only is the theory proposed in my paper to be used.

The same point may be further emphasised by reference to Mr. Smith's second difficulty. In reviewing the influence of temperature on the Hall effect in the ferromagnetic metals, I stated that as the temperature increases the Hall effect increases exactly parallel with the magnetic permeability, until the critical temperature is reached, when it decreases rapidly to a value more akin to that found in the simpler metals. I conclude that this is evidence for the explanation of the irregularities of these metals by the assumption of strong local magnetic fields proportional to the polarization. My use of the words "exactly parallel" was perhaps unfortunate, as I did not intend to imply that the constant of proportionality was independent of the temperature, or in other words that the magnetization and Hall effect depend on the temperature in the same way. The phenomena of magnetization and conduction are fundamentally distinct, and the average influence of the strong local molecular fields, which now seem to be generally regarded as surrounding the molecules of most substances, will in all probability be very different in the two cases. The proportionality of the strength of the local field with the polarization has been found to provide a generally sufficient explanation in most cases of the irregularities of the phenomena concerned; but it is to be remembered that it is an averaged field that is operative, and the type of average may be very different in the separate cases or at different temperatures, so that the mode of dependence of the various phenomena on the temperature need not necessarily be the same. This implies that the constant A in Mr. Smith's equation may quite consistently be a function of the temperature, without in any way affecting the general conclusions drawn in my theoretical paper.

Very truly yours,

G. H. LIVENS.

The University,
Sheffield.
May 26th, 1916.

XXI. *On the Velocity of Secondary Cathode Rays emitted by a Gas under the Action of High-Speed Cathode Rays.*
By M. ISHINO, *Rigakushi, of the Imperial University of Kyoto, Japan* *.

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§ 1. *Introduction.*

THE ionization of various substances in different states has been investigated by many observers. The initial velocity of the particles during ionization may have an important bearing on the theory of ionization and on that of the structure of the atom. Lenard† was the first to estimate the velocity of the secondary cathode rays from a metal. The metal was bombarded by slow cathode rays produced by the action of ultra-violet light on a metallic plate. He observed that the greater proportion of the secondary cathode rays left the metal with velocities less than "that acquired by the corpuscle under a fall of potential of 11 volts" (or merely for brevity "a velocity of 11 volts"). v. Baeyer‡, Gehrts§, and Campbell|| and others investigated the same problem. They used a retarding electric field to determine the velocity of the secondary cathode rays, *i. e.* so-called δ rays. The results showed that the velocity of the δ rays depended upon the velocity or the kinetic energy of the primary rays, provided that the kinetic energy was not large; but that the velocity did not depend very much on the kind of metal from which δ rays were liberated. They have shown that the greatest velocity of the δ rays did not exceed 30 volts, notwithstanding the fact that different velocities of the primary rays were used.

For the case of primary rays having very high speeds of

* Communicated by Sir J. J. Thomson, O.M., F.R.S.

† Lenard, *Ann. d. Phys.* xii. p. 449 (1903).

‡ v. Baeyer, *Phys. Zeits.* x. p. 176 (1909).

§ Gehrts, *Ann. d. Phys.* xxxvi. p. 995 (1911).

|| Campbell, *Phil. Mag.* xxv. p. 803 (1913).

several thousand volts, Fürchtbauer* observed, that for positive rays with velocity 2100 to 4500 volts the velocities of most of the δ rays from a metallic plate were between 27 and 30 volts, and were independent of the velocity of the primary rays. He found that this was also true when the positive rays were replaced by cathode rays with velocities of 1920–4400 volts. He used a magnetic field to determine the velocity of the δ rays. Campbell† found that the initial velocity of the δ rays was less than 40 volts, and was nearly independent of the velocity and the nature of the high-speed primary rays—which may be α or β particles,—and also of the nature of the metals which emitted the δ rays. Bumstead‡, however, showed that the δ rays produced by α rays from polonium have velocities which range from 0 to 2000 volts. He pointed out the existence of tertiary rays which were produced by the δ rays and had generally slow velocities, and he made use of a plan to eliminate them. Campbell§ suggested that the uniform nature of these δ rays from metals may be due to the ionization of a gaseous layer condensed on the surface of the metals, and not to the ionization of the metals themselves.

So far as I know, there are not many direct experiments on the nature of the secondary cathode rays ejected from the gaseous molecules themselves. Sir J. J. Thomson|| measured the velocity of secondary cathode rays ejected by slow cathode rays. He used a retarding electric field, and determined the maximum velocity possessed by those secondary rays whose velocities were sufficiently high to produce an appreciable effect upon the luminosity of the residual gases. Potential differences up to 1500 volts were applied to a Wehnelt cathode. He found the maximum velocity of the secondary rays to be about 40 volts, and that the energy of the rays was independent of the energy of the primary rays and of the nature of the atom emitting them.

The present investigation was undertaken with a view to determine the distribution of velocities of the secondary cathode rays produced from gases by high-speed cathode rays, and to find some relations between this distribution and the speed of the primary rays and nature of the gaseous atom.

* Fürchtbauer, *Phys. Zeits.* vii. p. 748 (1906).

† Campbell, *Phil. Mag.* xxii. p. 276 (1911), xxiv. p. 529 (1912), xxiv. p. 783 (1912).

‡ Bumstead, *Phil. Mag.* xxvi. p. 233 (1913).

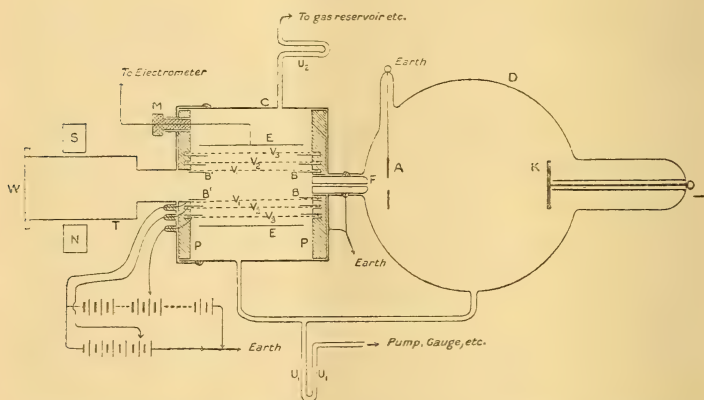
§ Campbell, *Phil. Mag.* xxviii. p. 286 (1914).

|| J. J. Thomson, *Camb. Phil. Soc.* xiv. p. 541 (1908).

§ 2. *Apparatus.*

The final apparatus used is indicated in fig. 1. The primary cathode rays are produced in an exhausted glass discharge-bulb D by means of an induction-coil. The high-speed

Fig. 1.



cathode rays are allowed to pass into an ionization-chamber C connected to the discharge-bulb by a narrow aluminium tube F of diameter 0.4 mm. The primary rays ionize the residual gas in C, and the gas molecules emit the secondary cathode rays.

To determine the intensities and velocities of the secondary rays, the method of retarding electric field due to Lenard * was used. Three concentric gauze cylinders of fine brass wire, V_1 , V_2 , and V_3 , are supported by two pieces of ebonite, P P, around the path of the primary rays. The gauzes are connected to secondary cells and can be raised to any potentials. A tubular brass electrode E is situated outside the gauzes and concentrically with them. The charge on those corpuscles which arrived at this electrode is measured by means of a Dolezalek electrometer. A stout brass wire, insulated from the ionization-chamber with a piece of amber M, holds the electrode E, and the amber is protected by an earthed guard-tube.

The ionization-chamber is 8 cm. long and 8 cm. in diameter. The innermost gauze is 1.6 cm. in diameter. The distances between the successive gauzes and that between the outermost gauze and the electrode are 3 mm. T is a Faraday cylinder with one end fitted to the ionization-chamber. The other end of T is closed by a glass window W

* Lenard, *Ann. d. Phys.* viii. p. 188 (1902).

covered with willemite. The direction of the primary cathode rays could be thus observed and adjustments made with a small magnet. A horseshoe magnet NS is placed around T to prevent the entering rays, and any rays excited inside, from escaping. It was experienced in preliminary investigations, that when the ebonite pieces were struck by the rays, some charge accumulated on the ebonite pieces and disturbed the electrometer. To get rid of this effect, short protecting tubes B, B' are fitted at each end of the gauzes, and any unnecessary surfaces of the ebonite pieces are covered with earthed foils. The gauzes and the electrode E are covered with candle-soot (§ 7).

To the ionization-chamber are connected two U-tubes, U_1 and U_2 . A Gaede pump, McLeod gauge, spectrum-tube, and P_2O_5 drying-tube are connected to U_1 , and a gas reservoir and drying-tubes to U_2 . The U-tubes are cooled with liquid air or a mixture of solid carbonic acid and alcohol. This cooling condenses any mercury vapour coming from the pump and also any wax vapour from the joints. The whole apparatus was arranged to get rid of wax-joints as much as possible.

§ 3. *Experimental Procedure.*

The separate parts of the apparatus before being put together were washed with water and alcohol, boiled with water for about one hour, again washed with distilled water, and finally dried. Glass parts before being thus treated were washed with nitric acid. The electrometer-electrode E, the gauzes, inside surfaces of the ionization-chamber and Faraday cylinder were covered with candle-soot.

The whole apparatus was exhausted by the Gaede pump and charcoal tube cooled with liquid air. To drive out any gas adhering to the electrodes of the discharge-bulb, the bulb was kept discharging while the apparatus was exhausted. A small quantity of air or hydrogen gas, previously dried by means of P_2O_5 and $CaCl_2$, was then introduced from a reservoir. The apparatus was exhausted to a desired pressure. The pressure was as low as possible, since, otherwise, the secondary cathode rays and the positive ions produced at the same time with the secondary corpuscles would ionize the residual gas by their passage to the electrometer-electrode. The same reason necessitated reducing the successive distances between the gauzes and the electrode to a value much smaller than the mean free paths of the corpuscles and the positive ions. The value, as mentioned above, was 3 mm. However, owing to the hardening of the discharge-bulb, the pressure had to be kept above a certain value.

The actual measurements were made at the pressures between 0.003 and 0.0008 mm. of mercury. At this low pressure, the current of the secondary corpuscles was diminished, but was quite measurable owing to the form and position of the electrometer-electrode and to the highly sensitive electrometer employed. The sensitiveness of the electrometer was 1500 scale-divisions per volt at 1.5 metre distance when the needle potential was 200 volts.

The different discharging potentials in the bulb were obtained by slightly varying the pressure, or by changing the primary current of the induction-coil. The potential was measured with a spark-gap micrometer having two brass spheres of equal diameters of 3.5 cm. connected with the discharge-bulb in parallel.

Hydrogen gas was prepared by the electrolysis of a solution of barium hydroxide, and the ionization-chamber was rinsed with a stream of the hydrogen for a considerable time. The purity of the hydrogen gas filling the chamber was examined. This was done by connecting a small spectrum tube to the ionization-chamber, passing a discharge through the tube, and measuring the wave-lengths of the spectral lines from the gas by a Hilger wave-length spectrometer. The examination of the spectrum showed that the ionization-chamber was filled with quite pure hydrogen gas. The same examination also proved that the cooling by means of U-tubes was absolutely necessary in order to free the gas in the chamber from any wax vapours. Measurement of the current due to the secondary rays was made to at least half an hour after applying the cooling material to the U-tubes.

A great difficulty experienced, however, was in getting a constant deflexion of the electrometer, which measured the ionization current inside under a definite condition. Much time was spent in trying to effect this. The induction-coil worked uniformly, and the electrometer very satisfactorily. The McLeod gauge showed that the pressure did not suffer marked change. But the variation of the electrometer deflexion amounted sometimes to 15 per cent. The cause of this variation might be either a small fluctuation of the induction-coil, or a small trace of wax vapour inside the ionization-chamber.

A method of taking readings of the electrometer deflexion illustrated in the following table was adopted to overcome the difficulty above mentioned. Here v was the potential applied to one of the gauzes, and was varied to any desired value, while the other gauzes were kept at

definite potentials. The reading for $v=0$ was taken both before and after taking the reading for $v=$ a desired value, v' say, and the mean of these two readings for $v=0$ was taken as representing the value for $v=0$ at the time of taking the reading for $v=v'$. The reading for $v=v'$ was expressed in a percentage of the mean reading for $v=0$.

v .	Deflexion in 20 secs.	Mean of Deflexion for $v=0$.	Deflexion as percent. of this mean.
0	13.6 cms.	13.7	100.0
0.3	13.2 "		96.3
0	13.8		
0.7	11.1	13.95	79.6
0	14.1		
0.5	12.0	14.2	84.5
0	14.3		

When the deflexion for $v=0$ changed too much, the exhaustion of the apparatus was renewed. The deflexions were observed for time varying from 10 seconds to 60 seconds. The final results selected for the percentages of the currents were the averages of a considerable number of observations.

§ 4. Experiments on the Positive Ions.

Positive ions are produced in the gas at the same moment as the negative corpuscles are ejected from the gas atoms under the bombardment of the primary rays. The initial velocity of the positive ions therefore must be due to the recoil caused by the corpuscles. Hence we have

$$MV = mv,$$

where M and m are the masses, and V and v the velocities of the positive ion and the corpuscle respectively. Since their kinetic energies are

$$\frac{1}{2}MV^2 = Ee, \quad \text{and} \quad \frac{1}{2}mv^2 = E'e,$$

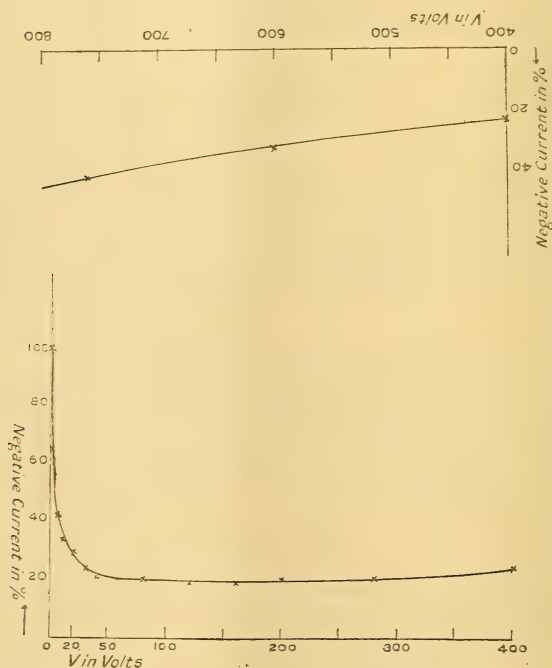
where E and E' are the potential falls corresponding to the velocities V and v respectively, we have

$$E = \frac{m}{M} E'.$$

Taking the initial velocity of the corpuscle, in accordance with Sir J. J. Thomson's result, as that corresponding to a potential fall of 40 volts, then in the case of a hydrogen atom, $m/M=1800$, and we have $E=0.02$ volt approximately.

Thus the initial velocity of the positive ions should be very small. Hence it was expected that the positive ions would not affect the measurement of the distribution of velocities of the secondary cathode rays. Measurements have been carried out with two gauzes. To them the following potentials were applied: $V_1=0$, $V_2=-v$ volts, where v was varied from 0 to several hundred volts. The currents arriving at the electrometer-electrode were measured. The results are shown in fig. 2.

Fig. 2.

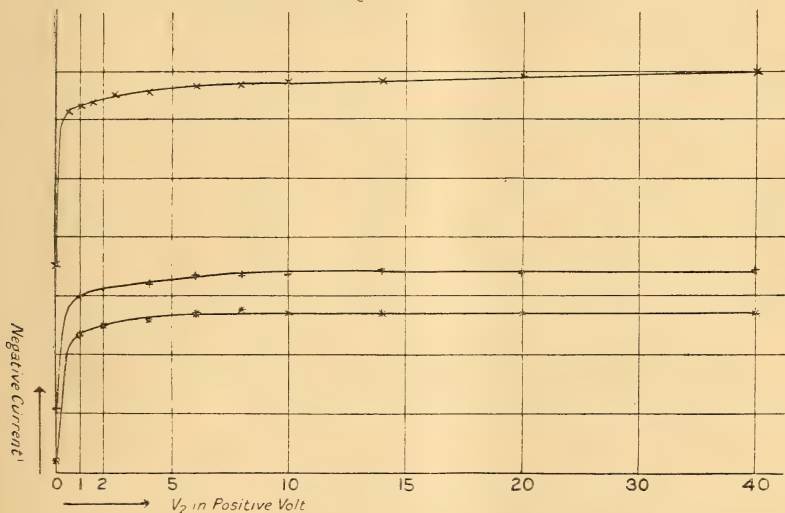


The form of the curve shows that ionization occurred for large values of v . This ionization should be attributed to the positive ions, because almost all the negative corpuscles must be stopped at the gauze V_2 by such high potentials. But the positive ions, diffusing into the region between V_1 and V_2 , are accelerated there, and get a velocity high enough to be able to ionize the residual gas; and the corpuscles thus produced are driven to the electrode E by the electric force between V_2 and E. Using different fields, such as $V_1=-10$

volts, $V_2 = -10 - v$ volts, where v is varied as before, a repetition of the experiments yielded similar results.

To see what potential difference is required to stop the positive ions, the following experiments were carried out with three gauzes. Keeping the potentials of V_1 and V_3 at some constant values, the potential of V_2 was varied from 0 to +30 volts, the idea being to stop the positive ions with the potential difference between V_1 and V_2 . The results for three cases, viz. ($V_1=0$, $V_3=0$), ($V_1=0$, $V_3=-10$ volts), and ($V_1=0$, $V_3=-30$ volts), are given in fig. 3.

Fig. 3.



The curves show that nearly all the positive ions are stopped by a potential difference of 1 volt. The current, however, increases very slowly till $V_2 = +8$ volts, and then in the case of curves II. and III. becomes constant, while it continues to increase in curve I.

Hence in order to stop the positive ions completely, we should use a potential difference greater than 8 volts against them.

§5. Experiments on the Action of the Metallic Gauzes.

Some of the secondary cathode corpuscles will hit the wires of the gauzes. The effect of the hits is to diminish the number of the secondary cathode rays which would otherwise arrive at the electrode. The hits would probably also cause the emission of tertiary cathode rays, and perhaps

of soft Röntgen rays. It might be expected that the hits would occur oftener when the gauzes are uncharged or charged positively, than when they are strongly negatively charged, as v. Baeyer * and Compton † suggested. As the intensity of the electric field at a point just outside a wire is $4\pi\sigma$, where σ denotes the surface density of the electricity on the wire, it is clear that, when the gauze is negatively charged, a negative corpuscle will be deviated from its path, so as to tend to pass around a wire, when it comes near it. Keeping constant potential differences between the gauzes, while the values of the potentials were changed, the following measurements of the current to the electrometer were made :—

V_1 .	V_2 .	V_3 .	Deflexion.
0 volt.	+6 volts.	-110 volts.	2.75 cm./15 sec.
-2	+4	-112	3.3
-4	+2	-114	3.6
-6	0	-116	3.75
-8	-2	-118	3.85
-10	-4	-120	3.9
-16	-10	-126	4.0

These results confirm the above idea, and show that it is necessary to charge the gauzes negatively, in order to prevent the corpuscles striking them.

§ 6. *Arrangement of the Potential Differences between the Gauzes.*

From what precedes it is clear that we must have two electric fields :

- (1) One variable electric field to measure the distribution of velocities of secondary cathode rays ; and
- (2) One electric field to stop completely all the positive ions.

The condition necessary to prevent hits of the secondary corpuscles on the gauzes is that the gauzes must be negatively charged.

Various arrangements of the fields between the gauzes were tried, and finally the following was selected :

$V_1 = -12$ volts, $V_2 = -2$ volts, $V_3 = -12 - v$ volts, where v was varied from 0 to 2000 volts. The first potential

* v. Baeyer, *loc. cit.*

† Compton, *Phil. Mag.* xxiii. p. 579 (1912).

difference between V_1 and V_2 stopped all the positive ions, and the second potential difference between V_2 and V_3 measured the number of the secondary cathode rays, this field being the main one. The third potential difference between V_3 and E served to prevent the reflexion of the corpuscles from the electrode E. This potential difference was already shown to be necessary by the experiments of v. Baeyer* and afterwards of Compton† and others.

Before describing the results with this arrangement, it might be interesting to give some results obtained by another arrangement, in which $V_1=0$, $V_3=+10$ volts, and V_2 was varied from 0 to -500 volts. The results are shown in curve I, fig. 4. The curve cuts the V_2 -axis at about

Fig. 4.



-120 volts, and then the positive current increases gradually with V_2 . With this arrangement, positive ions were not stopped in the region $V_1 V_2$, but in the region $V_2 V_3$. They were therefore accelerated in the first region, and acquired velocities large enough to ionize the residual gas. The positive currents for large values of V_2 are partly due to this ionization. Moreover, the positive potential of V_3 with

* v. Baeyer, *loc. cit.*

† Compton, *loc. cit.*

respect to the electrode E did not prevent the reflexion of the secondary rays from the electrode. This defect accounts in part for the positive current. This arrangement was abandoned as unsatisfactory.

§ 7. *Effects due to Röntgen Rays.*

Röntgen rays would be produced from the gas molecules under the bombardment of the high-speed primary cathode rays. Some effects due to the Röntgen rays were observed. With a brass electrode not covered with soot, the experiment described in § 6 was tried. The results are shown in curve II. fig. 4. The amount of the negative current is always less than that in the curve I. obtained with soot-covered electrode. This is explained to be due to the emission of some β rays, when the Röntgen rays hit the electrometer electrode.

The other effect caused by Röntgen rays will be considered in the discussion of the final results (§ 9).

§ 8. *The Results. The Distribution of the Velocities of the Secondary Cathode Rays.*

The results obtained for air are tabulated in Table I., the values of the potential differences applied to the gauzes being $V_1 = -12$ volts, $V_2 = -2$ volts, and $V_3 = -12 - v$ volts. The first column gives the values of v representing the variable retarding fields for the secondary cathode rays. These results are calculated by noting the number of secondary cells used, each cell being considered to have a potential difference of 2 volts. The kinetic energies of the primary cathode rays are written in volts in the first row. The figures in each column (in the second row and below) are the currents representing the relative numbers of the secondary cathode rays which have velocities greater than v volts in the corresponding rows.

As the table shows, the currents decrease very rapidly with increase of v up to 40 volts, then slowly, and after $v = 900$ volts they become almost constant and small, though not negligibly so. The currents observed in the case of the primary rays with a velocity of 10,500 volts are plotted in curve II., fig. 5, where the abscissæ indicate the values of v in volts.

TABLE I.—Air.

Velocity of Primary Rays. volts.	14,500 volts.	12,600 volts.	11,200 volts.	10,500 volts.	8100 volts.	7100 volts.
$v=0$	100.0	100.0	100.0	100.0	100.0	100.0
0.3	94.0	94.6	95.6	96.6	97.5	95.5
0.5	90.0	91.9	91.0	94.0	91.0	93.0
0.7	78.8	84.1	85.7	87.8	84.6	83.5
1.0	76.5	76.6	75.6	77.4	77.0	75.6
1.5	63.7	65.1	66.8	65.0	64.4	64.6
2.0	60.0	61.0	61.5	59.2	61.6	59.0
2.5	52.6	55.8	54.8	54.3	53.0	54.5
3	49.9	52.4	50.6	50.6	50.2	51.6
4	45.2	45.3	44.7	43.3	44.5	46.8
5	41.0	41.5	41.7	41.4	39.5	41.0
6	36.4	39.2	37.8	37.0	37.3	36.1
7	34.7	36.0	36.1	34.3	35.1	33.5
8	32.0	33.1	32.5	31.9	32.1	29.4
10	29.0	29.3	28.2	28.8	27.0	25.8
14	25.0	24.6	24.7	25.0	23.1	22.9
20	20.6	19.6	18.9	19.8	18.4	17.3
30	17.5	17.6	16.6	16.8	15.9	15.1
40	16.0	15.5	15.1	14.5	12.7	13.3
70	13.6	12.8	12.0	11.6	10.75	10.25
110	11.8	11.3	10.78	10.10	9.68	8.46
150	11.07	10.34	10.42	9.04	8.90	7.41
190	10.72	10.34	9.82	8.60	8.24	6.68
270	10.62	9.87	9.15	8.02	7.51	6.24
390	10.11	9.12	8.75	7.72	6.96	6.01
590	9.55	8.94	8.36	7.21	6.69	5.51
790	9.13	8.24	7.79	6.83	6.23	5.18
990	8.80	8.04	7.55	6.66	6.04	5.08
1190	8.85	8.00	7.53	6.71	6.02	5.04
1390	8.68	8.00	7.41	6.64	5.84	4.84
1590	8.50	8.00	7.29	6.71	5.93	4.84
1990	8.68	8.00	7.41	6.47	5.93	5.04

The results obtained for hydrogen gas are given in Table II.

§ 9. Examination of the Residual Currents.

The nature of the residual currents obtained for the values of v greater than 1000 volts was repeatedly examined. If the residual currents are due to the secondary cathode rays with very large velocities, the currents should decrease as v increases, because it seems quite improbable that a gap occurs between 1000 and 2000 volts in the distribution of the velocities. But there was no evidence of such decrease.

TABLE II.—Hydrogen.

Velocity of Primary Rays. } volts.	15,500 volts.	11,400 volts.	7500 volts.
$v=0$	100.0	100.0	100.0
0.3	95.1	95.5	97.0
0.5	84.5	85.0	85.5
0.7	76.1	79.0	77.6
1.0	66.0	66.2	66.0
1.5	58.4	56.0	57.9
2.0	52.3	51.5	52.1
2.5	48.8	47.7	45.8
3	41.3	44.3	42.8
4	39.4	39.3	39.1
5	35.6	35.8	36.2
6	34.2	32.2	34.4
7	31.6	30.0	31.3
8	29.1	28.2	28.5
10	25.4	20.6	23.6
14	22.1	20.6	20.2
20	17.6	17.4	17.4
30	16.5	14.8	13.4
40	14.2	12.7	11.4
70	11.0	9.62	8.50
110	9.53	8.41	7.25
150	9.15	7.58	6.34
190	8.52	7.24	5.83
230	8.37	6.91	5.68
270	8.05	6.74	5.47
310	7.93	6.67	5.23
390	7.77	6.35	5.04
490	7.59	6.21	4.91
590	7.45	6.04	4.71
690	7.20	6.03	4.50
790	7.05	5.68	4.40
990	6.90	5.53	4.20
1190	6.72	5.38	4.18
1390	6.75	5.54	4.16
1590	6.69	5.38	4.15
1990	6.70	5.38	4.15

Again, if they are due to the ionization of the residual gas by corpuscles of large velocities or Röntgen rays, they should increase with very large values of v ; which was also not observed.

The primary rays strike the gas molecules in their paths, and by these bombardments they may produce Röntgen rays of some hardness. The Röntgen rays would strike the

metallic gauzes, and might cause them to emit β rays with very high speed of the same order as that of the primary cathode rays. To test whether the residual currents are due to these β rays the following experiments were made.

The innermost gauze V_1 was covered with a thin celluloid film of a thickness of about 0.0015 mm. (0.2 mg. per sq. cm.), and the current to the electrometer was measured, taking

$$V_1 = -12, V_2 = -2, V_3 = -12 - v \text{ volts.}$$

$v.$ volts.	Deflexion cm./20 sec.	$v.$ volts.	Deflexion cm./20 sec.
0	1.4	400	1.35
10	1.35	600	1.3
40	1.3	1000	1.4
200	1.35	2000	1.35

The currents are thus practically constant through a wide range of the potential differences up to 2000 volts. Next, the same film was applied to the outermost gauze V_3 , and the currents disappeared.

According to Lenard's * law of absorption-density, we can assume that the absorption coefficient of celluloid (density 1.35 gr./cm.³) for cathode rays is nearly equal to that of paper (density 1.30 gr./cm.³), which is 2690 cm.⁻¹ * for β rays with a speed of 30,000 volts. Then, by the law of inverse fourth power of velocity †, the coefficient of celluloid for cathode rays with a speed of 10,000 volts (which is the average velocity of the primary cathode rays used in my experiments) is 2690×3^4 cm.⁻¹. Hence the intensity I_0 of the cathode rays of 10,000 volts is reduced to the value I on emergence from the celluloid film, where

$$I = I_0 e^{-\lambda s} = I_0 e^{-33}.$$

Therefore, with this film we can practically stop all the cathode rays with the velocity of 10,000 volts. The film, however, can be easily penetrated by Röntgen rays, even by those which have a hardness corresponding to 500 volts, as the experiment of E. Laird ‡ showed. Hence the residual currents cannot be attributed to the secondary cathode rays produced by the primary rays or the scattering of the primary rays themselves; but they are most probably due

* Lenard, *Wied. Ann.* lvi. p. 255 (1895).

† J. J. Thomson, 'Conduction of Electricity through Gases,' p. 379, 1906.

‡ E. Laird, *Ann. d. Phys.* xlv. p. 605 (1915).

to the β rays produced from the gauzes by the Röntgen rays.

Consequently, the currents due to the secondary cathode rays only can be obtained by subtracting the residual currents from the observed currents. The results so corrected are given in Table III. for air, the mean value of the currents over the range $v=1100$ to 1990 volts being taken as the residual current for each case. The current for each value of v indicates the relative number of the secondary corpuscles whose energies are greater than the v volts concerned, the total number of the corpuscles being taken as 100.

TABLE III.—Air.

Velocity of Primary Rays. volts.	14,500 volts.	12,600 volts.	11,200 volts.	10,500 volts.	8100 volts.	7000 volts.	Mean.
$v=0$	100.0	100.0	100.0	100.0	100.0	100.0	100.0
0.3	94.5	95.1	96.2	97.4	98.4	96.3	96.3
0.5	89.0	91.2	90.4	93.6	91.2	92.6	90.9
0.7	76.9	82.7	84.5	87.0	83.6	82.6	82.9
1.0	74.3	74.6	73.6	75.8	75.6	74.4	74.7
1.5	60.4	62.1	64.1	62.5	62.1	62.7	62.3
2.0	56.2	57.6	58.5	56.4	59.2	56.9	57.5
2.5	48.1	51.9	51.1	51.1	50.0	52.1	50.7
3	45.2	48.2	46.6	47.1	47.1	49.1	47.2
4	40.0	41.0	40.3	39.3	41.0	44.1	41.0
5	35.4	36.4	37.1	37.0	35.7	38.0	36.6
6	30.4	33.9	32.8	32.6	33.4	32.8	32.7
7	28.5	30.4	31.0	29.7	30.8	30.0	30.1
8	25.3	27.3	27.1	27.1	27.9	25.8	26.8
10	22.3	23.2	22.5	23.8	22.4	22.0	22.7
14	17.9	18.1	18.7	19.7	18.3	18.9	18.6
20	13.1	12.6	12.4	14.1	13.2	13.0	13.1
30	9.66	10.38	9.88	10.90	10.65	10.65	10.38
40	8.02	8.16	8.31	8.44	7.24	8.75	8.32
70	5.44	5.25	4.97	5.33	4.89	5.59	5.25
110	3.20	3.35	3.64	3.72	3.99	3.71	3.60
150	2.62	2.54	3.25	2.58	3.16	2.60	2.79
190	2.24	2.54	2.61	2.11	2.46	1.52	2.25
270	2.13	2.17	1.88	1.49	1.67	1.37	1.78
390	1.57	1.24	1.45	1.18	1.09	1.13	1.28
590	0.955	1.02	1.03	0.62	0.81	0.60	0.839
790	0.494	0.251	0.411	0.214	0.319	0.253	0.324
990	0.133	0.044	0.151	0.032	0.117	0.147	0.104
1190	0	0	0	0	0	0	0
1390	0	0	0	0	0	0	0
1590	0	0	0	0	0	0	0
1990	0	0	0	0	0	0	0

For the case of hydrogen, the same examination was tried, and the same correction was made as in the case of air. The corrected results are given in Table IV.

TABLE IV.—Hydrogen.

Velocity of Primary Rays. } volts.	15,500 volts.	11,400 volts.	7500 volts.	Mean.
$v=0$	100.0	100.0	100.0	100.0
0.3	94.8	96.0	96.8	95.9
0.5	83.4	84.2	84.8	84.1
0.7	74.5	77.7	75.8	76.0
1.0	63.5	64.3	64.4	61.1
1.5	55.5	53.5	56.1	55.0
2.0	48.9	48.8	50.0	49.2
2.5	45.1	44.7	43.4	44.4
3	37.1	41.1	40.3	39.5
4	35.1	35.9	36.4	35.8
5	31.0	32.1	33.4	32.2
6	29.5	28.3	31.6	29.4
7	26.7	26.0	28.3	27.0
8	24.9	24.1	25.4	24.8
10	20.1	21.4	20.3	20.6
14	16.5	16.1	16.7	16.5
20	11.7	12.7	13.8	12.7
30	10.41	9.90	9.65	9.99
40	7.96	7.70	7.55	7.74
70	4.59	4.44	4.53	4.52
110	3.08	3.07	3.22	3.12
150	2.60	2.29	2.28	2.38
190	1.93	1.92	1.74	1.87
230	1.73	1.58	1.59	1.63
270	1.45	1.40	1.37	1.41
310	1.30	1.32	1.12	1.25
390	1.13	0.985	0.919	1.01
490	0.935	0.835	0.783	0.84
590	0.785	0.656	0.575	0.67
690	0.515	0.645	0.355	0.505
790	0.354	0.275	0.251	0.293
990	0.193	0.116	0.042	0.120
1190	0	0	0	0
1390	0	0	0	0
1590	0	0	0	0
1990	0	0	0	0

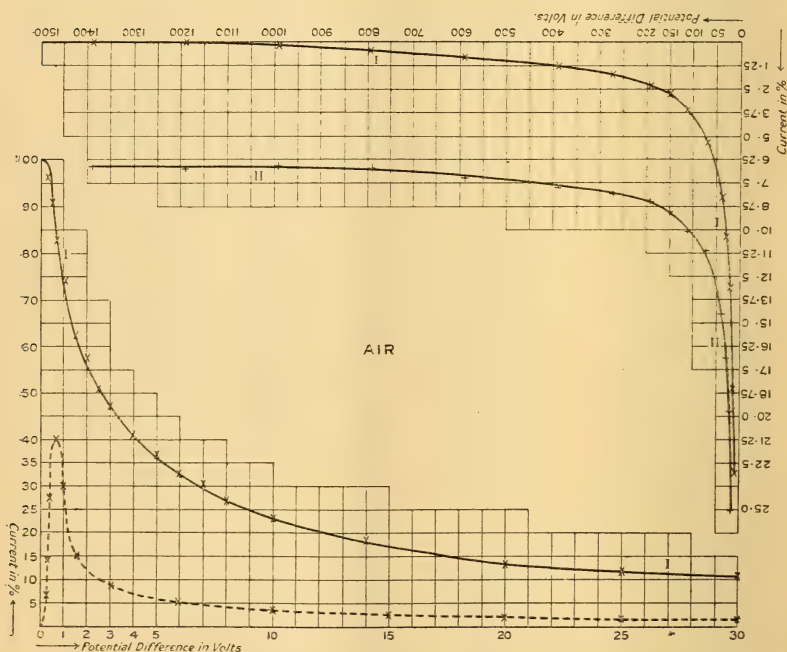
§ 10. General Conclusion.

As is seen from Table III., the currents for each value of v are nearly equal. Hence we can conclude that the distribution of the velocities in the secondary cathode rays from

the atoms in air is nearly independent of the energy of the primary rays. This result agrees with those obtained for δ rays from a metallic plate, and with that obtained for the secondary rays from gas.

The mean values of the corrected currents for each value of v were calculated. These are given in the last column in Table III. The mean values may be taken as giving the most probably true distribution of the velocities. These are

Fig. 5.



plotted in curve I., fig. 5. It is clearly seen that corpuscles, whose velocities are very large up to 1000 volts, exist in the secondary rays. This result does not agree with that obtained by Sir J. J. Thomson in the case of slow primary cathode rays (§ 1). It is noticeable that the number of the secondary corpuscles whose energies are greater than 40 volts is only about 8.3 per cent. of the total number of the secondary corpuscles. The fact that a small fraction of the secondary cathode rays have large velocities agrees with Bumstead's result obtained in the case of α rays.

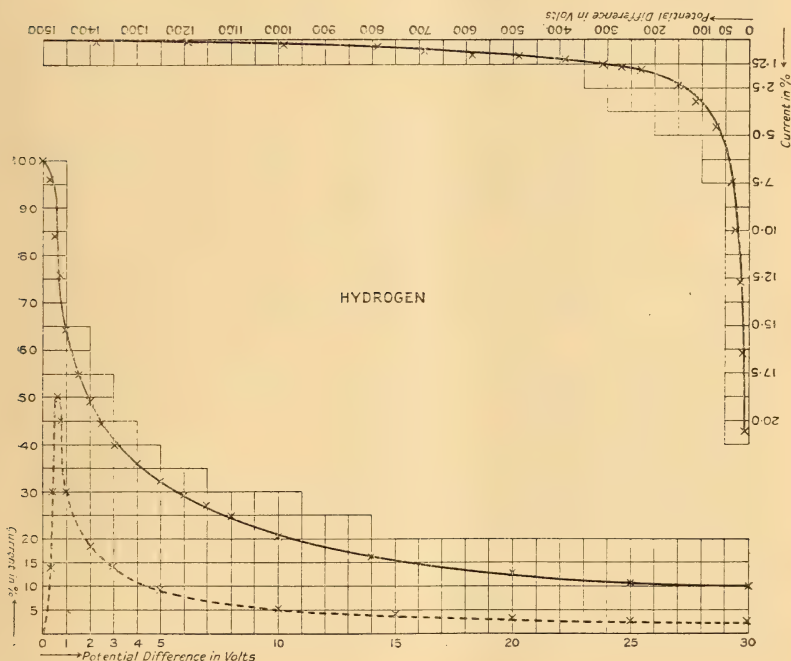
The curve I., fig. 5, resembles a hyperbolic curve over the

range $v=40$ to $v=700$ volts. Assuming an equation $y=Cx^{-a}$, calculation by the method of least square gives

$$y = 188 \cdot v^{-0.84}.$$

The Table IV. for hydrogen leads us to the same conclusions as in the case of air. The mean values in the last column in this table are plotted in fig. 6.

Fig. 6.



The mean values are very close to the corresponding values for air. Calculation gives an equation :

$$y = 224 \cdot x^{-0.91},$$

over the range $v=40$ to 700 volts. The difference between the two exponents of x in the equations for air and hydrogen, while not negligible, is not large. These results seem to support strongly the fact that the distribution of velocities with which these secondary rays are emitted does not vary much with the nature of the atoms emitting them. This property also agrees with that of δ rays from metals (§ 1).

For potentials lower than 30 volts, currents calculated from the above equations do not agree with those obtained

by experiment, the observed values being much greater than the calculated values. This is explained by supposing that there are some tertiary rays with slow velocities.

The relative number of the secondary rays which are emitted with any given velocity can be obtained by finding the differential coefficient of the curves of distribution at that corresponding point. The differential curves are plotted in the figures 5 and 6 in dotted lines. The maximum number of the secondary rays is found at $v=0.7$ volt for air, and at $v=0.6$ volt for hydrogen. These values cannot be determined very accurately, because the currents at small values of v may be the sum of the secondary and the tertiary rays.

According to Sir J. J. Thomson's * theory of ionization, the kinetic energy Q communicated to a corpuscle of mass m_1 at rest in an atom, by a particle of mass m_2 moving with velocity V , is given by the equation :

$$Q = \frac{\frac{4m_1m_2}{(m_1+m_2)^2}T}{1 + \frac{4d^2}{e^2E^2}T^2\left(\frac{m_1}{m_1+m_2}\right)^2},$$

where d is the perpendicular distance from the corpuscle m_1 to the initial direction of the moving particle m_2 , T is the kinetic energy possessed by the moving particle, and E and e are the charges on the particles m_2 and m_1 respectively. For the case in which the moving particles are the cathode rays, $m_1=m_2$ and $E=e$, and also the equation is simplified as follows :

$$Q = \frac{T}{1 + \frac{d^2}{e^4}T^2}.$$

The theory assumes that (1) the force which keeps the corpuscle to the atom is negligible compared with the action of the moving corpuscles, and (2) if the energy communicated to a corpuscle in an atom exceeds a certain value, which may be a characteristic for the atom in question, the corpuscle is liberated from the atom. Suppose the energy necessary to ionize the atom is q_0 , then the ionization will occur for the values of d less than d_0 given by the relation

$$d^2 \leq d_0^2 = \left(\frac{T}{q_0} - 1\right) \frac{e^4}{T^2}.$$

* J. J. Thomson, *Phil. Mag.* xxiii. p. 449 (1912).

Hence, if the moving corpuscle passes the stationary corpuscle at a distance d_1 , which satisfies an equation

$$d^2 = \left(\frac{T}{q_0 + q} - 1 \right) \frac{e^4}{T^2},$$

the stationary corpuscle will be ejected with a velocity corresponding to the energy q . The number of the corpuscles ejected per centimetre of the paths of the moving corpuscles will be proportional to d^2 . Therefore the ratio of the ejected corpuscles, whose kinetic energy is not less than q , to the total number of those ejected at the same time, is given by the equation

$$\frac{d_1^2}{d_0^2} = \frac{T - q - q_0}{T - q_0} \cdot \frac{q_0}{q + q_0}.$$

Now, if T is very large compared with q_0 and q , this ratio is approximately equal to $q_0/q + q_0$. Again, if q is large compared with q_0 , this ratio becomes $q_0/q = Cq^{-1}$.

The corrected currents in the Tables II. and IV. are the quantities corresponding to q . It is seen that the values of " a " in the equations representing the experimental results are nearly unity. The agreement between theory and experiment is thus fairly good.

Summary.

The distribution of the velocities of the secondary cathode rays emitted from air and hydrogen under the action of high-speed cathode rays over the range 7000 volts to 15,000 volts has been studied.

(I.) It was found that the corpuscles were emitted with velocities varying from 0 up to 1000 volts, but the great majority of the corpuscles—about 90 per cent. of them—had velocities less than 40 volts.

(II.) The distribution of the velocities of the secondary cathode rays appeared to be nearly independent of the velocity of the primary cathode rays, and also approximately the same for air and hydrogen.

(III.) It was found that the distribution curve of velocity can be expressed with an empirical formula $y = Cx^{-a}$ over a range of $x = 40$ to $x = 700$ volts, where y indicates the relative number of the secondary cathode rays emitted with velocities greater than x volts, and a is a constant differing slightly from unity. This is in accordance with the theory of ionization given by Sir J. J. Thomson.

(IV.) Some similarity between the secondary cathode rays from a gas and from a metal was pointed out.

(V.) The effect of the positive ions produced at the same time as the secondary corpuscles was examined, and the potential difference to stop them determined. The effects due to Röntgen rays produced from gaseous molecules by the primary rays were also investigated.

In conclusion, I desire to express my best thanks to Sir J. J. Thomson for his many suggestions during the course of the investigation, and also for his kind permission to carry out this research at the Cavendish Laboratory.

The Cavendish Laboratory,
University of Cambridge.

XXII. *The Straggling of α Particles.*

By W. MAKOWER, M.A., D.Sc.*

THE variation in the number of α particles from radium C and polonium near the end of their ranges in air was first studied by Geiger †. Later, Taylor ‡ repeated the observations with radium C and extended the investigation to the cases of hydrogen and helium. The method adopted by both investigators was to count the number of scintillations on a zinc sulphide screen kept at a fixed distance from the radioactive source and to vary the pressure of the gas through which the α rays passed. The results obtained by Geiger and Taylor for the variation in the number of α particles from radium C in air are in substantial agreement, but the straggling thus indicated near the end of the range is considerably larger than is predicted by theory §. It seemed, therefore, desirable to test the point further by an independent experimental method to see whether there is any defect inherent in the scintillation method giving rise to the discrepancy between theory and experiment.

The method adopted in this investigation consisted essentially in substituting for the zinc-sulphide screen a photographic plate which, after development, could be examined under a microscope. The density of the silver grains in the film gives a record of the number of α particles which fell upon the plate ||. It is therefore only necessary to count the number of grains visible in a definite area in the field

* Communicated by the Author.

† Geiger, Proc. Roy. Soc. A, vol. lxxxiii. p. 505 (1910).

‡ Taylor, Phil. Mag. Sept. 1913.

§ Flamm, *Sitzungsber. d. K. Akad. d. Wiss. Wien, Math.-nat. Kl.* cxxiii. II. A (1914); Bohr, Phil. Mag. Oct. 1915.

|| Kinoshita, Proc. Roy. Soc. A, vol. lxxxiii. p. 432 (1910).

of view of the microscope to determine the number of α particles which impinged on that area.

The main difficulty of the method consists in obtaining plates capable of development without showing under the microscope a number of blackened grains, even when the plate has not been exposed to light or any radioactive source; but it was found that this difficulty could be overcome by using Schumann plates, which possess the additional advantage of having such thin sensitive films that an α particle striking normally or at a small angle to the normal usually meets only a single silver grain in its passage through the film. Moreover, when viewed even under a high-power microscope, all the silver grains, being nearly in one plane, are simultaneously in focus, a circumstance which greatly facilitates counting. To obtain accurate results the grains affected by the α particles must not be too close together, and it is important not to over-develop the plates, for if the silver grains become too large, several neighbouring grains may be fused into a single large particle, and if this takes place the counting becomes inaccurate.

Fig. 1 (a).

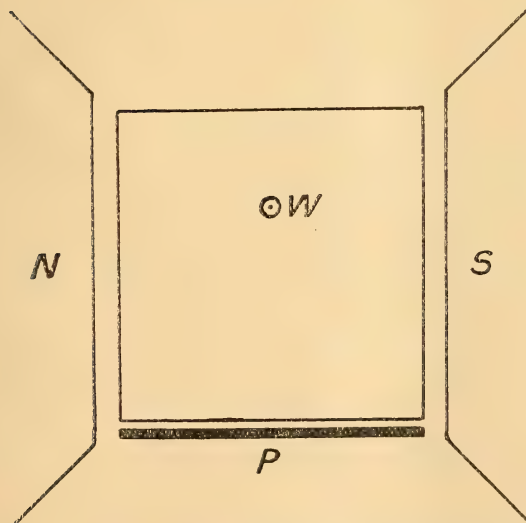
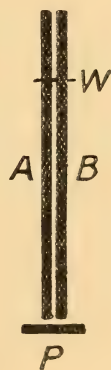


Fig. 1 (b).

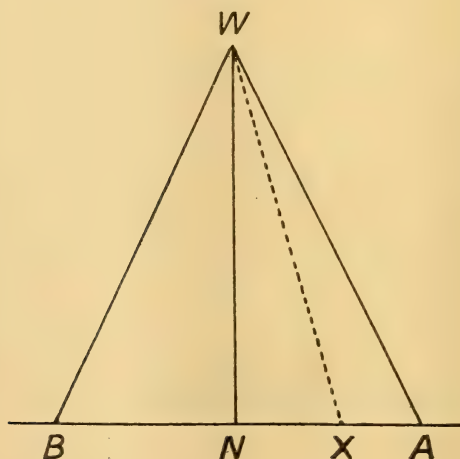


The experimental arrangement used is shown in fig. 1 (a) and (b), drawn in plan and section. The thin platinum wire *W*, rendered active by exposure for several hours to radium emanation and freed from adhering emanation in the usual way by heating, was passed through holes bored in the two

thick brass plates, A and B, which were parallel and fixed slightly less than a millimetre apart. The wire was held in position by wax. The apparatus was placed in a magnetic field sufficiently intense to deflect β rays, but too weak to deflect α rays appreciably. Thus only α rays could reach the Schumann plate P, which was 8.5 cm. long and 1.2 cm. wide and situated 6.2 cm. below the wire. After a suitable exposure the plate was removed and developed, and mounted on a movable microscope stage in such a way that it could be displaced parallel to its length. Cross-wires, suitably placed in the eyepiece of the microscope, defined a definite area within which the number of silver grains could be counted, and by altering the position of the plate the number of α particles reaching different sections of the film could be compared.

In order to deduce from such observations the variation in the number of α particles with the distance from the screen, it is necessary to apply certain corrections. For consider the active wire W (fig. 2) at a distance $WN = 6.2$ cm. from the photographic plate. The whole photographic effect must be confined between the points A and B,

Fig. 2.

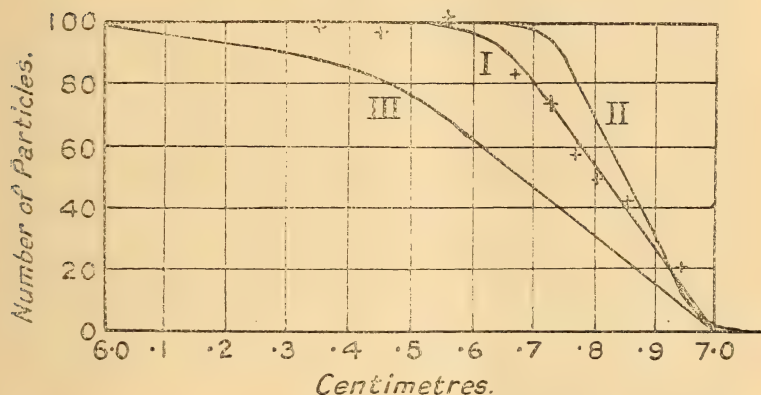


where $WA = WB$ = the range in air of the α particles from radium C. But quite apart from any stoppage of α particles by air, the density of the silver grains on the photographic plate at any point X will diminish as that point recedes from N, both because WX continually increases and because the obliquity of the rays to the photographic plate changes.

It is easy to see that on account of these two causes the density of the silver grains will be inversely proportional to WX^3 . If, therefore, N be the number of grains observed at any point X , the variation of the product $N(WX)^3$ gives the required falling off in the number of α particles in their passage through the air intervening between the wire and photographic film.

The results of a series of experiments is shown in fig. 3, curve I., which has been arbitrarily drawn to give an extreme

Fig. 3.



range of 7 cms. The absolute measurements of the distances involved have not been made with sufficient accuracy to improve upon the accepted value of the range for radium C. Each point on the curve represents the counting of a large number of silver grains, varying from about 7000 for the points at the maximum to some 400 near the end of the range. The number of grains outside the α -ray photograph which amounted to about 3 per cent. of the maximum intensity was subtracted from each observation.

In the same figure (curve II.) is given the theoretical curve calculated from Bohr's formula

$$W(s)ds = \frac{1}{\rho \sqrt{\pi}} e^{-\left(\frac{s}{\rho}\right)^2} ds,$$

which gives the probability that the range R has a value between $R_0(1+s)$ and $R_0(1+s+ds)$. The value of ρ has been taken equal to 1.6×10^{-2} . Curve III. represents the experimental results of Geiger. Those obtained by Taylor have not been plotted, for he states that his results obtained for air agree with those of Geiger, though it should be stated

that the curves given by Taylor appear to be slightly steeper.

It will be seen that the results of the investigation approach considerably more closely to the requirements of theory than those previously obtained, but, as might be expected, the agreement between theory and experiment is not perfect. The facts that the photographic film, though thin, is still of finite thickness, and that there may be imperfections of the surface emitting the radiation, lead to the anticipation of straggling greater than is demanded by theory. Moreover, as the end of the range is approached the silver grains become feebler and feebler and consequently more difficult to count, and it seems not unlikely that the same difficulty in the case of the scintillation method may lead to even more serious errors. For, whereas a photographic film can be studied again and again at leisure, scintillations have to be counted at the moment of impact of the α particles. If, when giving only feeble scintillations near the end of their range, some of these escape detection, erroneous results will be obtained. But, however this may be, it will be seen that the photographic method gives a more rapid falling off in the number of α particles at the end of their range than was previously indicated by the scintillation method and tends to confirm the law given by Bohr's theory.

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XXIII. *The Efficiency of Recoil of Radium D from Radium C.* By W. MAKOWER, M.A., D.Sc.*

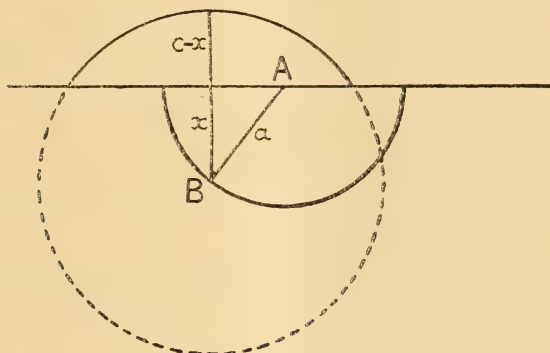
THERE is evidence that when radium A is deposited on a platinum surface half of the radium B formed is shot into the metal so that the radium C produced from it is situated at varying depths below the surface. The recoil stream of radium D subsequently emerging traverses different thicknesses of platinum, and therefore escapes with all possible velocities †. On this account the efficiency of recoil of radium D from radium C should be low, and it is of interest to determine its value.

It has been pointed out to me by Dr. N. Bohr that the problem is a very simple one, for if the recoil-streams have

* Communicated by the Author.

† Wood and Makower, *Phil. Mag.* Dec. 1915.

definite ranges *, and if radium A is distributed uniformly on the surface of a flat plate, the radium B and therefore the radium C subsequently formed will be uniformly distributed through a depth a equal to the range of the recoil-stream from radium A in the plate. For, consider atoms of radium A on the surface of the plate at A (see figure). The



radium B produced will be shot downwards into the plate and distributed uniformly over the surface of a sphere of radius a . The fraction confined within the surface of a segment of the sphere of thickness δx at a depth x below the surface is $\frac{\delta x}{a}$, since this is the ratio of the area of the

segment to that of the hemisphere. Radium C is now produced *in situ* from the radium B, and if c is the range of the recoil-stream from radium C, the fraction of atoms coming from the segment considered which emerges from the plate

can be seen to be $\frac{c-x}{2c}$; for this represents the ratio of the area of the cap of the sphere of radius c outside the plate to the total area of the sphere. Consequently, the probability that an atom of radium B will be projected from the surface of the plate to a depth x and then give rise to an atom of radium D which escapes from the plate will be $\frac{c-x}{2c} \cdot \frac{\delta x}{a}$.

The efficiency of the recoil, as usually defined, will have half this value, since half of the atoms of radium D must of necessity be projected away from the surface of the plate.

* Wertenstein, Thèses présentées à la Faculté des Sciences, Paris, 1913.

Hence the total efficiency E of the recoil is given by

$$E = \int_0^a \frac{c-x}{c} \cdot \frac{dx}{a} = 1 - \frac{1}{2} \frac{a}{c}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

provided that a is less than c , an assumption which is certainly correct, since the range of the α particles from radium A is less than that of the α particles from radium C.

When $a=0$, $E=1$, and when $a=c$, $E=\frac{1}{2}$, below which value the efficiency cannot fall unless the surface is rough or contaminated with grease.

The determinations of the efficiency of the recoil of radium D were carried out as follows:—A clean platinum plate was exposed for a short time as a cathode in an electric field to a large quantity of radium emanation and then removed without passing through mercury or in any way contaminating the surface. The platinum was mounted with wax on a piece of mica with a hole in the middle, and after sufficient time had elapsed for the radium A to decay, the mica was placed on a piece of aluminium foil. In this way the active platinum surface was brought near the aluminium without touching it. A potential difference of 100 volts was maintained between the platinum and the aluminium which was the negative pole. The electric field was maintained for several hours, during which the radium C decayed. Some of the radium D thus produced remained on the platinum, whereas some reached the aluminium by recoil. By measuring after some months with an α -ray electroscope the amounts of polonium on each plate, the efficiency of recoil could be deduced.

Two separate experiments were made, in both of which the platinum plates were exposed to the emanation for 10 minutes. The recoil was started in one case after another 25 minutes, and in the other after 28 minutes. It is easy to calculate the fraction δ of the maximum quantity of radium D found before the recoil began under the special conditions of the experiments. If P is the quantity of polonium subsequently detected on the platinum plate, and A the amount on the aluminium plate, the efficiency is given by

$$E = \frac{2A}{(P+A)(1-\delta)} \cdot \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Two sets of measurements of the activities of the plates were made at an interval of three months and the results

were found to agree well with each other. They are given below :—

Date of Exposure to Emanation, 21st Sept., 1915.									
Date of testing.	Plate 1.				Plate 2.				Mean E.
	P in div./min.	A in div./min.	δ .	E.	P in div./min.	A in div./min.	δ .	E.	
31.1.16	2.18	.83	.161	.657	1.93	.78	.186	.707	.687
1.5.16	3.54	1.46	.161	.696	3.37	1.315	.186	.690	

Inserting the value $E = .687$ in equation (1) we obtain

$$\frac{a}{c} = .626.$$

The ratio of the ranges of the α particles from radium A and radium C in platinum seems not to have been determined, but the value of this ratio in gold can be deduced from the experiments of Marsden and Richardson to be .72 * ; and since the atomic weights of gold and platinum are nearly equal, this number must apply with considerable accuracy to the case of platinum. There is therefore more difference between the ranges of the recoil-streams from radium A and radium C than of the corresponding α particles, and there can be no doubt that this difference is real, for experimental errors such as contamination of the platinum surfaces would tend to reduce the efficiency of recoil and so increase the value of the observed ratio a/c .

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XXIV. *A Method of Deriving Planck's Law of Radiation.*

By GEORGE GREEN, D.Sc., *Lecturer on Natural Philosophy in the University of Glasgow* †.

THE following method of deriving Planck's law was suggested by Lord Rayleigh's paper on "The Dynamical Theory of Gases and of Radiation" ‡. We take the radiating body to be a gas contained in a perfectly reflecting enclosure in thermal equilibrium at temperature θ ; and the investigation is based on the following assumptions :—

* Marsden and Richardson, Phil. Mag. Jan. 1913.

† Communicated by Professor A. Gray, F.R.S.

‡ Scientific Papers, vol. v. p. 248.

Let the gas consist of molecules having internal modes of vibration which are identical with the modes of vibration of unit volume of æther.

Let no molecule emit radiation in any mode until the total energy in that mode is hf ; h being a constant, and f the frequency of the mode. When the limiting energy is reached let the whole quantity hf or any portion of it be emitted at a discharge.

Let Maxwell's law of distribution of velocities hold throughout the gas. This implies an equilibrium condition within the enclosure maintained by collisions, statistically regular absorption, and sudden emission in each mode when the limiting energy is reached.

Consider now the energy distribution amongst the N molecules per unit volume in a single vibrational mode. If we adopt the view that each internal mode corresponds to the motion of an electron in an orbit whose plane is fixed relative to each molecule, the velocities to be considered in estimating the kinetic energy of a molecule in a given mode are clearly velocities *relative* to the centroid of each molecule. Hence, in assigning for any mode the number of vibrators having a given velocity, we have only to consider all phases in a plane orbit as equally possible—and we are not concerned with the directions of the velocities in space or with the orientation of the orbital planes. On the above view, or on any other which allows us to deal with the velocities in any internal mode as a two-dimensional system, we have according to Maxwell's law

$\left(A'e^{-\frac{3mv^2}{k\theta}} v dv \right)$ for any assigned vibrational mode,

Number of molecules having energy in range $(\epsilon - d\epsilon, \epsilon)$

$$= A e^{-\frac{\epsilon}{k\theta}} d\epsilon, \quad . \quad . \quad (1)$$

where A depends on the number of molecules and on the temperature of the enclosure, and $k\theta$ is the mean total energy of a molecule in any mode, assuming all velocities from zero to infinity to be possible in that mode. In the above equation, since in each mode the potential energy is on the average equal to the kinetic, we may take ϵ to represent total energy, instead of kinetic energy alone. This involves only an alteration in the value of A , which we now determine to suit the case where ϵ represents total energy of a molecule in a given mode. Integration of (1) gives

Number of molecules in energy range $(0, \epsilon)$

$$= Ak\theta \left(1 - e^{-\frac{\epsilon}{k\theta}} \right). \quad . \quad . \quad (2)$$

Hence for the mode whose limiting energy is hf we have

$$A = \frac{N}{k\theta \left(1 - e^{-\frac{hf}{k\theta}}\right)}, \quad \dots \quad (3)$$

the N molecules per unit volume being all in energy range $(0, hf)$.

From (3) and (1), for the total energy of the N molecules in unit volume in mode f we easily obtain the value

$$Nk\theta - \frac{Ne^{-\frac{hf}{k\theta}} hf}{\left(1 - e^{-\frac{hf}{k\theta}}\right)}. \quad \dots \quad (4)$$

But the "law of equipartition of energy" (assumed applicable to the combined system of matter and æther in unit volume, the free modes of each being identical,—and not to matter or æther separately) requires that the total energy per unit volume in each mode be independent of f . Accordingly the radiant energy contained in unit volume of æther must be

$$\frac{Ne^{-\frac{hf}{k\theta}} hf}{\left(1 - e^{-\frac{hf}{k\theta}}\right)}. \quad \dots \quad (5)$$

No additional terms involving θ alone can enter, in accordance with Wien's law. The same point is clear from the consideration that the total kinetic energy in any mode must equal one-third of the kinetic energy of the translational motion of the centroids of the N molecules, $\frac{1}{2}(Nk\theta)$, as stated in Lord Rayleigh's discussion. Hence the mean energy per molecule contained in mode f in unit volume of æther is

$$\frac{hf}{\left(e^{\frac{hf}{k\theta}} - 1\right)}, \quad \dots \quad (6)$$

and the number of modes of vibration in the frequency range df , as given by Lord Rayleigh's paper referred to above with the correction indicated by Professor Jeans, is $8\pi f^2 df/V^3$, where V is the velocity of light in æther. We have therefore as the total energy in æther in the frequency range df

$$\frac{8\pi}{V^3} f^2 df \cdot \frac{hf}{\left(e^{\frac{hf}{k\theta}} - 1\right)}; \quad \dots \quad (7)$$

or, in terms of the wave-length λ ,

$$\frac{8\pi Vh}{\lambda^5} \frac{d\lambda}{\left(e^{\frac{hf}{k\theta}} - 1\right)}, \quad \dots \dots (8)$$

in agreement with Planck's law.

In the second assumption above the essential requirement is the existence of a limiting energy, hf , in each mode. For this it is not necessary that emission of energy should take place suddenly when the limiting energy is reached. Emission and absorption may be continuous processes, and the attainment of the limiting energy condition may involve some other action than that postulated, such as the disruption of the molecular system with possibly very little transference of energy to the æther.

XXV. *Bessel Functions of Equal Order and Argument.* By
G. N. WATSON, M.A., Assistant Professor of Pure
Mathematics at University College, London.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

MY attention has been directed to a paper under the title “Bessel and Neumann Functions of Equal Order and Argument” by Dr. J. R. Airey (Phil. Mag. June 1916, pp. 520–528). The ultimate object of that paper is to obtain a number of numerical results concerning Bessel functions; with these results I am not at present concerned. But in the theoretical work which Dr. Airey gives at the beginning of his paper, on which his numerical results are based, there are a number of statements for which more complete investigation appears to be necessary. Some of these statements contain errors which seem to me to be serious from the theoretical point of view, while others are merely liable to be misinterpreted by the casual reader.

Before enumerating the statements to which exception may be taken, I wish to emphasize the fact that these errors do not, so far as I can see, affect the numerical results contained in the paper.

(I.) Graf's approximation for $J_n(n)$ is incorrectly given (p. 520) as

$$n^{\frac{1}{2}} J_n(n) = \frac{\Gamma(\frac{1}{3})}{2\pi\sqrt{3}};$$

the correct formula is

$$n^{\frac{1}{2}} J_n(n) \sim \frac{\Gamma(\frac{1}{3})6^{\frac{1}{3}}}{2\pi\sqrt{3}},$$

but this is obviously an oversight.

(II.) Dr. Airey's formula (2) seems to me to be misleading. In it he equates the Bessel function $J_n(z)$ to an asymptotic expansion which is not that of the Bessel function but is the asymptotic expansion of Airy's integral; it is well known that this integral is only an approximation to $J_n(z)$, and, in fact, the asymptotic expansions of $J_n(z)$ and of Airy's integral agree in their first two terms. This fact appears to be implied by Dr. Airey's statement: "The third and following terms are correctly given in (14)"; I think that a rather more definite statement would have been desirable, but this is a comparatively trivial matter.

(III.) In finding the asymptotic expansion of $J_n(n)$, Dr. Airey uses Bessel's integral, and proceeds thus*:

$$\begin{aligned} J_n(n) &= \frac{1}{\pi} \int_0^\pi \cos \{n(\sin w - w)\} dw \\ &= \frac{1}{\pi} \int_0^\infty \cos \left\{ n \left(\frac{w^3}{6} - \frac{w^5}{120} + \frac{w^7}{5040} - \dots \right) \right\} dw, \end{aligned}$$

and gives a reference to Lord Rayleigh's work (Phil. Mag. Dec. 1910). Now the last expression is really

$$\frac{1}{\pi} \int_0^\infty \cos \{n(w - \sin w)\} dw,$$

and this last integral is not even oscillatory but is quite divergent; at any rate when n is an integer, as is supposed to be the case. For, if we take k to be a (large) integer, we have

$$\frac{1}{\pi} \int_0^{2k\pi} \cos \{n(w - \sin w)\} dw = \frac{k}{\pi} \int_{-\pi}^\pi \cos \{n(w - \sin w)\} dw,$$

because $\cos \{n(w - \sin w)\}$ is a periodic function of w with period 2π ; since, in addition to being periodic, the function under consideration is an even function, we have

$$\int_{-\pi}^\pi \cos \{n(w - \sin w)\} dw = 2 \int_0^\pi \cos \{n(w - \sin w)\} dw,$$

and so

$$\begin{aligned} \frac{1}{\pi} \int_0^{2k\pi} \cos \{n(w - \sin w)\} dw &= \frac{2k}{\pi} \int_0^\pi \cos \{n(w - \sin w)\} dw \\ &= 2kJ_n(n), \end{aligned}$$

and this obviously tends to infinity with k .

* It is implied that the infinite integral is an approximation to $J_n(n)$. The fact that the *exact* expansion (11) is derived from this result which purports to be merely *approximate* indicated to me the desirability of making this investigation.

Since we have just proved that

$$\frac{1}{\pi} \int_0^{2k\pi} \cos \{n(w - \sin w)\} dw = 2kJ_n(n),$$

it follows that in no sense can

$$\frac{1}{\pi} \int_0^\infty \cos \{n(w - \sin w)\} dw$$

be said to be an approximation to $J_n(n)$.

The fact is that Dr. Airey has made an error of the same nature as one which occurs in the course of Lord Rayleigh's work; but, as Lord Rayleigh's error appeared in only a single line of his work, his final result was not vitiated.

Lord Rayleigh's analysis was as follows:—

“If $z = n$ absolutely, we may write ultimately

$$\begin{aligned} J_n(n) &= \frac{1}{\pi} \int_0^\pi \cos \{n(w - \sin w)\} dw \\ &= \frac{1}{\pi} \int_0^\infty \cos \{n(w - \sin w)\} dw \\ &= \frac{1}{\pi} \int_0^\infty \cos \frac{nw^3}{6} dw = \frac{1}{\pi} \left(\frac{6}{n}\right)^{\frac{1}{3}} \int_0^\infty \cos \alpha^3 d\alpha \\ &= \Gamma\left(\frac{1}{3}\right) 2^{-\frac{2}{3}} 3^{-\frac{1}{3}} \pi^{-1} n^{-\frac{1}{3}}. ” \end{aligned}$$

[Of course, at the present time, most mathematicians would employ a notation which has been introduced in the last few years and would replace the symbols $=$ in the second and third lines by \sim .]

For the reasons explained above, $\int_0^\infty \cos \{n(w - \sin w)\} dw$ is not an approximation to $\int_0^\pi \cos \{n(w - \sin w)\} dw$. But, in the range $0 \leq w \leq \pi$, it is permissible, for purposes of approximation, to replace $w - \sin w$ by $\frac{1}{6}w^3$; though it must be stated that it is somewhat difficult to give a completely rigorous proof of this. And, since $\int_0^\infty \cos(nw^3/6) dw$ is convergent, it is easy to prove that it is an approximation to $\int_0^\pi \cos(nw^3/6) dw$ when n is large. Consequently, Lord Rayleigh's work may be made strictly accurate by replacing

the expression (which occurs in the second line of the analysis quoted above)

$$\frac{1}{\pi} \int_0^{\infty} \cos \{n(w - \sin w)\} dw$$

by

$$\frac{1}{\pi} \int_0^{\pi} \cos (nw^3/6) dw.$$

To return to Dr. Airey's paper, his formula (9), which is obtained by taking a new variable $x \equiv n(w - \sin w)$ in Bessel's integral, becomes an *exact* result if the ∞ in the upper limit of the integral is replaced by $n\pi$, so that the formula reads

$$J_n(n) = \frac{1}{\pi} \int_0^{n\pi} \left\{ \frac{1}{3} \left(\frac{6}{n} \right)^{\frac{1}{3}} x^{-\frac{2}{3}} + \frac{1}{60} \left(\frac{6}{n} \right) + \frac{1}{840} \left(\frac{6}{n} \right)^{\frac{2}{3}} x^{\frac{2}{3}} + \dots \right\} \cos x dx.$$

Next Dr. Airey substitutes $\Gamma(p) \cos \frac{1}{2} p\pi$ for $\int_0^{\infty} x^{p-1} \cos x dx$ wherever it occurs, quite disregarding the fact that this integral diverges except when $0 < p < 1$, and that it is only when $0 < p < 1$ that the equation

$$\int_0^{\infty} x^{p-1} \cos x dx = \Gamma(p) \cos \frac{1}{2} p\pi$$

is true.

In point of fact, by integrating by parts, it can be shown that $\int_0^{n\pi} x^{p-1} \cos x dx$ differs from $\Gamma(p) \cos \frac{1}{2} p\pi$ by an expression whose asymptotic expansion, when n is an integer *, is

$$\{(p-1)(n\pi)^{p-2} - (p-1)(p-2)(p-3)(n\pi)^{p-4} + \dots\} \cos n\pi,$$

and so the integral of each term in the expansion given above for $J_n(n)$ differs from the expression which Dr. Airey substitutes for it by a term which may be written

$$\cos n\pi \times O(1/n^2).$$

Now it so happens that the aggregate of these terms cancel one another. But it seems to be impracticable to prove this by employing Bessel's integral alone.

To prove the correctness of Dr. Airey's expansion (11), it

* When n is not an integer, a second expansion of which $\sin n\pi$ is a factor has to be added to this.

seems necessary to make use of Schlömilch's contour integral

$$J_n(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{\frac{1}{2}z(t-1/t)} t^{-n-1} dt,$$

or the derived Bessel-Schlöffli integral

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - z \sin \theta) d\theta - \frac{\sin n\pi}{\pi} \int_0^\infty e^{-n(\theta + z \sinh \theta)} d\theta,$$

from which can be derived the formula

$$J_n(z) = \frac{1}{2\pi} \int e^{ni\theta - iz \sin \theta} d\theta,$$

the integral being taken round a contour consisting of three sides of a rectangle whose corners are

$$-\pi + \infty i, \quad -\pi, \quad \pi, \quad \pi + \infty i.$$

As θ describes this contour, $x \equiv n(\theta - \sin \theta)$ describes a similar contour whose corners are

$$-n\pi + \infty i, \quad -n\pi, \quad n\pi, \quad n\pi + \infty i;$$

to avoid a branch point at the origin, it is convenient to suppose that the θ contour passes above it, and then the x contour encircles it one and a half times.

But a contour of this nature is easily deformable into the contour used by Debye in his investigations of Bessel functions of high order; and the problem of obtaining the range of validity of the expansion seems to be much more difficult with the contour consisting of three straight lines than with Debye's contour.

The fact is that Bessel's integral is extremely ill-adapted for obtaining the asymptotic expression of $J_n(z)$ when n is large; while Debye's beautiful contour integrals, which (like Bessel's integral) are special cases of Schlömilch's contour integral, yield theoretically complete expansions without great difficulty. The real reason for this lies in the fact that the integrands in Debye's integrals are positive and monotonic while the integrand in Bessel's integral is oscillatory.

I might remark that I have obtained the first two terms of the expansions of $J_n(n)$ and $J'_n(n)$ by using Bessel's integral, though it was necessary to use Tannery's theorem* for integrals in the course of the analysis. To obtain the

* Bromwich, 'Infinite Series,' p. 443.

third and following terms, exceedingly intricate algebra would be necessary.

(IV.) The explanation of the fact, mentioned by Dr. Airey on p. 523, that all the formulæ for $J_n(n)$ etc. are true when n is not an integer will be seen from (III.) above; for these formulæ really have to be derived from the Bessel-Schläfli integral, or some such result, which holds whether n is an integer or not.

(V.) It would have added considerably to the interest of Dr. Airey's paper if he had given some indication as to how formula (16) on p. 526 for $J_{-n}(n)$ is obtained without making use of the methods employed by Debye.

I am, Gentlemen,

Yours faithfully,

G. N. WATSON.

Trinity College, Cambridge.

June 17th, 1916.

XXVI. *Bessel Functions of Equal Order and Argument.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

AS Prof. Watson surmises, the object of my paper was to obtain formulæ for the calculation of $J_n(n)$, $Y_n(n)$, etc. for integral values of the order and argument and was not intended in any way as a contribution to the theory of these functions.

Formulæ for $J_n(n)$, $J_{-n}(n)$, $Y_n(n)$, etc. had already been given, and for the purposes of the paper, the calculation of further terms was all that was required. I can corroborate Prof. Watson's statement that intricate algebra is necessary for the computation of the third and following terms. However, Bessel's Integral appeared to afford a simple method of deriving the asymptotic expansion of $J_n(x)$ in the case under consideration. Although divergent integrals were employed and errors of small order introduced thereby, it is interesting to note that, owing to these errors cancelling one another, results were obtained which were in agreement with those derived from the Bessel-Schläfli integral and the contour integrals of Sommerfeld and Schlömilch.

From the Bessel-Schläfli integral, Prof. Nicholson found that, to order $\frac{1}{n}$, $J_n(x)$ can be equated to the asymptotic expansion of Airy's integral as in formula (2). Debye's formulæ for $J_n(x)$ and $J_{-n}(x)$ derived from Sommerfeld's

integral are given in (7) and (16). The following expressions were obtained by Graf from Schlömilch's integral :

When $x < 1$,

$$J_a(\alpha x) \sim \frac{e^{-a(\gamma - \tanh \gamma)}}{\sqrt{2\pi\alpha \tanh \gamma}}, \text{ where } \cosh \gamma = \frac{1}{x}.$$

When $x = 1$,

$$J_a(\alpha) \sim \frac{\Gamma(\frac{1}{3})}{\pi\alpha^{\frac{1}{3}}2^{\frac{2}{3}}3^{\frac{1}{6}}} \sim \frac{\Gamma(\frac{1}{3})}{2\pi\sqrt{3}}\left(\frac{6}{\alpha}\right)^{\frac{1}{3}}.$$

When $x > 1$,

$$J_a(\alpha x) \sim \frac{2 \cos \left[\alpha(\tan \delta - \delta) - \frac{\pi}{4} \right]}{\sqrt{2\pi\alpha \tan \delta}}, \text{ where } \cos \delta = \frac{1}{x}.$$

Graf proceeded to throw the last expression into the form

$$J_a(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left[x - \left(\alpha + \frac{1}{2} \right) \frac{\pi}{2} \right],$$

and found that β_n , the n th root of $J_a(x)$, is represented approximately by

$$\beta_n = \left(2n + \alpha - \frac{1}{2} \right) \frac{\pi}{2},$$

overlooking the simple formula

$$\beta_n = \alpha \sec \phi,$$

where

$$(\tan \phi - \phi) = \frac{(4n-1)\pi}{4\alpha},$$

and which is especially useful in calculating the roots of the functions $J_a(x)$ of high order. This result is given in detail in the *Phil. Mag.* July 1916, where, as in the paper on "The Bessel Functions of equal order and argument," the integral

$$\int_0^\infty \theta^{x-1} \cos \theta d\theta = \Gamma(x) \cos \frac{\pi x}{2}$$

was employed for values of x outside the limits $0 < x < 1$.

Perhaps, in this case also, the correct result may be due to the errors being mutually destructive. In the expression

(14) for $\tan \epsilon$, r is equal to $\frac{1}{n \tan \phi}$, not $n \tan \phi$ as stated in the paper.

I am, Gentlemen,

Yours faithfully,

JOHN R. AIREY.

XXVII. *On the Absorption of Gases in Vacuum-Tubes.**To the Editors of the Philosophical Magazine.*

GENTLEMEN,—

WE shall be much obliged if you will permit us to add a note to our paper on "Absorption of Gases in Vacuum-Tubes, &c.," published in the May issue of the *Philosophical Magazine*. After the publication of our paper we received copies of a paper by Dr. L. Vegard entitled "*Über die elektrische Absorption in Entladungsröhren*," *Videnskaps-selskapets Skrifter*, I. *Mat.-naturv. Klasse*, 1913, No. 7, Kristiania. We regret that we had no access to this paper, as Dr. Vegard to a certain extent anticipated some of our results. The preliminary account of Dr. Vegard's work was published in the *Philosophical Magazine*, xviii. 1909, p. 465. However, owing to the form of the title—"Electric Discharge through HCl, HBr, and HI,"—the possibility that the paper might have any reference to our work escaped our notice.

As affecting the subject of our paper, we should like to mention that Dr. Vegard found that there was no absorption—except for a little which he calls "non-conservative"—with normal cathode-fall, whilst with abnormal fall the rate of absorption increased with increasing fall for nitrogen, oxygen, and helium, but not for hydrogen. Dr. Vegard also suggests that there is some connexion between absorption and disintegration of the cathode. We have much pleasure in acknowledging Dr. Vegard's priority in these results.

Dr. Vegard suggests that the absorbed gas enters into combination—"due to a new kind of combining power called 'electric affinity'"—with the disintegrated particles of the cathode.

Yours faithfully,

Physics Department,
University of Bristol.
May 22nd, 1916.

S. BRODETSKY,
B. HODGSON.

XXVIII. *On the Electric Absorption of Gases in Vacuum-Tubes.**To the Editors of the Philosophical Magazine.*

GENTLEMEN,—

IN the number of the *Phil. Mag.* for May this year Dr. S. Brodetsky and Dr. B. Hodgson have published an interesting paper on the absorption of gases in vacuum-tubes and allied phenomena. In this connexion I should like to call the attention of the authors to some papers

dealing with the same subject, which I have published some years ago.

As these papers seem to have been overlooked by the authors, I should like to state briefly some of my results.

In a paper appearing in the Phil. Mag. for Oct. 1909 I published some experiments on absorption in oxygen and hydrobromic acid, showing that the absorption was mainly a function of the cathode-fall. *As long as the cathode-fall was small the absorption was very small indeed; but if in any way—either by increase of current or diminution of pressure—the cathode-fall was raised the absorption rapidly increased until it reached a maximum of the order of the electrochemical equivalent.*

It was concluded that *the absorption took place at the cathode and was closely connected with the disintegration of the cathode*, and to explain these results I put forward the hypothesis that *the absorption was produced by the rapidly moving positive ions (positive rays) before the cathode.*

Further experiments were carried out at the Cavendish Laboratory in 1909 on the absorption in oxygen, helium, and hydrogen, and the results given in a paper which was put to the care of the Society of Science at Christiania, in 1910. In this paper I introduced a distinction between two kinds of absorption. One quite definite and lasting, which I called the *conservative absorption*, and another which I called *non-conservative*, and which is to designate a number of absorption phenomena which may have various causes and essentially depend on the state of the tube.

When the bulb has been run for some time the *conservative* absorption becomes definite and can for the same tube very nearly be regarded as a single-valued function of current and pressure. This function was studied, and it was found that the absorption per coulomb was essentially dependent on the cathode-fall in the way shown in my first paper, and set in when the cathode-fall was raised above a critical value.

For a cathode-fall C which did not greatly exceed the critical value C_0 and a constant current, the absorption per coulomb was found to be approximately given by the expression

$$q = k(C - C_0). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

I also calculated the number (μ) of atoms absorbed per coulomb.

In the table are given the values $\frac{\mu}{2N}$, where N is the number of molecules in a cubic centimetre of gas at atmospheric pressure and 0°C .

$$\mu/2N.$$

Electrodes.	Absorption.	Disintegration.
Gold	1.25×10^{-4} (C-320)	1.95×10^{-4} (C-495)
Platinum	0.90×10^{-4} (C-400)	0.81×10^{-4} (C-495)

The absorption was determined for gold and platinum electrodes. In the last column are given the corresponding disintegration values for discharge in air as calculated from results of Holborn and Austin*. The table shows the intimate relation between absorption and disintegration.

The considerable difference between the absorption with gold and platinum electrodes was used to prove, by alternately changing the metal of the anode and cathode, that the conservative absorption was connected with the cathode.

The relation (1) only holds approximately for cathode-falls up to about 1000 volts. When the cathode-fall is increased still more the absorption increases less rapidly and reaches, as already mentioned, a maximum value.

In helium the absorption was smaller; but there was a *distinct conservative absorption to be observed*.

In hydrogen, however, *no conservative absorption was found*, but only absorptions of the more conservative type, which very much depended on the state of the bulb and its previous history of running. This want of conservative absorption in hydrogen gave an explanation of the effect of the residual gases of the bulbs to reduce the velocity of absorption.

Regarding the view taken about the mechanism of the process, I give the following statements from my paper.

"Whatever may be the view we take about the mechanism of the process, we cannot escape from the assumption that gas is attached to the metal and carried away with the metallic deposit." And further, *"The previous results suggest that the electric discharge imparts to the gas a new kind of combining power, 'electric affinity,' which probably is a function of the cathode-fall and which is a power attached to ions moving with a high velocity."*

The attachment of the gas to the metal was considered as something between an occlusion and a real chemical combination, and was compared with the attachment of helium to the radioactive minerals.

* *Wissenschaftl. Abh. d. Phys. Techn. Reichsanst.* B. iv. p. 101 (1904).
Phil. Mag. S. 6. Vol. 32. No. 188. Aug. 1916. R

An extensive series of experiments were further undertaken in Christiania in 1910. The absorption was studied in pure nitrogen with a number of electrodes of various metals, and found to follow essentially the same law as already stated for oxygen. Further, the rate of disintegration of the cathode was measured with the same tube as that used for the absorption experiments and for gold and platinum electrodes.

The results of my experiments on electric absorption were published in the Proceedings of the Christiania Academy*.

Reading the paper of Dr Brodetsky and Dr. Hodgson, I was very glad to notice that they have confirmed the results which I obtained in 1909. As regards the view taken about the mechanism, we also agree in the essential points. Thus they assume the absorption to be due to the velocity (energy or momentum) of the positive ions before the cathode, and they suppose the gas to be attached to the disintegrated metal.

Still I do not quite agree with the assumptions on which they base their theoretical deduction of the absorption curve, for both assumptions would for a constant current make the velocity of absorption increase continually with increase of cathode-fall, while all experimental evidence goes to show that there is an upper limit for the velocity of absorption.

Yours faithfully,

L. VEGARD.

University of Christiania,
May 12, 1916.

XXIX. *The Cause of Lowered Dielectric Strength in High Frequency Fields.* By W. M. THORNTON, D.Sc., D.Eng., Professor of Electrical Engineering in Armstrong College, Newcastle-upon-Tyne †.

1. **T**HE breakdown strength of dielectrics falls with rise of frequency of the field, up to the frequencies used in wireless telegraphy. The experimental evidence for this is not extensive, but it is found in solid, liquid, and gaseous media. For example, glass having a strength of 320 kilovolts per centimetre at 50 cycles a second bears only 128 kilovolts at 8500 cycles ‡. A light petroleum oil failing at 170 kilovolts at 60 cycles, fails at 67 kilovolts at 90,000

* "Über die elektrische Absorption in Entladungsröhren." *Kristiania Vid. Selsk. Skr.* Nr. 7 (1913).

† Communicated by the Author.

‡ Moscicki, *E. T. Z.*, 1904, p. 527.

cycles a second *. In air the change is less, the breakdown voltages at 60 and at 40,000 cycles a second being respectively 22.5 and 19.5 kilovolts †. The effect is not to be accounted for by temperature change, for during the test this does not rise appreciably. An explanation ‡ has been given in general terms based on the fact that on the application of the field there is polarization at the velocity of electric waves in the medium, followed by a further polarization caused by the interattraction of the separated molecular charges. This secondary polarization is not instantaneous, and the breakdown conditions are determined by the superposition of the two fields, the elastic restoring force being, as always in dielectrics, proportional to the displacement.

2. Let F be the intensity of the applied field, and x the electrical displacement. When, as in paraffin-wax, interattraction is negligibly small, $b x = F$. When present it has been shown § to be approximately proportional to x^2 , so that, when fully established,

$$b x = F + c x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

During the growth of the internal field || we may write, when F is unidirectional,

$$b x = F + c x^2 (1 - e^{-\alpha t}), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where α is constant and small relative to the time of test, or to the periodic time when the field is alternating. From this

$$\left\{ b - 2c x (1 - e^{-\alpha t}) \right\} \frac{dx}{dt} - \alpha c x^2 e^{-\alpha t} = 0$$

$$\text{or} \quad \frac{dx}{dt} = \frac{\alpha c x^2 e^{-\alpha t}}{b - 2c x (1 - e^{-\alpha t})} \quad . \quad . \quad . \quad . \quad (3)$$

This is infinite, and the material breaks down, when

$$x = \frac{b}{2c(1 - e^{-\alpha t})} \quad . \quad . \quad . \quad . \quad (4)$$

* 'Dielectric Phenomena in High Voltage Engineering,' F. W. Peek, Jr., McGraw-Hill, N.Y., p. 165.

† Peek, *loc. cit.* p. 107.

‡ "The Electric Strength of Solid Dielectrics," Phil. Mag. vol. xxx. July 1915, p. 133.

§ "The Electric Strength of Solid Dielectrics," *loc. cit.* p. 130.

|| See "The Polarization of Dielectrics in a Steady Field of Force," Phil. Mag. March 1910, p. 401.

The field at which this occurs is

$$F_1 = \frac{b^2}{4c(1 - \epsilon^{-at})}, \quad \dots \dots \dots (5)$$

and if time is given for interattraction to be complete $F_1 = \frac{b^2}{4c}$.

When the applied field is alternating let $F = F_0 \sin pt$; then

$$F_0 \sin pt_1 = \frac{b^2}{4c(1 - \epsilon^{-at})}, \quad \dots \dots \dots (6)$$

where pt_1 is the phase at which failure occurs. Since $p = 2\pi/T$, where T is the period, and α is always much less than T , take $t_1 = q/\alpha$, q being a small fraction. Then

$$F_0 \left\{ \frac{pq}{\alpha} - \frac{1}{3!} \left(\frac{pq}{\alpha} \right)^3 + \dots \right\} = \frac{b^2}{4c(1 - \epsilon^{-q})} \dots \dots \dots (7)$$

When $\frac{pq}{\alpha}$ is small,

$$F_0 p = \frac{b^2 \alpha}{4cq(1 - \epsilon^{-q})} \dots \dots \dots (8)$$

This is constant, so that if breakdown occurs, as it always does, in an early stage of the maximum possible polarization (that is, the maximum which would be reached in a steady field if the structure did not break down), $F_0 p$ is constant, and the breakdown voltage is inversely proportional to the frequency.

3. Polarization at a voltage well below that of breakdown takes many hours to reach its final value of equilibrium in the field, and this value is independent of the external field *. The influence of time of application of the field on breakdown strength is very largely controlled by the rate at which interattraction is established.

A full experimental examination of the influence of time of application on breakdown voltage has been made recently by Mr. F. W. Peek, Jr. †. He has arrived at the empirical formula

$$F_1 = A + BT^{-\frac{1}{4}},$$

where F_1 is the breakdown gradient and T the time of its

* Phil. Mag. loc. cit. 1910, p. 398.

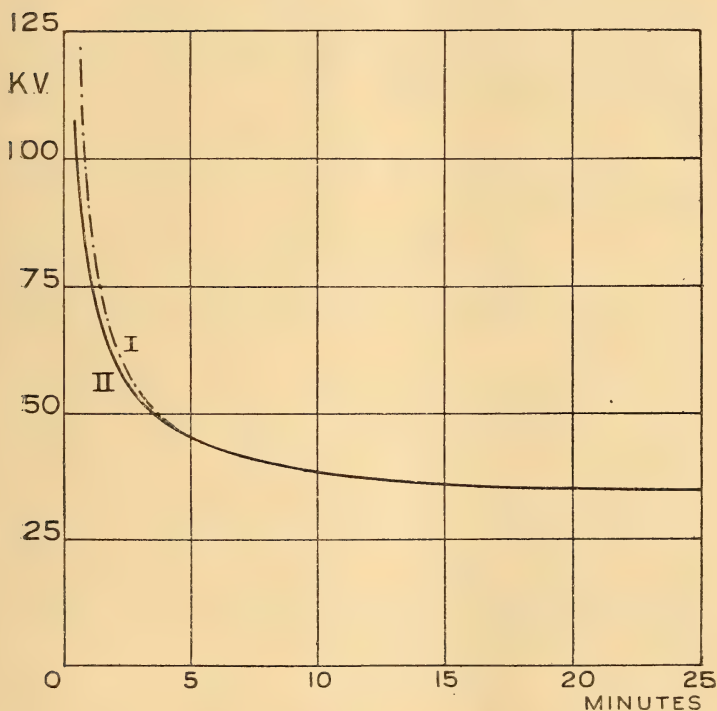
† Loc. cit. p. 178.

application. This may be compared with equation (5) above, and the graphs of both are given below, taking

$$A = b^2/4c, \quad \alpha = .33.$$

$$\text{Curve I.} \quad F_1 = \frac{b^2/4c}{1 - e^{-\alpha T}}$$

$$,, \text{ II.} \quad F_1 = A + BT^{-\frac{1}{4}},$$



The close resemblance of the curves would appear to justify the view that the observed influence of time of application on breakdown strength is largely, if not entirely, caused by the rate of growth of interattraction, and that Peek's formula might be replaced by a simpler exponential type.

4. The nature of dielectric hysteresis has long been under discussion. That it is a time and not a static phenomenon was recognized from the first, but it would appear that the use of the term "viscosity" instead of hysteresis, or as indicating the cause of it, is not tenable. The retarding force

due to viscosity, if taken as usual to be proportional to rate of displacement, leads to results not in keeping with the experimental facts.

Writing equation (1) with a viscosity term

$$a \frac{dx}{dt} + bx - cx^2 = F_0 \sin pt. \quad . \quad . \quad . \quad (9)$$

This equation was submitted to Professor A. R. Forsyth, who very kindly gave the following note upon it:—

“The equation is not integrable in ‘finite terms.’ It belongs to the Riccati type, and its integral can best be expressed by means of series. Take

$$x = -\frac{a}{c} \cdot \frac{1}{u} \cdot \frac{du}{dt} + \frac{b}{2c}, \quad . \quad . \quad . \quad (10)$$

the equation becomes

$$\frac{d^2u}{dt^2} + \left(\frac{cF_0}{a^2} \sin pt - \frac{b^2}{4a^2} \right) u = 0. \quad . \quad . \quad (11)$$

Let $pt = \frac{1}{2}\pi + 2\theta$, and write $\sin^2 \theta = z$; the equation becomes, first

$$\frac{d^2u}{d\theta^2} + \left(\frac{4cF_0}{a^2p^2} \cos 2\theta - \frac{b^2}{a^2p^2} \right) u = 0, \quad . \quad . \quad (12)$$

and then

$$z(1-z) \frac{d^2u}{dz^2} + \left(\frac{1}{2} - z \right) \frac{du}{dz} + \left(\frac{cF_0 - \frac{1}{4}b^2}{a^2p^2} - \frac{2cF_0z}{a^2p^2} \right) u = 0. \quad (13)$$

Of this equation there are two distinct integrals proceeding in ascending powers of z , which is positive and not greater than 1. One of these is of the form

$$1 + k_1z + k_2z^2 + \dots = T_1, \quad . \quad . \quad (14)$$

and the other

$$z^{1/2} + l_2z^{3/2} + l_3z^{5/2} + \dots = T_2, \quad . \quad . \quad (15)$$

and the most general value is

$$u = AT_1 + BT_2, \quad . \quad . \quad . \quad (16)$$

where A and B are arbitrary constants. Thus the value of x would be

$$x = \frac{b}{2c} - \frac{a}{c} \cdot \frac{A \frac{dT_1}{dt} + B \frac{dT_2}{dt}}{AT_1 + BT_2}, \quad . \quad . \quad (17)$$

which really contains only one arbitrary constant...”

Writing $\frac{cF_0 - \frac{1}{4}b^2}{a^2p^2} = m$, and $\frac{2cF_0}{a^2p^2} = n$, and substituting for u in equation (13) the series $A(1 + k_1z + k_2z^2 + \dots)$, we obtain

$$\begin{aligned} \frac{k_1}{2} + m + z \left\{ \frac{2 \cdot 3}{2} k_2 + (m-1)k_1 - n \right\} \\ + z^2 \left\{ \frac{3 \cdot 5}{2} k_3 + (m-1+3)k_2 - nk_1 \right\} \\ + z^3 \left\{ \frac{4 \cdot 7}{2} k_4 + (m-1+3+5)k_3 - nk_2 \right\} \\ + \dots = 0. \quad (18) \end{aligned}$$

Each term vanishes identically and the first series becomes

$$\begin{aligned} T_1 = 1 - 2mz + \frac{2}{3} \left\{ (m-1)m + n \right\} z^2 \\ - \frac{2}{3 \cdot 5} \left[(m-4) \cdot \frac{2}{3} \left\{ (m-1)m + n \right\} + 2mn \right] z^3 + \dots \quad (19) \end{aligned}$$

In the same manner the second series gives

$$\begin{aligned} z^{1/2} \left\{ \frac{1 \cdot 3}{2} \cdot l_2 + \left(m - \frac{1}{2^2} \right) + 0 \right\} \\ + z^{3/2} \left\{ \frac{2 \cdot 5}{2} \cdot l_3 + \left(m - \frac{3^2}{2^2} \right) l_2 - n \right\} \\ + z^{5/2} \left\{ \frac{3 \cdot 7}{2} \cdot l_4 + \left(m - \frac{5^2}{2^2} \right) l_3 - nl_2 \right\} \\ + \dots = 0. \quad (20) \end{aligned}$$

In this $l_2 = -\frac{2}{3} \left(m - \frac{1}{2^2} \right)$ and

$$\begin{aligned} T_2 = z^{1/2} - \frac{2}{1 \cdot 3} \left(m - \frac{1}{2^2} \right) z^{3/2} - \frac{2}{2 \cdot 5} \left\{ \left(m - \frac{3^2}{2^2} \right) l_2 + n \right\} z^{5/2} + \dots \quad (21) \end{aligned}$$

Making the necessary substitutions and finally obtaining a series for $\frac{dx}{dt}$, the conditions for breakdown are given by equating to zero the denominators of successive terms, each of these then making $\frac{dx}{dt}$ infinite.

This gives, in order of approximation,

$$F_0 = \frac{b^2}{4c}, \quad F_0 = \frac{b^2}{4c} + \frac{a^2 p^2}{4c}, \quad F_0 = \frac{b^2}{4c} + \frac{11a^2 p^2}{4c} \pm \frac{ap}{c} (3b^2 + 28a^2 p^2)^{\frac{1}{2}}. \quad (22)$$

In these p never occurs in the denominator, so that, as might be expected, viscosity of *displacement* of charge would increase the field required for breakdown as the frequency is raised, and this is not the case.

Dielectric hysteresis must therefore be considered as neither a static nor a viscous retardation of displacement, but as a quasi-elastic adjustment of equilibrium of the field of interattraction between opposite charges in molecules whose (fixed) axes are in perfect confusion of alignment; the molecules being partly polarized, and interattraction started, by the application of the external field.

XXX. *Image Formation by Crystalline Media.* By H. F. DAWES, *Professor of Physics, McMaster University, Toronto, Canada* *.

IN 1877 Stokes † discussed the question of the position of the image (the "apparent depth" of an object-point) due to the refraction of light from a crystal to air through a cleavage-plane or other plane refracting surface. This paper was written with the object of developing a method of testing crystals by applying the then recently discovered microscope method of determining the index of refraction, and was tested out for a great many specimens by Sorby ‡.

Stokes's investigation does not seem to have been very generally followed up or applied from an optical point of view, although the results are particularly interesting in the study of the optical properties of crystals. In fact, the only record of the application of the principles developed which I have been able to find is in Clay's admirable collection of *Experiments in Light*. Independent of these sources, however, the writer developed in 1908 a simple laboratory experiment of the type under consideration for use in his laboratory classes at the Physical Laboratory of the University of Toronto. In addition to the phenomena considered by Stokes there are further cases of image formation which are interesting and important, and the writer proposes to consider certain of these.

* Communicated by Professor J. C. McLennan, F.R.S.

† G. G. Stokes, *Proc. Roy. Soc.* xxvi. p. 386 (1877).

‡ Sorby, *ibid.* p. 384.

General Considerations.

The ordinary light in uniaxial crystals is propagated in precisely the same way as in ordinary isotropic substances, so that the laws for image-formation by this light are well known and need no further discussion. It is the extraordinary images in these crystals and both images in biaxials which follow more complex laws,—due to the more complicated shape of the wave-front and the fact that the rays are not necessarily at right angles to the waves. In investigating the laws for the image-formation in such cases, therefore, it is necessary to take into account the curvatures of the incident wave as well as the nature of the refracting surface and the change of velocity on refraction. The image is in general astigmatic, so that the investigation involves the determination of the positions and directions of two focal lines.

As an example of the method which must be employed, consider the formation of the image of a point-source within a uniaxial crystal by the extraordinary light refracted into air through a plane surface which is normal to the optic axis of the crystal. Such an image is intended to be seen by the eye or a low-power microscope, so that the pencil of rays coming from object or image-point is of small angular aperture. For this reason the standard approximations of the “first order” discussion of image-formation are justified and will be assumed without comment. The procedure here followed is not the same as that of Stokes, although it amounts to the same thing in the end.

As is well known *, this extraordinary wave in a uniaxial crystal travels out from a point-source in the form of a spheroid whose axis of revolution is the optic axis of the crystal. The following are geometrical and optical properties of such a wave-front.

The principal indices of refraction are n_1 equal to V/a , and n_2 equal to V/c , (1), where V is the velocity of light in air (or vacuum), a is the velocity of the extraordinary wave along the axis, and c its velocity at right angles to the axis; a is also the velocity of the ordinary wave.

If a wave has travelled a distance x along the axis it has also travelled a distance y at right angles to the axis such that $y = cx/a$, (2), so that the semi-axes of the wave-surface

* Drude, ‘Theory of Optics,’ Properties of Transparent Crystals, from (55) setting b equal to c . Further relations are quoted or are deduced very directly from results in the same chapter; they will be referred to by the number of the equation which represents them.

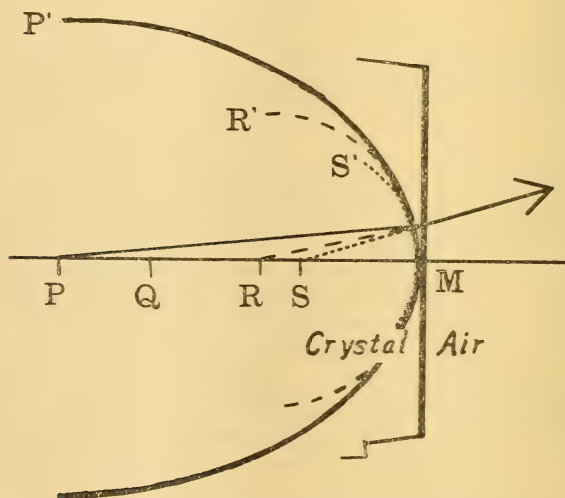
are x, y, y . The radius of curvature of the spheroid in any normal section at its pole is

$$r = y^2/x = c^2 x/a^2 = n_1^2 x/n_2^2, (3).$$

The diagrams are drawn to correspond to a crystal of principal indices $n_1=4/3$ and $n_2=2$. The trace of the wave is shown at incidence and just after refraction; also the circle of curvature of the elliptic wave is shown in certain cases. The traces for actual wave-positions are indicated by continuous lines and for "virtual" wave-positions by dotted lines.

For the example under consideration the source of light P' lies within the crystal at a distance p from the refracting surface at M (Fig. 1). The ordinary waves form an image

Fig. 1.



at a distance q from M such that p/q is equal to n_1 , (4), as in ordinary isotropic substances. The extraordinary waves also form an image, but it is not situated at Q , although the two sets of waves travel with the same velocity along PM . (This is contrary to the statement usually made in discussions of this subject, viz., that only one image is seen on looking along the axis into a crystal. It is true that the eye seems to see only one image, but this is because the one image is vertically beneath the other and the depth of focus of the eye is very great compared with the distance between the images. The presence of the two images may, however, be

readily demonstrated by means of a microscope ; for example, in the case of calcspar it is necessary to move the microscope along its axis a distance of about one-seventh of the depth of the object-point in the crystal in order to change the focus from the one image to the other.)

The formation of this second image arises from the fact that although the waves have equal velocities along the axis they have not the same curvatures, and the position of the image is determined not only by the change of velocity on refraction but also by the change of curvature. Since the wave has travelled a distance p along the axis, the radius of curvature of the wave-front is c^2p/a^2 as it reaches the refracting surface, by (3). Now, when refraction takes place at a plane refracting surface, the ratio of the curvatures of the incident and refracted waves is equal to the ratio of the velocities of light in the two media. Hence the radius of curvature of the wave as it enters the air is given by

$$\frac{s}{c^2p/a^2} = \frac{a}{V}, \text{ i. e. } s = \frac{c^2p}{aV};$$

and since the image in air is situated at the centre of curvature S of the waves coming from it, the distance s must be the required distance from the refracting surface to the image. In the diagram, $P'M$ is the incident wave, $R'M$ its circle of curvature, and $S'M$ the refracted wave.

These results may be used to measure the principal extraordinary index of the crystal by the microscope method. Thus, s/p which is equal to c^2/aV is also equal to n_1/n_2^2 , (5), by (1). The image is therefore at the same depth as it would be in a substance of index n_2^2/n_1 , and the value of n_2 is given by

$$n_2 = \frac{n_1}{s/p} \text{ or } \frac{p^2}{qs}, \text{ by (4).}$$

The values of p , q , and s will then be found by focussing the microscope in turn on P , Q , and S , and finally on M .

No one appears to have called attention to the fact that the two images cannot be distinguished from one another by the usual polarization test if the eye or microscope is placed symmetrically over the axis of the crystal through the source. For, in the wave-front of the extraordinary pencil the vibration directions radiate out equally in all directions from the axis, so that an analyser will transmit the same amount of light in all azimuths and the source will appear to be of the same brightness for all positions of the analyser. On the other hand, in the wave-front of the ordinary pencil

the same as if the light were being refracted into a substance of index n_2^2/n_1 .

2. *Image formed by direct refraction through a plane parallel plate of uniaxial crystal of thickness t , the faces being normal to the axis.*—The light is refracted into the crystal in accordance with the law developed in the preceding section, and is refracted out according to that of the first example considered. Thus, if the source of light is at P in fig. 2 at the distance p from the first surface M, the virtual source of the refracted pencil is R at the distance $n_2^2 p/n_1$ from that surface (as in Section 1) and consequently at the distance $t + n_2^2 p/n_1$ from N at the second surface of the plate. The image formed by the second refraction is therefore by (5) situated at S at the distance $p + t n_1/n_2^2$ from N.

The distance from the final image to the original source is $t(1 - n_1/n_2^2)$, (7), so that the image is "drawn up" toward the observer by this amount, which is the same as for an isotropic plate of index n_2^2/n_1 and is independent of the distance of the source.

3. *Image formed by direct refraction from one uniaxial crystal to another through a plane surface of separation which is normal to the axis of each crystal.*—The set of waves forming the image is extraordinary in both crystals, since the plane of incidence and refraction is a principal section in each.

One may deduce the law for the image position in this case from the laws developed for refraction from air to crystal and from crystal to air. Suppose for the moment that the two crystals are separated by a plane parallel air-space of thickness t , and that the light is refracted from crystal 1 into this air-space and then into crystal 2. According to (5) the image formed by the first refraction is situated at the distance $n_1' p/n_2'^2$ from the first refracting surface, and therefore $(t + n_1' p/n_2'^2)$ from the second. Since this image forms the source with reference to the second surface, it follows by (6) that the distance of the final image from this surface is $(t + n_1' p/n_2'^2) n_2''^2/n_1''$.

Now the *direction* of the final part of any of the rays forming this image is independent of the thickness of the air-space, and is therefore the same as if the thickness were indefinitely diminished, *i. e.*, as if the crystals were in optical contact. Hence the required image position, which is determined by the direction of the rays for optical contact, is the limiting position of the above image when the value of t is reduced to zero. Its distance from the surface of contact will therefore be

$$p \frac{n_2''^2}{n_1''} \bigg/ \frac{n_2'^2}{n_1'}.$$

As in other cases considered, this distance is just the same as if the light were refracted from a substance of absolute refractive index $n_2'^2/n_1'$ to one of index $n_2''^2/n_1''$.

4. *Image formed by direct refraction through a series of plane parallel plates of uniaxial crystal all having optic axes in the common normal to the surfaces.*—The extraordinary wave in the first crystal will be extraordinary in all successive plates and the ordinary will remain ordinary, so that only two images will be formed at each stage.

By the argument of Section 3 the image-position at each stage and therefore the final image-position will be the same as if the light were refracted into and out of a thin air-film of negligible thickness separating each pair of plates. By (7) the image is "drawn up" toward the observer a distance $t(1-n_1/n_2^2)$, on account of refraction through any plate of thickness t and indices n_1 and n_2 , and this "drawing up" process will be repeated for each plate of the series. Hence the image will be nearer to the observer than the original source by the sum of all such quantities as $t(1-n_1/n_2^2)$. The case of some plates being isotropic is of course included in this, the corresponding displacement terms being obtained by setting n_1 equal to n_2 .

5. *Image-formation by "direct refraction" through a lens of uniaxial crystal with optic axis lying along the geometrical axis.*—Recall that for refraction from a medium of index n' to one of index n'' through a surface of curvature $1/r$, the object and image distances are related by the law

$$\frac{n''}{v} - \frac{n'}{u} = k, \quad (8),$$

where k is the dioptric power of the surface and is equal to $\{n''-n'\}/r$, (9). Note that this is fundamentally a relation between the curvature $1/u$ of the incident wave and the curvature $1/v$ of the refracted wave.

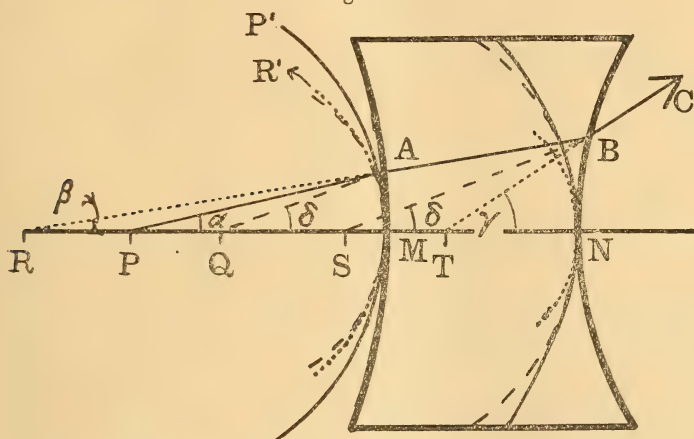
In the case under consideration, the light-wave P'M from a point P on the axis (fig. 3) is incident on the first surface M; is refracted as a spheroidal wave R'M whose centre of curvature is a certain point Q and centre of radiation the centre of the spheroid R; is incident on the second surface N as a wave with centre of curvature at a point S; is finally refracted into air as a homocentric pencil of centre T. T is the required image-point.

Since the light travels along the axis of the crystal with the velocity corresponding to the index n_1 , the powers of the first and second surfaces will be k' and k'' , equal to $(n_1-1)/r'$ and $(1-n_1)/r''$, by (9). In accordance with (3)

$$RM = n_2^2 \cdot QM/n_1^2 \text{ and } RN = n_2^2 \cdot SN/n_1^2, \quad (10).$$

Draw the ray PABC whose successive directions make the angles α , β , γ with the axis; the line QA makes the angle δ with the axis. These angles, much exaggerated in the diagram, are sufficiently small to admit of the ordinary approximations of the standard "first order" discussions of the principles of image-formation.

Fig. 3.



The line SB is parallel to QA. For the time required to travel from any one wave-position to any other is the same for all rays; and for any given ray this time is proportional to the distance between the waves. Hence, since A lies on the wave-front R'M from R and B lies on the corresponding wave-front through N, it follows that $RA : RB :: RM : RN$. But $QA : SB$ in this same ratio by (10), so that the triangles RQA and RSB are similar.

The distances from A and B to the axis are

$$y' = PM \cdot \alpha = QM \cdot \delta = RM \cdot \beta,$$

$$\text{and } y'' = RN \cdot \beta = SN \cdot \delta = TN \cdot \gamma, \quad (11);$$

and, in view of these values, standard relations for the first and second refractions may be written from (8) as follows:

$$n_1 \delta - \alpha = k' y', \quad (12),$$

$$\text{and } \gamma - n_1 \delta = k' y'', \quad (13).$$

The problem now resolves itself into the elimination of the quantities which are involved in the use of particular rays and of the intermediate image formed within the medium of the lens. By this process the law for the image-position

is determined in terms of the optical constants of the system and the position of the object. The method corresponds to the standard one of dealing with a thick lens. Thus :

$$\begin{aligned}\gamma - \alpha &= k'y' + k''y'', \text{ by addition, from (12, 13),} \\ &= k'y' + k''y' + k''t\beta, \text{ by (11),} \\ &= k'y' + k''y' + k''tn_1^2\delta/n_2^2 \\ &= k'y' + k''y' + k''tn_1k'y'/n_2^2 \\ &\quad + k''tn_1\alpha/n_2^2, \text{ by (12).}\end{aligned}$$

Hence

$$y/\alpha = K \cdot PM + (1 + k''tn_1/n_2^2), \text{ (14),}$$

where K is the power of the system and is equal to

$$k' + k'' + k'k''tn_1/n_2^2, \text{ (15).}$$

Similarly,

$$\alpha/\gamma = -K \cdot TN + (1 + k'tn_1/n_2^2), \text{ (16).}$$

(14) and (16) state the fundamental general law for the image-position by expressing the distances from the refracting surfaces to object and image in terms of the ratio γ/α , which may be considered an arbitrary parameter. These formulæ are similar to the standard formulæ for a lens of isotropic substance*, the modification consisting essentially in that the distance travelled within the crystal is divided by n_2^2/n_1 , corresponding to division by the index in ordinary lenses. The positions of the cardinal points of the lens and the lens formulæ of standard type may therefore be written down on this basis.

In addition to the image system formed by the light which traverses the lens as the extraordinary wave-front, there is an image system produced by the ordinary wave-front, and although the light traverses the lens in the neighbourhood of the optic axis the two image-systems are not coincident. The fundamental formulæ corresponding to (14, 16, 15) are for this system,

$$\gamma/\alpha = K' \cdot PM + (1 + k''t/n_1), \text{ (17),}$$

$$\alpha/\gamma = -K' \cdot TN + (1 + k't/n_1), \text{ (18),}$$

where K' the power of the lens is

$$k' + k'' + k'k''t/n_1, \text{ (19).}$$

It will be seen that the difference in the two sets of formulæ lies in the terms which involve the thickness of the lens, so that the two images of an object coincide when and

* See Hermann, 'Geometrical Optics,' p. 57.

only when the thickness is negligible. In consequence of this it will not be possible with thick lenses of crystal to focus sharply either on a photographic plate or in the focal plane of an eyepiece. The two most important cases in which this may affect the efficiency of the lens are in its use as objective of telescope or collimator. In these cases it is the separation of the one set of principal foci which is important. This separation may be estimated as follows:—

The distances from the lens to the second principal foci are determined by setting the arbitrary parameter equal to zero in (16) and (18), giving

$$0 = -K \cdot FN + (1 + k'tn_1/n_2^2),$$

$$0 = -K' \cdot F'N + (1 + k't/n_1),$$

from which the distance between the foci is equal to

$$k'^2t(n_1^2 - n_2^2)/n_2^2n_1KK'.$$

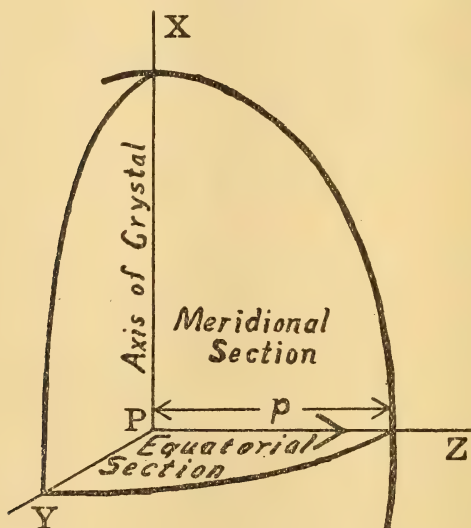
In a quartz lens of say about 20 cm. focal length and surfaces of equal curvature, this amounts to about .001 of the thickness, and in a calcite lens of the same power to about .037 of the thickness. The separation is thus unimportant in quartz lenses of less than about .5 cm. thickness except for the highest magnifications.

This separation of the second principal foci disappears when the first surface of the lens is plane, so that plane waves are unchanged in shape until the second surface is passed, but in this case the separation of the first principal foci is proportionately greater on account of the fact that the whole of the dioptric power is produced at the one surface. Similar considerations would show that the separation of the first principal foci would be zero for a lens with the second surface plane. The fact of double refraction requires, therefore, that the objectives of collimator and telescope be plano-convex and that the lenses be placed with plane faces out.

6. *Image-formation by direct refraction by uniaxial crystals cut so that the optic axis is at right angles to the geometrical axis of the refracting system.*—For such examples of refraction of the extraordinary wave, the part of the spheroid presented to the refracting surface lies in the neighbourhood of zero latitude. The curvatures in normal sections in this part of the wave-front are unequal, so that on refraction by a symmetrical system an astigmatic image will be formed. Since the principal curvatures lie in and at right angles to

the plane of the meridian, the focal lines, which are determined by these curvatures, will be at right angles to and parallel to the axis of revolution, and hence at right angles to and parallel to the optic axis of the crystal. Thus in fig. 4, which represents the extraordinary wave from a

Fig. 4.



point P, the optic axis of the crystal is parallel to the x -axis and the geometrical axis of the refracting system lies along the z -axis. The wave travels with the velocity a along the optic axis and with the velocity c at right angles to it, so that the section by the zy -plane is the equatorial circle and that by the zx -plane the meridional ellipse. The focal line corresponding to the former section is parallel to the x -axis and intersects the z -axis at the image-point, as determined for a spherical wave of the same curvature travelling with the same velocity in an isotropic substance. Similarly, the focal line corresponding to the meridional section is parallel to the y -axis, and its point of intersection with the z -axis is the same as if this wave were spheroidal about this line as axis of revolution. The position of this point of intersection will therefore be determined in accordance with the principles which we have been considering in previous sections; the necessary modification of the formulæ will consist simply in interchanging the indices n_1 and n_2 , since the velocities a and c are now interchanged with reference to the incident wave-front.

Keeping these observations in mind, the following conclusions may be easily deduced :—

(1)* *For an object-point within a crystal at a distance p from a plane refracting surface.*

Distance from surface to :

- (a) focal line parallel to axis . . . p/n_2 ;
- (b) focal line at right angles to axis, pn_2/n_1^2 ;
- (c) ordinary image p/n_1 .

(2) *For a lens of crystal.*

The powers of the first and second surfaces and of the lens as a whole are :—

(a) for pencils of ordinary light :

$$k_1 = \frac{n_1 - 1}{r}, \quad k_2 = \frac{1 - n_1}{s}, \quad K = k_1 + k_2 + k_1 k_2 t / n_1;$$

(b) for focal lines parallel to the optic axis of the crystal, as governed by the circular section of the wave-surface:

$$k_1' = \frac{n_2 - 1}{r}, \quad k_2' = \frac{1 - n_2}{s}, \quad K' = k_1' + k_2' + k_1' k_2' t / n_2;$$

(c) for focal lines at right angles to optic axis, governed by elliptic section of wave-front :

$$k_1'' = \frac{n_2 - 1}{r}, \quad k_2'' = \frac{1 - n_2}{s}, \quad K'' = k_1'' + k_2'' + k_1'' k_2'' t n_2 / n_1^2.$$

Knowing the values of the powers for these cases, the positions of the sets of cardinal points of the lens and the formulæ relating object to image positions may be written down in accordance with the standard formulæ for these quantities,—it was shown in the discussion of Section 5 that these formulæ in a modified form apply to the refraction of the elliptic wave.

It will be seen that the powers of the individual refracting surfaces are the same for both the principal sections of the wave-front. The powers of the lens as a whole differ, however, in the term which involves the thickness, so that the amount of the astigmatism is governed by the thickness; if the thickness is negligible the focal lines degenerate into a point and the emerging pencil remains homocentric. The powers for the ordinary and extraordinary waves are, however, essentially different for each surface and for the lens as a whole. There are therefore always two image series for a lens of crystal cut as described in this section.

* Included in Stokes's discussion.

Crystalline Components in a Lens System.

Whenever a crystalline component,—lens or plane-parallel plate,—which produces two image series is incorporated into a lens system, the system as a whole will of course produce two series of images. For example, for a doublet consisting of a lens of crystal of powers K' and K'' and a lens of glass of power K , the combination will have the powers $K' + K$ and $K'' + K$. It is evident that the relative values of the powers of such a combination may be made almost anything one wishes; one of the values may even be made zero by choosing the glass lens of power equal and opposite to that of the selected crystal power. In this case the other power will be the difference $K' - K''$ of the powers of the crystal lens.

Again, it will not be possible to produce an achromatic doublet which will at the same time correct the chromatic aberration of both powers of a lens of crystal. If ω' , ω'' , ω , are the dispersions for the ordinary and extraordinary powers of the crystalline lens and of the glass lens combined with it, and K' , K'' , K the corresponding mean powers, the chromatic aberrations of the powers of the combination are

$$\omega'K' + \omega K, \text{ and } \omega''K'' + \omega K.$$

If the values of ω and K are so chosen that one of these is zero in accordance with the ordinary principles of Geometrical Optics, the value of the other will be

$$\omega''K'' - \omega'K',$$

and this will not likely happen to be zero. In the case of a lens of Iceland spar, of which the principal indices are about 1.66 and 1.49 and the dispersions for the D-F lines .014 and .0088, this outstanding chromatic aberration will be $-.0051 (1/r - 1/s)$, or about .02 of the power of the combination if the glass lens has the index 1.62 and the dispersion .02.

Summary.

Laws for the image position are developed for a number of typical cases of image formation by the extraordinary rays in uniaxial crystals.

It will be noticed that in the examples of light travelling along the axis of the crystal, the thickness of the crystal traversed enters into the expressions for image position as if the index were n_2^2/n_1 as compared with isotropic substances; n_1 and n_2 are the principal indices of the crystal. This quantity may be considered a sort of pseudo-index applicable to such cases.

It is shown that lenses of crystal will in general form two

series of images corresponding to the two sets of waves ; the laws for the image series due to the extraordinary waves correspond to those for lenses of isotropic substance, and the expressions are found for the dioptric power. The question of the use of crystal components in lens systems is also considered briefly.

Physical Laboratory,
University of Toronto,
May 2, 1916.

XXXI. *Notices respecting New Books.*

An Introduction to the Mechanics of Fluids. By EDWIN H. BARTON, D.Sc., F.R.S.E. Pp. xiv+249. With diagrams and examples. Longmans, Green & Co. : London. 1915. 6s. net.

IT is somewhat difficult to write a book on elementary mechanics which shall present any very novel feature. But there are traits of this one which are certainly novel, and, which is still better, novel in an attractive way. One of these characteristics is that while the calculus is avoided it is replaced, not by the very artificial methods with which we are all so familiar, but by a method of summation which forms one of the best possible introductions to the calculus itself. This is a very great gain. The most interesting part of the book is that dealing with applications of the mechanics of fluids in technical practice. We have never come across so complete and satisfactory an account of the different forms of suction and compression pumps, rams, Bourdon and other gauges, water-wheels, presses, lifts, cranes, turbines, drills, brakes, &c. We commend this part in particular to school teachers ; nothing could be better calculated to stimulate and at the same time to instruct the youthful mind.

The introduction to mechanics is in general very logical. One or two very minor points made the reviewer hesitate for a moment. The first page scarcely brings out the fact that a liquid opposes *no permanent* resistance to change of shape. To talk of *very small* resistances scarcely meets the case. In consequence of this inadequate statement it will not be evident (as it is stated to be) that viscosity "need not enter into our account when considering any fluids at rest in equilibrium."

Again, is it a matter of such indifference what definitions are given of mass and force ? The inadequacy of Newton's treatment of mass arises from the fact that although the masses (or quantities of matter) of the same substance may be taken as proportional to their volumes, we cannot pass from these cases to cases in which the material is different.

It must further be remembered that those who are driven to define force with reference to our muscular sensations are not only illogical, but are wrong ; because, as every psychologist knows, our sensations of force are *not* proportional to the equivalent forces measured mechanically.

We wish to emphasize, however, that the method of definition given in detail by Professor Barton, and which apparently is preferred by himself, is perfectly logical. He has moreover put the matter very clearly—at least considering that he only professes to give an introduction to the subject.

We recommend the volume very strongly to every teacher, and we are confident that it will be well received.

Fundamental Conceptions of Modern Mathematics. By ROBERT P. RICHARDSON and EDWARD H. LANDIS. Chicago and London. The Open Court Publishing Company. 1916.

Numbers, Variables, and Mr. Russell's Philosophy. By R. P. RICHARDSON and E. H. LANDIS. (Reprinted from the 'Monist' of July 1915.) The Open Court Publishing Company. 1915.

THE book and pamphlet naturally go together, part of the criticism in the latter finding its place in the former. The purpose of the book is best explained by some quotations from the preface. It is—

"to examine critically the fundamental conceptions of Mathematics as embodied in the current definitions.... In expounding our own views we have often been obliged to find fault with those of others; but we have not gone out of our way for the sake of mere criticism; we have merely cleared away false doctrine preparatory to replacing it with true.... The keynote of our work is the distinction we find it necessary to make between quantities, values, and variables, on the one hand, and between symbols and the quantities or variables they denote or values they represent, on the other."

The theme is attractive, and is to be worked out in thirteen volumes, of which this is the first. It will be useful, meanwhile, to see how the authors put their maxims into practice.

Consider, for example, the meaning of Variable. We are practically told that all mankind since Newton have, in their endeavour to define a variable, fallen into an estate of mathematical sin and misery. After much criticism of these endeavours our Pennsylvanian authors say (pp. 154-5) that "while we have made plain what a variable is not, and have described in what manner it is constituted, care has been taken to avoid any statement as to what a variable is.... Any attempt to give a precise account of the definition of the term 'variable' would require a somewhat lengthy consideration of the philosophical theory of the categories, which cannot be given in this place." We had hoped for bread, and they give us a loaf-ticket! After all, did it never strike them that the mathematicians they criticize were in the same difficulty, and that their so-called definitions were intended to be simply first notions which experience would amplify and even correct as the mathematical student advanced in knowledge?

Consider, also, their discussion of quaternions and vectors. Their view, we are told, is not precisely that of Hamilton. No objection can be taken to such a position. But what of their treatment of Hamilton? They quote on p. 47 his definition that

"A right line considered as having not only *length* but also *direction*, is said to be a VECTOR." But this, they say, is obviously inadequate, "since a linear velocity at a point is a vector, and of course a velocity is not a line. Further, not all straight lines possess the attributes requisite to vectors." These reasons are quite irrelevant. For Hamilton never said that a velocity was a line in their assumed sense of the term; nor did he even say that a velocity was a vector. Also Hamilton never said that all straight lines, in any sense in which other writers might be pleased to regard them, were vectors. Nor does his definition imply it. He said that right lines *considered as having not only length but also direction* were vectors—a totally different statement. Obviously, however, our critics are psychologically incapable of understanding the plain meaning of Hamilton's words; for two pages further on they make the following remarkable utterance: "Even so high an authority as Hamilton states that a right line is a vector; a view which is quite untenable." Hamilton certainly never made such a statement, except when he carefully guarded himself by saying that the right line was *considered as having not only length but also direction*. Following this second emasculated statement wrongly ascribed to Hamilton, the authors go on to declare that a right line remains the same so long as its length remains constant, even though it rotate like the spoke of a wheel into a new position. Now—so runs the argument—the vector of this rotating "right line" does not remain the same. Therefore, they conclude, this "right line" cannot be a vector. But whoever said it was? Their statement that Hamilton said so is absolutely ludicrous. In the course of two pages these critics of Hamilton, with wondrous wisdom, drop out of the definition certain all-important words, and then, with superb simplicity, develop their attack by first giving to the term right line a meaning which, as their own previous words prove, was not Hamilton's meaning. They commit the double sin of misquotation and bad logic.

We are nowhere told distinctly what they themselves consider a vector to be. They seem to follow the ordinary custom of making it apply as a class name to certain very different physical quantities, such as displacement, velocity, acceleration, force, angular velocity, moment of momentum, and the like—a use which compels them to talk profoundly of vectors of different sorts. For example, on page 76 they say: "In Quaternions, vectors of described straight lines and of points constitute one sort of applicate quantities; linear velocities at points, another; linear accelerations at points, a third; and so on." In Hamilton's Quaternions this is not so. Must we repeat it? A vector is a right line *considered as having not only length but also direction*. This is the opening statement of the Elements of Quaternions, and is the foundation of the whole calculus. Vectors, so defined, are found to satisfy an important law, the law of vector addition. Hamilton does not speak of a velocity or an acceleration as being a vector. When he applies his calculus to dynamics and physics he finds certain quantities which can be represented or symbolized

by right lines regarded as having not only length but also direction, and which, when so represented, satisfy the law of vector addition. These representative lines are Hamilton's vectors. Analytically there is only one sort. He speaks of the vector of a point, the vector of a velocity, the vector of a force; but he nowhere speaks of the point, or the velocity, or the force, as being the vector. For example, what is commonly called the vector product of two vectors has various meanings according to the kind of quantities symbolized by the vectors. Thus, if \mathbf{b} be the vector of the force acting at the point whose vector is \mathbf{a} , then in Hamilton's view, \mathbf{ab} is a quaternion, whose vector part, $V\mathbf{ab}$, represents the vector of the moment of the force about the origin, and whose scalar part, $S\mathbf{ab}$, measures what Clausius subsequently called the virial. Again, if \mathbf{a} is the vector of the angular velocity, the expression $V\mathbf{ab}$ is the vector of the velocity of the point whose vector is \mathbf{b} . In their misapprehension and misrepresentation of Hamilton's meaning and method are not our authors confusing symbol and quantity?

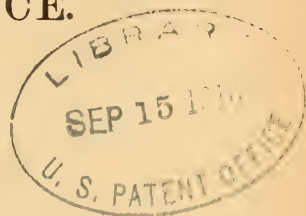
On page 132 we read: "Quaternions is defective in that it is not possible to multiply a vector by a quaternion. Thus, if the relation of one vector to another is the quaternion q , it does not necessarily follow that we can multiply a third vector by q and obtain a vector." The second clause of the first sentence is simply not true. As regards the second sentence, which is probably meant to explain the first, why should we expect, or why should we desire, that any quaternion multiplying any vector should give a vector? The truth is that a quaternion multiplying a vector gives, in general, another quaternion. Why should a calculus be called *defective* because by the fundamental laws of its being it cannot always satisfy a law which it satisfies occasionally and under specially restricted conditions? If the quaternion is admitted at all, why desire to get rid of it by setting it in product combination with any vector? In uniplanar quaternions this law is satisfied; but then uniplanar quaternions is so defective a form of the calculus as to cease to be quaternions at all.

Whatever important additions this volume may make to mathematical thought, the authors' claim to be the children of the light is seriously menaced by their misquotation and misrepresentation of the meaning of the words of the author they criticize, and by their elaboration of an argument which is either irrelevant or a begging of the question. Nevertheless they have a real appreciation of the value of the quaternion and speak warmly of the great services rendered by Hamilton. That they are also capable of doing good things is shown in their discussion of the meaning of Function, which occupies a comparatively small part of the volume, and of which more will be given in the succeeding parts. Here, we are glad to say, they give a definition, which in regard to length and style recalls Euclid's famous definition of proportion. In this discussion also, as in the discussion on variables, almost every mathematician who is named is found sadly lacking in accuracy; but the ordinary mathematical reader will be cheered by the thought that should he likewise fall into error he will still be in the company of the immortals.

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XXXII. *On Fluorescent Vapours and their Magneto-optic Properties.* By L. SILBERSTEIN, Ph.D., Scientific Adviser to Adam Hilger, Ltd.*

1. *General Remarks.*

THE fluorescence of vapours, such as iodine vapour, excited by light of a particular, appropriately chosen, frequency N , gives what Wood calls a *resonance spectrum*, consisting of a series of lines, whose frequencies are $n_0 = N$ and, say, n_1, n_2, n_3 , etc., where the latter stand in some relation to the fundamental frequency N .

Now, an ordinary oscillator or resonator, of free period $T = 2\pi/N$ and relaxation- or extinction-time $\tau = 2/k$, *i. e.* a system obeying the familiar differential equation

$$\ddot{x} + k\dot{x} + N^2x = 0, \quad (H)$$

when acted upon by an external force of frequency n , gives only oscillations of the same frequency n , and of no other in addition. (And, assuming T/τ small, the resonator answers vigorously only when $n \doteq N$.) Since this simple behaviour is due to the assumption of Hooke's law for the restitutive elastic force, *i. e.* force proportional to displacement x , we can appropriately call a system obeying the equation (H) a *Hookean* resonator. On the other hand, if N^2x is replaced by a non-linear function of x , an exciting force of frequency N will generate oscillations of frequency N and also an infinite variety of other frequencies.

* Communicated by the Author.

Keeping this in mind, we can mathematically describe the excitation and emission of fluorescent spectra by saying either

1st, that the atoms of the vapour behave so as if each contained a Hookean resonator under the simultaneous action of forces of all the frequencies $n_0 = N, n_1, n_2, n_3$, etc., *i. e.* by writing

$$\ddot{x} + k\dot{x} + N^2x = c_0e^{iNt} + c_1e^{in_1t} + c_2e^{in_2t} + \dots, \quad (1)$$

or, 2nd, that each atom contains an appropriate non-Hookean resonator, acted upon by c_0e^{iNt} only, *i. e.* that

$$\ddot{x}^2 + k\dot{x} + N^2x + f(x) = c_0e^{iNt}, \quad (2)$$

where $f(x)$ is some non-linear function of the displacement.

The equivalence of the two methods of treatment is manifest. In fact, the non-Hookean resonator will be the "appropriate" one when, and only when, the supplementary term $-f(x)$ ultimately reduces to $c_1e^{in_1t} + c_2e^{in_2t} + \dots$

If the frequencies n_1, n_2 , etc., accompanying the fundamental one N , and the ratios $c_1 : c_0$, etc., are all assumed to be known (from experience), then we do not require the form (2) at all, and we can study the properties of *each line of the spectrum separately* by writing $x = x_0 + x_1 + \dots$ and splitting (1) into

$$\ddot{x}_i + k\dot{x}_i + N^2x_i = c_ie^{in_it}, \quad i = 0, 1, 2, \dots, \quad (3)$$

and introducing into each of these equations, and of the analogous ones for y_i, z_i , the well-known supplementary terms due to a superposed magnetic field. It is only when we wish to make a guess as to the law of sequence of the lines in the spectrum that we require the form (2). This will occupy our attention for a few moments in the next section, in which we shall assume hypothetically a particular form of $f(x)$, and test it experimentally; but in all the remaining sections of this paper we shall avail ourselves only of the equation (3) and of ordinary electromagnetism, so that all results obtained in these sections will be entirely independent of that hypothesis.

2. Law of Constant Frequency Intervals.

The simplest case of a non-Hookean resonator is well-known from acoustical problems in which use is made of a quadratic supplementary term, $\text{const.} \times x^2$. Such a term would, obviously, not answer our purpose, since it gives, in

addition to N , its octave $2N$ as the next frequency, then $3N$, etc., while the lines* of Wood's iodine spectra, excited by green or yellow light, follow upon one another in intervals of 60 to 85 Å.U. only. What we require is a fractional power of x , but little differing from unity.

Let us write, therefore, $f(x) = a \cdot x^p$, where p is a real, positive, constant, whose value is for the moment undetermined. Then (2) will become

$$\ddot{x} + k\dot{x} + Nx^2 = c_0 e^{iNt} - ax^p. \quad (4)$$

Whatever the value of p , the difference $1-p$ will be a certain measure of departure of the resonator from Hookean elasticity, and it will be responsible for the series of lines accompanying the fundamental one. The coefficient a is a real constant. If symmetry around the position of equilibrium is required, then x^p is to be taken as $\pm |x^p|$, according as $x > 0$ or $x < 0$.

As to c_0 , if the exciting agent be (polarized) light, c_0 is the product of the amplitude of the electric force in the incident wave, and of q/m , the ratio of the charge and mass of the resonator.

Now, proceeding by the well-known method of successive approximations, write, first,

$$x = be^{iNt},$$

and neglect ax^p . Then $\ddot{x} + Nx^2 = 0$, and

$$b = -i \frac{c_0}{kN} \dots \dots \dots (5)$$

Substitute the first approximation in x^p , so that (4) will become

$$\ddot{x} + k\dot{x} + N^2x = c_0 e^{iNt} - ab^p e^{ipNt};$$

thus, as a second approximation,

$$x = be^{iNt} + b_1 e^{ipNt},$$

where b is as above, and b_1 is determined by

$$b_1 [N^2(1-p^2) + ikNp] = -ab^p. \quad (5_1)$$

Thus, b_1 is a complex magnitude, say $b_1 = \rho_1 e^{i\theta_1}$; θ_1 is the phase-difference with respect to the exciting oscillation, and ρ_1 the amplitude of the oscillations of frequency $n_1 = pN$. The former is of no interest here, and as regards the latter

* To speak only of the chief lines of groups or "orders," each of which contains in addition a number of fainter lines.

it is enough to note that ρ_1 is proportional to $|b^p|$, and therefore, by (5),

$$\rho_1 \sim \left(\frac{c_0}{kN} \right)^p.$$

Otherwise the coefficients b , b_1 , and the following ones are of no interest in the present connexion, since we are concerned here with the emitted frequencies only. Substituting the second approximation in the term ax^p in (4) we have

$$(be^{iNt} + b_1e^{iNpt})^p = b^pe^{ipNt} + pb^{p-1}b_1e^{iN(2p-1)t} \\ + \binom{p}{2}b^{p-2}b_1^2e^{iN(3p-2)t} + \dots \quad (6)$$

Thus, as a third approximation,

$$x = be^{iNt} + b_1e^{ipNt} + b_2e^{i(2p-1)Nt} + b_3e^{i(3p-2)Nt} + \dots, \quad (6a)$$

where b , b_1 are as above, and b_2 , b_3 , etc., are easily determined. With regard to these coefficients it will be enough to remark here that as long as $|b_i|$ contains a positive power of c_0/kN , the corresponding line will be strong enough to be visible; now, if

$$p > \frac{r-1}{r}, \quad \dots \quad (7)$$

where r is a positive integer, the last condition will be still fulfilled for the r th line of the spectrum (the fundamental line being the zeroth line). Thus, if p satisfies (7), we can have, theoretically, as many as r lines, not counting the fundamental one.

Returning to (6a), we see that the third approximation gives a series of lines whose frequencies are

$$n_0 = N, \quad n_1 = pN, \quad n_2 = (2p-1)N, \quad n_3 = (3p-2)N, \text{ etc.} \quad (8)$$

Pushing the process to the fourth approximation we should obtain, in addition to (8), such frequencies as p^2N , etc., satellites of the above, whose interest will be shown in a future communication. For the present we shall confine ourselves to the third approximation, *i. e.* to the series of frequencies (8).

This series can, obviously, be written

$$n_i = N - i(1-p)N, \quad i=0, 1, 2, 3, \dots \quad (9)$$

so that its members should succeed one another in *constant frequency-intervals*,

$$\delta n = (1-p)N, \quad \dots \quad (9a)$$

the series extending from the fundamental line either towards

the red or the violet end of the spectrum according as $p \leq 1$ *.

This result seemed at first discouraging, since the discoverer of the resonance spectra, Prof. R. W. Wood, speaks in his Clark University Lecture (of 1909, published 1912) and in his paper in *Physik. Zeitschr.* xii. p. 1204 (1911) of constant wave-lengths, and not frequency-intervals. But a glance at Wood's numerical table, given in the last-mentioned paper (p. 1207), for iodine vapour has sufficed to notice the tendency of $\delta\lambda$ to increase towards the red, so that a detailed comparison of the above law (9) with Wood's measurements has seemed worth undertaking. The result is given in the following Table, in which A, B, C are Wood's three resonance series of iodine vapour excited by light of wave-length $\lambda = 5461.0, 5769.5, 5790.5$ respectively. The n -columns contain the observed frequencies, *i. e.* the reciprocals of Wood's wave-lengths reduced to vacuum, and δn are their differences. Since Wood's wave-lengths are, according to himself, correct only within about $\pm 1 \text{ \AA.U.}$, the last figures in our n -columns are unreliable. It may be well to remark also that the lines made use of for this Table seem to be the chief lines of small groups or "orders" given by Wood in a later publication (*Phil. Mag.* xxvi. p. 828, 1913). To the fainter lines accompanying the chief ones in each "order," I hope to return at a later opportunity.

The fundamental or exciting frequencies N , at the head of each of the three series, are printed in clarendon. At the bottom of the Table are given the mean values of δn .

The constancy of δn along each series is quite satisfactory. Since the observed λ 's are uncertain almost within $\pm 1 \text{ \AA.U.}$, the deviations from $\bar{\delta n}$ are well within the limits of experimental error.

Using the above mean values for δn in (9a), we have for $1-p$, which in some way is a measure of deviation from a perfectly Hookean resonator, for the three series,

$$1-p_A = .0111, \quad 1-p_B = .0118, \quad 1-p_C = .0119, \quad . \quad (10)$$

and for p itself

$$p_A = 0.9889, \quad p_B = .9882, \quad p_C = .9881. \quad . \quad (10a)$$

* Wood's series of iodine, A, B, C, consist of 17, 13, and 13 lines, respectively, on the red side of the corresponding fundamentals, and only of *two* ultra-N lines, in each case, thus showing a marked tendency to a unilateral development. The ultra-N lines can easily be accounted for by an extra term x^{-2} , but since they are but two, we can avoid complication by disregarding them for the present.

<i>i.</i>	Series A.		Series B.		Series C.	
	<i>n.</i>	$\delta n.$	<i>n.</i>	$\delta n.$	<i>n.</i>	$\delta n.$
0	18307		17328		17265	
1	18100	207	17118	210	17049	216
2	17890	210	16916	202	16842	207
3	17673	217	16705	211	16637	205
4	17468	205	16488	217	16641	206
5	17261	207	16283	205	16227	204
6	17049	212	16080	203	16022	205
7	16845	204	15879	201	15812	210
8	16636	209	15687	192	15616	196
9	16431	205	15476	211	15404	212
10	16231	200	15273	203	15199	205
11	16033	198	15091	182	15011	188
12	15837	196	205	208
13	15633	204		205		208
14	194	14681		14595	
15		194				
16	191					
17	191					
	14864					
	Mean, $\overline{\delta n}=202.53$		$\overline{\delta n}=203.61$		$\overline{\delta n}=205.38$	

Thus, the required deviations from unity are small and very nearly equal for the three series of iodine.

Finally, let us remark that, p being greater than 0.98, we could have theoretically, by what has been said in connexion with formula (7), more than 90 lines, and since δn is roughly 200, each of the series could extend even to the very end of the spectrum ($n=0$, or $\lambda=\infty$). This would, obviously, be the case also for smaller values of p . Thus, the condition connected with (7) contains in reality no limitation, and a non-Hookean resonator with the above p proves to be competent to emit the observed resonance-series.

Notwithstanding this we shall not use it any further, but shall avail ourselves of the form (1), as split into the equations (3) for each line separately. The two procedures are equivalent to one another. The coefficients c_1, c_2 , etc., are complex, say, $c_1 = \rho_1 e^{i\theta_1}$, etc.; but since we are not concerned with the phase-difference between the higher and the fundamental line, we can imagine θ_i thrown on $e^{in_i t}$, by shifting the origin of t for each i separately.

Thus, our further developments will be based on the simple equations

$$\ddot{x}_i + k\dot{x}_i + N^2 x_i = c_i e^{in_i t}, \quad i=0, 1, 2, \dots,$$

where c_i are real constants. For the sake of simplicity we shall henceforth drop the suffix i and write, for each line of the spectrum separately,

$$\ddot{x} + k\dot{x} + N^2 x = c \cdot e^{int}, \quad . \quad . \quad . \quad (11)$$

with similar equations for y, z , keeping in mind that $n_0 = N$, $n_1 = pN$, and so on. At first sight it would seem that these simple equations can hardly yield anything new or important; but the following sections will prove the reverse.

3. Emission of Resonance Spectra in a Magnetic Field.

Let the fluorescent vapour be placed in a uniform magnetic field of intensity H . Let us take the z -axis along the lines of the field, and the x and y axes *equally inclined* to the electric force of the exciting light, which is supposed to be rectilinearly polarized. Then, for each line of the spectrum, the coefficients c in (11) will be the same for the x - and the y -equation. Rigorously speaking, the values of the c 's themselves will be slightly changed by the magnetic field*; but, for the present at least, these modifications can be disregarded. Thus, the presence of the magnetic field will give only the

familiar supplementary terms $-\frac{q}{mV}H\dot{y}$, $+\frac{q}{mV}H\dot{x}$ for the right sides of the x - and the y -equations respectively. The equation for z will remain unaltered. It will be enough to consider the case of exciting electric oscillations perpendicular to the magnetic field. Then we can write, simply, $z=0$, and we have, for each line of the spectrum taken separately, the two equations

$$\left. \begin{aligned} \ddot{x} + N^2 x + k\dot{x} - Z\dot{y} &= ce^{int}, \\ \ddot{y} + N^2 y + k\dot{y} + Z\dot{x} &= ce^{int}, \end{aligned} \right\} \quad . \quad . \quad . \quad (12)$$

where $Z = \frac{q}{mV}H$. Remember that n stands here for the

* Especially if we would look on (11) as the equivalent of a non-Hookean system.

frequency of the given spectral line in ordinary circumstances, *i. e.* in absence of the magnetic field. Now the solution of (12) is, apart from damped vibrations, independent of c ,

$$x = Ae^{int}, \quad y = Be^{int}, \quad . \quad . \quad . \quad . \quad (13)$$

where A, B are complex constants.

Thus we have, to begin with, the negative but nevertheless noteworthy result *that there should be no splitting of lines of the fluorescent spectrum, i. e. no ordinary Zeeman effect.*

As far as I know, this conclusion is not contradicted by experience. In Wood and Ribaud's paper on "The Magneto-optics of Iodine Vapour" (Phil. Mag. xxvii. pp. 1009-1018, 1914), which to my knowledge is the only publication bearing on the subject, there is no mention of any splitting of the lines of fluorescent iodine, although the authors have disposed of a field of 20,000 gauss*.

The reader will notice that the ordinary equations of Lorentz's elementary theory of the Zeeman effect for common spectrum-lines differ from (12) by the absence of ce^{int} (and of the terms $k\dot{x}$, $k\dot{y}$). This is the reason why Lorentz's equations give a splitting, while our equations refuse to give any trace of it.

However, although none of the lines of a resonance-spectrum can be expected to be split by the magnetic field, each of them will undergo a change of intensity and of character, *i. e.* of its state of polarization. In fact, substituting (13) in (12) we have, for the constants A, B ,

$$\left. \begin{aligned} A &= \frac{c}{D} [N^2 - n^2 + in(k + Z)], \\ B &= \frac{c}{D} [N^2 - n^2 + in(k - Z)], \\ \text{where } D &= (N^2 - n^2 + ikn)^2 - Z^2 n^2. \end{aligned} \right\} \quad . \quad . \quad . \quad (14)$$

These are the (complex) values to be inserted in (13). The intensities will be given by $|A|$ and $|B|$, and this will occupy our attention a little later (*cf.* Section 6). The state of polarization, which interests us here, is entirely determined by the quotient $y/x = B/A$, and this is, by (14),

$$\frac{y}{x} = \frac{N^2 - n^2 + in(k - Z)}{N^2 - n^2 + in(k + Z)} = \rho e^{i\theta}, \text{ say.} \quad . \quad . \quad (15)$$

* No direct mention is made, in the quoted paper, of the absence or presence of splitting of the fluorescent lines. Wood and Ribaud's statement (p. 1015) "that if the Zeeman effect exists, it is less than 0.01 Å.U. for a field of 20,000 gauss" refers to the *absorption* lines of iodine vapour.

The meaning of this equation is obvious. Let the vector \mathbf{E} denote $\frac{q}{m} \times$ incident electric force, apart from the factor e^{iNt} , and let \mathbf{r} be the fluorescent light vector $(x, y, z)^*$. Then, in absence of the magnetic field, we have $y/x=1$, i. e. $\theta=0$ and $\rho=1$, which means rectilinear polarization and \mathbf{r} parallel to \mathbf{E} . Owing to the magnetic field these rectilinear fluorescent oscillations become, in general, *elliptic*, and the major principal axes of the ellipses are turned away from \mathbf{E} through certain angles ζ , in the plane x, y . These angles ζ are easily expressed by ρ alone, and the ratio of axes, say $b:a$, by θ alone. In fact, if ζ be the angle contained between \mathbf{E} and the a -axis, we have

$$\tan \zeta = \frac{1-\rho}{1+\rho}, \quad \frac{b}{a} = \left| \tan \frac{\theta}{2} \right|. \quad . \quad . \quad . \quad . \quad (16)$$

And since, by (15), both ρ and θ are functions of n ($n=n_0, n_1, n_2$, etc., for the fundamental, first, second, etc., lines), *the ellipticity and the rotation ζ will be different for the different lines of the resonance-spectrum*, having $n_0=N$ for its fundamental. The sense of the rotation round the

magnetic field is easily ascertained. Since $Z = \frac{q}{mV} H$, it is one way or the other according as the resonator, which need not necessarily be an electron, is positively or negatively electrified. Details concerning ζ and b/a , at least for the zeroth and the first line of the series, will be given presently.

Such then should, according to our theory, be the behaviour of the resonance lines in a magnetic field. As far as I know, nothing of the kind has yet been observed, but I have reasons to expect that the predicted rotations ζ will be detected even with moderate fields H .

The reader may be tempted to simplify (15) by neglecting k , —at least, for the first and the higher lines. I must warn him, however, not to do so, unless he is willing to neglect Z as well. In fact, we shall see further on that, say, for $H=10,000$ gauss, $|Z|$ and k are not only of the same order but very nearly equal to one another.

* In our present case $z=0$.

4. *Magnetic Rotation of the Fundamental Line.*

For the fundamental line of the resonance spectrum we have $n=N$ and therefore, by (15),

$$\frac{y}{x} = \frac{k-Z}{k+Z} = \rho, \text{ and } \theta = 0,$$

and, by (16), $b/a=0$ and

$$\tan \zeta = \frac{Z}{k}. \quad . \quad . \quad . \quad . \quad . \quad (16_0)$$

Thus, the fundamental line will continue to be *rectilinearly polarized*, but its plane of polarization will be *rotated round the magnetic field through an angle* $\zeta_0 = (\mathbf{r}, \mathbf{E}) = \arctan (Z/k)$.

It will be remembered that $Z = \frac{q}{mV} H$, where H is the absolute value or the intensity of the magnetic field. Thus Z is positive or negative according as $q > 0$ or $q < 0$. This settles the question of the sense of rotation. Whether the charge q is positive or negative, can only be decided by experiment.

Let us, henceforth, write shortly

$$q/mV = \eta,$$

so that $Z = H\eta$, and

$$\tan \zeta_0 = H \frac{\eta}{k}. \quad . \quad . \quad . \quad . \quad . \quad (16_0)$$

If ζ_0 exists, and is great enough (for H of the order 10^4) to be measured, the ratio η/k , and therefore the product

$$\eta\tau,$$

will be found experimentally (where $\tau = 2/k$ is the relaxation- or the extinction-time of the resonator). In order to find η and τ separately, some further connexion is required, which might be obtained from the first or the higher lines of the resonance-spectrum.

Before passing on to these it will be convenient to distinguish here a certain, particular, intensity H , viz. that intensity which makes $\zeta_0 = \pm 45^\circ$, *i. e.* that intensity which makes

$$\begin{aligned} Z &= k \quad \text{for } q > 0, \quad . \quad . \quad . \quad . \quad . \quad (17) \\ Z &= -k \quad \text{for } q < 0. \end{aligned}$$

Let us denote this particular intensity of the field by H_c , so that

$$H_c = k : |\eta|. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

We shall see later that, for iodine vapour, H_c is of the order of 10^4 gauss.

5. *Magnetic Rotation and Ellipticity of the First and the Higher Lines.*

For any line of the resonance series, and for any field H , ρ and θ are determined by the general formula (15), which gives at once

$$\rho = \frac{\{[(N^2 - n^2)^2 + (k^2 - Z^2)n^2]^2 + 4Z^2n^2(N^2 - n^2)^2\}^{\frac{1}{2}}}{(N^2 - n^2)^2 + (k + Z)^2n^2}, \quad (18)$$

$$\tan \theta = -\frac{2Zn(N^2 - n^2)}{(N^2 - n^2)^2 + (k^2 - Z^2)n^2} \dots \dots (19)$$

Here we have to insert $n = Np$ for the first, $n = N(2p - 1)$ for the second line of the spectrum, and so on. Introducing (18) and (19) into (16), we obtain the corresponding rotations ζ and the ratios b/a of the axes of the ellipses described by the end-point of the fluorescent vector r , for any line and for any field H .

It will be enough to develop the final formulæ for ζ , $\frac{b}{a}$, for the special intensity $H = H_c$ of the magnetic field, when the above formulæ are considerably simplified. In fact, if the resonator carries, say, a negative charge *, then, by (17), $Z + k = 0$, and (18), (19) become

$$\tan \theta = \frac{2nk}{N^2 - n^2}, \quad \rho = \frac{1}{|\cos \theta|}.$$

Substituting these values in (16) we have, finally, with a magnetic field H_c (which would produce a rotation of the fundamental line through $\zeta_0 = 45^\circ$) for any of the higher lines of the spectrum :

$$\left. \begin{aligned} \tan \zeta &= -\frac{1 - |\cos \theta|}{1 + |\cos \theta|}, \quad \frac{b}{a} = \left| \tan \frac{\theta}{2} \right|, \\ \text{where} \quad \tan \theta &= \frac{2nk}{N^2 - n^2}. \end{aligned} \right\} \dots \dots (20)$$

Thus, all the lines, beginning with the first, will be *elliptically polarized*, and their a -axes will make with E the angles ζ_1 , ζ_2 , etc., expressed by (20) with $n = Np$, $N(2p - 1)$, etc., respectively. All of these angles will differ more or less from $\zeta_0 = 45^\circ$.

If either ζ or b/a could be measured on any one of these lines, in the field H_c , then θ , and therefore by the third of

* If it is positively charged, $Z = k$, and we obtain the same final formulæ if the rôles of x , y are exchanged, as will be seen at once from (15). Thus, the amount of rotation and the shape of the ellipse will be the same, but with reversed sense.

equations (20), the value of $k = \frac{2}{\tau}$ would be found. And since $\eta\tau$ can be measured on the fundamental line, we should have η as well as τ .

To illustrate (20) a little further, take the first line of the resonance-spectrum, $n = pN$. Let T be the fundamental period of oscillation, and τ the extinction-time, as before. Then $\theta = \theta_1$ is given by

$$\tan \theta_1 = \frac{2p}{\pi(1-p^2)} \cdot \frac{T}{\tau} \quad (21)$$

For the second line we have only to replace p by $2p-1$, and for the third line, by $3p-2$, etc. Thus

$$\tan \theta_2 = \frac{2(2p-1)}{\pi[1-(2p-1)^2]} \cdot \frac{T}{\tau}, \quad (21')$$

$$\tan \theta_3 = \frac{2(3p-2)}{\pi[1-(3p-2)^2]} \cdot \frac{T}{\tau}, \quad (21'')$$

and so on. Thus in the case of the iodine series A, for which $p = 0.9889$,

$$\tan \theta_1 = 28.51 \frac{T}{\tau}; \quad \tan \theta_2 = 14.18 \frac{T}{\tau}; \quad \tan \theta_3 = 9.396 \frac{T}{\tau},$$

and so on. Now, T/τ is certainly a small fraction, so that θ_1 is a small angle, θ_2 nearly one-half of θ_1 , and so on. Thus, by (20), the angles ξ_1, ξ_2 , etc., become, apart from the sign, $\frac{1}{4}\theta_1^2, \frac{1}{4}\theta_2^2$, and so on, *i. e.* of the order $10^3(T/\tau)^2, 3 \cdot 10^2(T/\tau)^2$, etc. And as for the ratio b/a , this can now be written

$$b/a = \frac{\theta}{2} = \frac{1}{2} \tan \theta,$$

i. e.

$$\frac{b_1}{a_1} = 14.26 \frac{T}{\tau}; \quad \frac{b_2}{a_2} = 7.09 \frac{T}{\tau}; \quad \frac{b_3}{a_3} = 4.70 \frac{T}{\tau}, \text{ etc.,} \quad (22)$$

and may, if τ does not greatly exceed 10^3T , be just detectable. The decision must be left to the experimental physicist. At any rate, b/a would be for the second line only about one-half, and for the third line one-third of what it is for the first line, and so on.

It will be remembered that (20), and therefore also (21), (22) are valid for $H = H_c$, as defined above, and this field is, as we shall see in the next section, very likely equal to about 10^4 gauss. For other field intensities of this order the behaviour of the lines n_1, n_2 , etc., will be very much the same. For *any* field we have the rigorous equations (18), (19) with (16), determining completely the "character" of each line of the resonance-spectrum.

6. *Destruction of Fluorescence by Magnetic Field.*

I have taken the liberty of copying the title of this section, literally, from Wood and Ribaud's paper already quoted *, first, because it is a short title and, second, in order to better recall the reader's attention to that experimental paper. It will be well to quote also a few lines from Wood and Ribaud's paper. On p. 1017 the authors say :—

“Under these conditions we estimated the reduction of intensity [of iodine vapour fluorescence] *to amount to fully 90 per cent. with a field of 30,000 gauss*, and it is probable that with a field of 50,000 the fluorescence would be practically destroyed. . . . *No obvious explanation of the effect of the field in reducing the intensity of and ultimately practically destroying the fluorescence suggests itself*” †.

Now, such a dependence of intensity of fluorescence upon the magnetic field has already been hinted at, just after the equations (14). We shall now see that these equations not only account for the observed effect, but, bringing it into close connexion with the magnetic rotation ζ (and ζ_0 especially), make the explanation so plain that it can be seen almost immediately. In fact, let I be the intensity of any line (taken separately), when the vapour is placed in a field H . Then, disregarding ellipticity,

$$2I = |A|^2 + |B|^2,$$

where A, B are given by (14). It will be enough to consider here, in detail, the fundamental line only. Writing, in (14), $n = N$, we have

$$|A| = \frac{c_0}{N} \cdot \frac{k+Z}{k^2+Z^2}, \quad |B| = \frac{c_0}{N} \cdot \frac{k-Z}{k^2+Z^2},$$

whence, for any H ,

$$I = \frac{c_0^2}{N^2(k^2+Z^2)},$$

and, therefore, for $H=0$, $I = c_0^2/N^2k^2$. Thus, the ratio of the two intensities, $J_0 = I_{(H)} : I_{(0)}$,

$$J_0 = \frac{1}{1+(Z/k)^2}, \quad . \quad . \quad . \quad . \quad (23)$$

where the suffix $_0$ is to remind us that the *fundamental* line is concerned. Thus, with increasing intensity of the magnetic field the fundamental fluorescence tends to nothing.

* “Destruction” stands, of course, in general, for “diminution of intensity.”

† The few words are italicized by the present author.

This seems very simple by itself. But the matter becomes even more simple when it is remembered that, by (16₀), $Z/k = \tan \zeta_0$, where ζ_0 is the rotation or the angle \mathbf{E} , \mathbf{r} , so that (23) becomes

$$J_0 = \cos^2 \zeta_0. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Now, this is as familiar as the most common of our everyday-life mechanical experiences. In absence of the rotating-agent (field \mathbf{H}) the force \mathbf{E} pulls \mathbf{r} in its own direction, and is thus most effective; and when these make with one another an angle ζ_0 , then $\mathbf{E} \cos \zeta_0$ only is operative, whence, and so on. The limit of magnetic rotation is $\zeta_0 = 90^\circ$, but before this is reached, the fluorescence vanishes. We might even have started the whole investigation by such a line of reasoning.

At any rate, it seems now indubitable that the magnetic "destruction" of fluorescence (at least as far as the fundamental line is concerned) is intimately connected with a magnetic rotation, and since the former is fully established, I have but little doubt that the rotation will also be found by appropriate experiments. In fact, if $J_0 = \frac{1}{10}$, as in Wood and Ribaud's experiments*, with $H = 30,000$, then, by (24), the angle of rotation can be expected to be as huge as

$$\zeta_0 = 71^\circ.565.$$

Again, by (23), we should have, for the same magnetic field, $(Z/k)^2 = 9$, *i. e.*

$$Z = \pm 3k,$$

and therefore for $H = 10,000$ gauss,

$$Z = \pm k.$$

Thus, our above H_c would be 10,000 gauss, or, at any rate, very nearly so. The signs correspond to $\eta \gtrless 0$, as in (17).

For the relative intensity J_1 of the *first* line of the series we have, by (14), writing $n = Np$, the rather complicate formula

$$J_1 = \frac{[N^2(1-p^2)^2 + k^2][N^2(1-p^2)^2 - k^2 - Z^2]}{N^4(1-p^2)^4 - 2N^2(1-p^2)^2 \cdot (Z^2 - k^2) - (k^2 + Z^2)^2}, \quad (25)$$

for any magnetic field H . The corresponding values of J_2 , J_3 , etc., for the second, third, etc., lines are obtained from

* Properly speaking, Wood and Ribaud's experiments, dealing with the whole fluorescence, authorize us to assume not $J_0 = 1/10$, but

$\frac{\Sigma I_{(H)}}{\Sigma I_{(H=0)}} = \frac{1}{10}$; but in absence of better information we assume here

that the fundamental line taken by itself is reduced, nearly, in the same ratio. *It would be interesting to investigate spectrophotometrically the diminution of intensity of each line separately.*

(25) by writing, instead of p , $(2p-1)$, $(3p-2)$, and so on. In particular, for $H=H_c \doteq 10,000$, *i. e.* for $Z=\pm k$, we have, neglecting $(1/\tau)^4$,

$$J_1 = 1 - \frac{1}{\pi^2(1-p^2)^2} \cdot \left(\frac{T}{\tau}\right)^2, \quad . \quad . \quad . \quad (25 a)$$

and similar expressions for J_2 , etc., with $(2p-1)$, etc., instead of p . Returning to (20), and remembering that θ_1 , θ_2 , etc., are small angles, we can put (25 a), etc., into the more suggestive form

$$J_1 = 1 - \frac{\sin^2 \zeta_1}{p^2}, \quad J_2 = 1 - \frac{\sin^2 \zeta_2}{(2p-1)^2}, \text{ etc.} \quad . \quad (26 a)$$

If the angles of rotation, ζ_1 , ζ_2 , etc., are small (as it would follow from our above considerations) then J_1 , J_2 , etc., would differ but little from unity. That is to say, a field which reduces the fundamental line to about $\frac{1}{10}$ of its intensity would weaken but little the remaining lines of the spectrum, while Wood and Ribaud's experiments seem to indicate that the fluorescence is strongly reduced as a whole. Thus, our theory seems to be just to the fundamental line, but not so to the higher lines of the resonance-spectrum. The reason of this outstanding difficulty seems obvious. In fact, formulæ (25), etc., for the first and higher lines have been deduced from the equations (12) under the assumption that the coefficients c_1 , c_2 , etc., have the same values in the presence of H as in the absence of any magnetic field*. Now, as has already been mentioned, this is only approximately true, and may well require a considerable correction when such strong fields as $H_c=10,000$ gauss are in question. The results, however, of my theoretical investigations concerning this matter are better postponed until spectrophotometric measurements of the magnetic reduction of the separate lines of the series are available.

Such measurements would also enable us to find the time τ , and since ζ_0 gives $\eta\tau$, we should be able to estimate $\eta=q/mV$, which need not, thus far, be the electronic ratio. Meanwhile, it may be interesting to see what τ is like *if* η is of the order of the electronic ratio. Now, returning to (23), and taking again for J_0 Wood and Ribaud's value $\frac{1}{10}$, for $H=30,000$ gauss, we have $10,000|\eta|=k=2/\tau$, *i. e.*

$$\tau = \frac{2 \cdot 10^{-4-7}}{1.86} \doteq 1.1 \cdot 10^{-11} \text{ sec.}, \quad . \quad . \quad . \quad (27)$$

* The reader will notice that this concerns only c_1 , c_2 , etc., but not c_0 , the coefficient for the fundamental line.

and if T be the fundamental period of the iodine series A ($\lambda = 5461 \text{ \AA.U.}$),

$$\frac{T}{\tau} = 1.7 \cdot 10^{-4}, \text{ if } \eta \text{ electronic. . . . (27a)}$$

On the other hand, if η approaches more the value of the electrolytic ratio, say, for hydrogen, then T/τ would be of the order 10^{-8} . In the former case τ would amount to about 5900 T , and in the latter to a hundred million T . Intermediate and other values are also possible. The decision must be left to the experimental physicists.

In connexion with this subject one more remark. In Wood and Ribaud's paper we read, p. 1016, *loc. cit.*: "The effect of the magnetic field in reducing the intensity of the fluorescence becomes more marked as the vapour-pressure of the iodine is diminished." This would mean, by (23), that with *decreasing pressure the time τ is lengthened* (or the "friction"-coefficient diminished), and this seems quite plausible. In fact, there are reasons to believe that extinction is due not only to emission but also due to the encounter of the molecules or atoms, and then τ ought to contain a term proportional to the mean time elapsing between successive encounters. Now, the latter time is certainly lengthened when, *ceteris paribus*, the gas or vapour is rarefied.

7. Short Note on the Amount of Polarization.

In the above treatment of the subject we have tacitly assumed that the atoms or molecules of the fluorescent vapour, in which the resonators are embedded, are fixed, *i. e.* non-rotating, with respect to the apparatus, and therefore with respect to the direction of the exciting electric vector \mathbf{E} . The result has been that rectilinearly polarized incident light has given, in absence of magnetic field*, fluorescent oscillations \mathbf{r} parallel to \mathbf{E} , *i. e.* totally polarized. On the other hand, Wood has found experimentally a partial polarization only (*Phys. Zeitschr.* xii. p. 1209), viz. for iodine vapour only 17 per cent. and for sodium vapour 20 per cent., "and in a very rarefied vapour and at a low temperature even 30 per cent."

Now, it occurred to me that this state of things could be accounted for by assuming that the atoms or molecules carrying the resonators are endowed with rotational motion,

* To this simple case will our attention be confined here. The same problem with a superposed magnetic field will be taken up at a later opportunity.

provided that the mean time of revolution is comparable with τ , and, of course, that the axes of rotation are haphazardly distributed.

Without entering upon the details of my calculations, which in vector language have assumed a very simple form, I shall quote here the final result only. Let, for the sake of simplicity, the absolute value of the angular velocity, ω , be the same for all atoms or molecules, but let all directions of axes of rotation be equally represented. Then the amount of polarization P , ranging from 0 to 1, is *nil for rays parallel to the ray n of the exciting (unpolarized) light*, and, for any ray *perpendicular to n* , for the fundamental line of the spectrum,

$$P = \frac{1 + \frac{1}{3}\alpha^2}{1 + \frac{5}{3}\alpha^2 + \frac{4}{15}\alpha^4}, \quad \alpha = \omega\tau, \quad . \quad . \quad . \quad (28)$$

where τ has the same meaning as throughout the paper, and α is the angle through which the atoms are turned during the time τ . [In reality, the angles α will not be equal but distributed round their average $\bar{\alpha}$, say, according to the error-law. But, for the present, we shall content ourselves with the above formula, keeping in mind that $\omega\tau$ is not precisely the average $\bar{\omega\tau}$]. For $\omega = 0$ we have $P = 1$, *i. e.* full polarization. When ω (or, better, $\omega\tau$) increases, P decreases, tending to nothing. If, as in one of Wood's more recent experiments with iodine vapour, $P = 0.64$, then, if the whole of the missing 93.6 per cent. can be thrown upon molecular rotation, formula (28) would require that

$$\omega\tau = 4.072 \text{ radians.} \quad . \quad . \quad . \quad (29)$$

This is a huge angle (warning us not to neglect α^4 in the above formula). If η is of the electronic order, or τ equal to about 5900 T, as in our example of section 6, then $\omega T = 0^\circ 2' 23''$. That is to say, during one period of luminous oscillation, of the fundamental period, the atoms would turn round through less than $2\frac{1}{2}$ angular minutes, so that their revolutions would still be very slow in comparison with the fluorescent oscillations, although comparable with the quickness of their extinction. Owing to the small value of ωT , the equations leading to (28) have been considerably simplified. Further details concerning this subject will be given in a later publication. Here but one more remark. If the ratio q/m has the electronic value, then, by (27) and (29),

$$\omega = 3.81 \cdot 10^{11} \text{ radians per second.}$$

This seems a prodigious angular velocity. Now, it is very

remarkable that the law of *equipartition of energy* of the kinetic theory of gases gives an angular velocity of the same order. In fact, taking for the mean translational velocity of iodine molecules at 0° C. the value $1.64 \cdot 10^4$ cm./sec. and for the molecular radius $2 \cdot 10^{-8}$ cm., and attributing to the molecules two exchangeable rotational momentoids, against three translational ones, I find for the average angular velocity

$$\bar{\omega} = 1.59 \cdot 10^{12},$$

which is well comparable with the above value.

I gladly take the opportunity of expressing my thanks to Professor R. W. Wood, who has drawn my attention to these beautiful phenomena and has encouraged me to take up the subject of resonance-spectra from the theoretical standpoint.

July 27, 1916.

XXXIII. *The Diurnal Variation of Atmospheric Electrical Quantities.* By E. H. NICHOLS, B.Sc., A.R.C.Sc.*

THE diurnal variation of the electrical conductivity has been investigated recently by Kahler† and Dorno‡ by means of a continuously recording dissipation apparatus, while Gockel§ has obtained eye observations of conductivity and electric charge. For the investigations here detailed, two Ebert Electrometers|| have been used to record the respective positive and negative electric charges, while the Wilson Compensating Gold Leaf Electroscope¶ measured the conductivity and air-earth current, the potential being obtained from a Kelvin Water-Dropper apparatus. The method employed for the Wilson instrument was a variation of the original method adopted by C. T. R. Wilson. There are three operations involved:—

(a) The charge on the test-plate is first found by removing the cap, and replacing it quickly after momentarily earthing

* Communicated by the Director of the Meteorological Office.

† *Ergebnisse der Met. Beob. im Potsdam*, 1909, 1911.

‡ "Studie über Licht und Luft des Hochgebirges," 1911 (Davos).

§ Mache u. von Schweidler, *Atmosphärische Elektrizität*, 1909, p. 90; A. Gockel, *Met. Zeit.* xxiii. pp. 53, 339 (1906); xxv. p. 9 (1908); *Archives des Sc. phys. et nat.*, Sept. 1913.

|| *Phys. Zeit.* viii. 8. p. 246; viii. 16. p. 527; x. 8. p. 251.

¶ *Proc. Camb. Phil. Soc.* xiii. p. 184 (1906). M. O. Geophys. Mem. No. 7.

the plate. The reading of the gold leaf is taken (g_1), and also after again earthing (g_0). During this operation the compensator must be fully out. The charge on the plate is represented by ($g_1 - g_0$).

(b) The charge lost in five minutes from the test-plate is obtained by removing the cap and adjusting the compensator until the reading of the gold leaf is again g_0 , and remains at g_0 for the five-minute period. The cap is then replaced and the compensator fully drawn out. The gold leaf is again read (g_2) and also after earthing (g_0'). The leakage in the test-plate due to the air-earth current is represented by ($g_2 - g_0'$).

(c) The first operation (a) is repeated to find the charge on the plate ($g_3 - g_0''$).

The conductivity is given by

$$\lambda = 5.89 \{ (g_2 - g_0') / (g_1 + g_3 - g_0 - g_0'') \} \times 10^{-25} \text{ e.m.u.,}$$

where 5.89 is a universal constant, and no instrumental calibration is involved*. The formula may be proved as follows. The surface density of the charge on the test-plate is given by

$$\sigma = \text{electric force} / 4\pi = \frac{1}{1200\pi} \text{ e.s.u.}$$

for a potential gradient of 100 volts per metre (1 volt per cm.). Supposing the loss of charge on the test-plate is p per cent. per minute, then the current is equal to

$$\frac{1}{1200\pi} \times \frac{p}{100} \times \frac{1}{60} \text{ e.s.u. per second} = 1.473p \times 10^{-17} \text{ amp./sec.}$$

For a potential of v volts per cm., the current

$$= 1.473pv \times 10^{-17} \text{ amp./cm.}^2$$

The percentage loss per minute (for five-minute intervals)

$$\begin{aligned} &= p = 100(g_2 - g_0') / [5 \times \frac{1}{2}(g_1 + g_3 - g_0 - g_0'')] \\ &= 40(g_2 - g_0') / (g_1 + g_3 - g_0 - g_0''). \end{aligned}$$

* This assumes the constancy in scale value for the divisions in the eyepiece.

Therefore the current

$$= 5.89v\{(g_2 - g_0') / (g_1 + g_3 - g_0 - g_0'')\} \times 10^{-16} \text{ amp./cm.}^2,$$

and the conductivity

$$= 5.89\{(g_2 - g_0') / (g_1 + g_3 - g_0 - g_0'')\} \times 10^{-25} \text{ e.m.u.}$$

The mobility, or the ionic velocity under the influence of unit gradient of 1 volt per cm., can be calculated from the conductivity (λ) and the positive charge (E_+). It is considered that the conductivity measured by the Wilson electrometer is that due to the positive ions*, so that $\lambda_+ = E_+ \mu_+$, where μ_+ is the mobility of the positive ions.

Observations of the diurnal variation of the electrical elements were commenced at Kew Observatory in May 1914, and continued during June and July on four days of each month (Table I.), with an additional day in August (Table III.). During May and June hourly observations were taken, from 7 A.M. to 8 P.M., each extending for about 15 minutes, while during July and August the times were from 4 A.M. to 8 P.M. The weather conditions were exceptionally favourable, only a very few days being affected by rain. As the instruments must be suitably mounted in the open, any rain beyond a slight drizzle prevents continued observation. Synchronous values of the potential were obtained from the Kelvin water-dropper, a factor being applied to convert the values to the potential gradient in the open.

A few hourly results are missing for the different elements. In these cases the mean value for the preceding and following hour has been used, while in one or two instances hourly values have been extrapolated by the usual method.

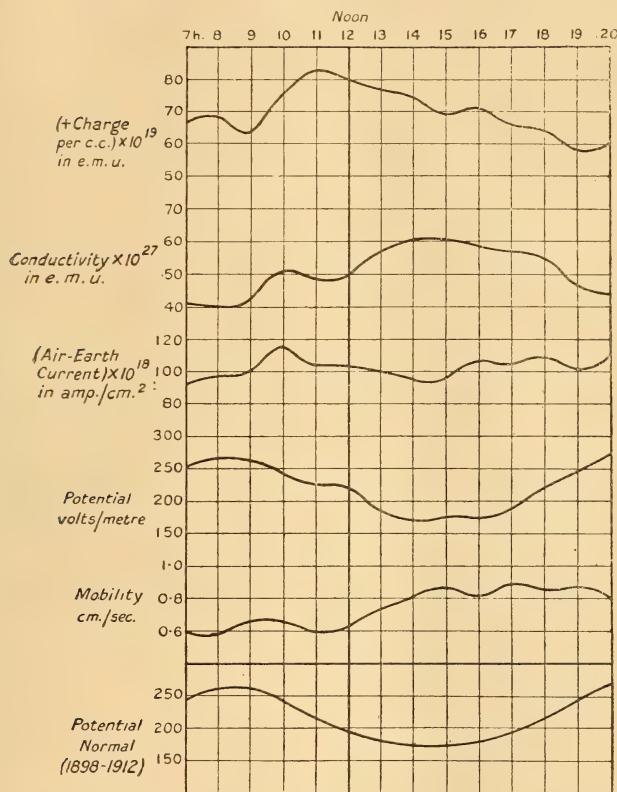
The mean summer diurnal variation for Kew obtained from twelve days in May, June, and July is shown in fig. 1. There is a maximum for the positive charge at 11 A.M. and for the conductivity at 2 P.M. The air-earth current shows no important variation, but the mobility gives a decided increase after noon. The potential curve indicates a definite minimum in the afternoon, and agrees closely with the normal for 1898-1912 † for those months, thus proving that the variation of the elements obtained should be generally applicable to electrically quiet days.

* Lutz, *Luftelektrische Messungen am München*, 1911.

† C. Chree, *Phil. Trans. A.* ccxv. p. 141.

Fig. 1.—Kew, 1914.

Mean Diurnal Variation (May, June, July) (unsmoothed).



Monthly Means for May, June, and July, 1914.

From the results for May given in Table I., there is a minimum for the positive charge at 9 A.M. and two almost equal maxima at 11 A.M. and 2 P.M., while the conductivity shows a well-marked minimum at 8 A.M. and maximum at 4 P.M. The air-earth current shows a minimum at 8 A.M., which corresponds with a maximum of potential. The potential is low during the afternoon with a minimum at 1 P.M., while the mobility increases gradually during the day. For June the positive charge has a maximum value at noon, and two minima at 9 A.M. and 7 P.M., which correspond with the times for minima in the conductivity. As usual, the maximum for the positive charge precedes that for conductivity by a few hours. The air-earth current varies somewhat similarly to the potential, and the mobility increases during the day to a maximum at 5 P.M.

The results for July give a more detailed account of the electrical conditions, as the negative charge was also measured and observations commenced three hours earlier. The range for the conductivity is considerably less than for the other months, there being a maximum at 1 P.M. and a minimum during the night. There is a curious well-marked minimum at 11 A.M. which corresponds to a maximum electric charge. Individual results show a minimum at 11 A.M. for July 23, 28, and 30, while for July 21 there is one at 9 A.M. The minimum at 11 h. appears in the 3-monthly means, and the depression from 10 A.M. to 1 P.M. is more pronounced still in the Eskdalemuir results discussed below, so that it is probably a real phenomenon.

Although the data for the individual days cannot be discussed in detail, a few points may be noticed. There appears to be no definite effect of precipitation on the electrical quantities as a general rule, but there are cases in which a considerable decrease in electric charge or conductivity has been observed to follow rain (see Table III.). In the few measurements made before and after thunderstorms, the electrical values obtained show nothing abnormal. Usually there is a similarity between the variation of the conductivity and the positive and negative charges, the two latter showing a close agreement. There is also a definite minimum about midday in the ratio of the positive and negative charges $\left(\frac{E_+}{E_-}\right)$, which is obviously associated with the solar effect.

A partially successful attempt was made on June 29-30, 1914, to obtain the electrical variation for the complete 24 hours, observations being commenced at 7 A.M. on June 29 and continued hourly till 7 A.M. June 30 (see Table II.). After 9 P.M., however, the Wilson electroscope failed owing to insulation trouble, which is a great difficulty in night work. For such work an electric torch must be used for reading, oil lamps being fatal to results, because of the production of conducting gases. The diurnal variation of electric charge gave some indication of a double period with primary maximum and minimum about 4 P.M. and 4 A.M. respectively. The values for both charges were lower at night than during the day. The ratio of the charges $\left(\frac{E_+}{E_-}\right)$ shows a remarkable maximum at 3 A.M. and a minimum at 12 noon. Such an outstanding maximum might appear fortuitous, but similar examples of high maxima occurred on July 23 at 4 A.M., July 28 at 5 A.M., and August 4 at 5 A.M.

TABLE I.—Kew, 1914. Mean Monthly Values for Diurnal Variation.
(Positive charge per c.c.) $\times 10^{19}$ in e.m.u.*

No. of Days.	4 h.	5	6	7	8	9	10	11	12	(noon)				16	17	18	19	(8 p.m.) 20
										13	14	15	16					
May	4	73	59	54	74	99	85	91	99	90	91	78	79	63	56	
June	4	65	82	58	78	71	87	84	78	81	79	64	67	51	58	
July	4	52	46	62	64	77	75	80	68	56	48	37	44	49	47	44	50	
<i>(Negative charge per c.c.) $\times 10^{19}$ in e.m.u.</i>																		
July	4	14	13	24	33	40	48	57	52	38	31	21	30	23	20	22	18	
Ratio of Charges (July)	3.7	3.4	2.6	1.9	1.9	1.6	1.4	1.3	1.5	1.6	1.8	1.5	2.2	2.3	2.0	2.7	
<i>Conductivity $\times 10^{27}$ in e.m.u.</i>																		
May	4	31	25	29	41	45	47	55	52	59	60	48	49	44	32	
June	4	41	48	42	55	52	49	60	74	74	74	80	74	55	70	
July	4	40	44	50	46	57	55	50	55	59	56	50	42	44	41	41	30	
<i>Potential, volts/metre.</i>																		
May	4	290	310	310	260	210	200	180	180	190	200	250	300	340	300	
June	4	260	270	250	250	250	260	200	180	180	200	170	170	230	300	
July	4	130	140	200	220	220	220	220	210	180	140	160	120	150	220	170	220	
<i>(Air-Earth current) $\times 10^{18}$ in amp./cm.²</i>																		
May	4	79	75	80	96	82	85	87	86	101	109	116	124	122	89	
June	4	105	118	101	131	116	114	108	116	106	133	127	117	114	173	
July	4	52	67	89	102	120	117	110	111	102	84	78	78	69	86	68	66	

* It should be noticed that 10^{19} is the factor employed here instead of 10^{20} as used in the "Geophysical Journal" of the Meteorological Office.

TABLE II.—*Kew*. Observations for June 29–30, 1914.
June 29.

Time →	7 h.	8	9	10	11	12	13	14	15	16	17	18	19
						(noon)							
(+ Charge) $\times 10^{19}$ in e.m.u.....	97	135	97	101	84	97	116	82	133	144	112	137	103
(- Charge) $\times 10^{19}$ in e.m.u.....	43	53	48	40	52	67	59	56	59	53	58	51	26
Conductivity $\times 10^{27}$ in e.m.u.	60	73	57	69	81	69	95	112	130	105	125	115	67
Potential, volts/metre	240	250	210	180	140	130	100	80	70	80	80	70	90
(Air-Earth Current) $\times 10^{18}$ in amp./cm. ² ...	140	180	120	130	110	90	90	90	100	90	100	90	60

June 30.													
Time →	20	21	22	23	(Mid ⁿ .)		1	2	3	4	5	6	7.
					24	25							
+ Charge	148	93	74	91	95	70	76	84	61	72	61	59	
- Charge	37	23	51	23	21	18	16	6	10	12	11	15	
Conductivity	121	121	
Potential	100	80	80	70	110	110	120	130	150	140	150	170	
Air-Earth Current	120	100	

TABLE III.—*Kew*. Observations for August 4, 1914.

	4 h.	5	6	7	8	9	10	11	12	(noon)	13	14	15	16	17	18	19	20	(8 p.m.)
(+ Charge) $\times 10^{19}$ in e.m.u....	87	51	42	53	61	61	84	80	93	97	106*	80	137	116	112	68	59		
(- Charge) $\times 10^{19}$ in e.m.u....	23	7	12	8	42	57	57	63	67	69	57	48	47	56	49	37	26		
Conductivity $\times 10^{27}$ in e.m.u.	74	49	73	35	83	124†	36	54	55	72	83	76	67	90	88	68	42		
Potential, volts/metre.....	140	130	140	190	180	80	190	180	130	190	160	80	130	130	130	180	320		
(Air-Earth Current) $\times 10^{18}$ } in amp./cm. ²	110	60	100	70	150	100	70	90	70	140	130	60	90	110	110	120	130		

† Slight drizzle.

* Shower at 14.30.

TABLE IV.—*Eskdalemuir*. August 1914. Mean Values (unsmoothed). (Units as in Table III.)

	No. of Days.	4 h.	5	6	7	8	9	10	11	12	(noon)	13	14	15	16	17	18	19	20	(8 p.m.)
+ Charge	7	73	58	75	65	54	54	62	66	60	62	60	52	53	59	50	43	33		
- Charge	7	26	17	17	15	18	21	27	25	31	37	32	28	22	34	22	20	15		
Conductivity	6	69	61	65	73	71	106	127	111	108	116	114	96	97	99	75	75	67		
Potential	7	250	330	340	340	240	180	150	150	150	140	130	130	130	130	160	280	320		
Air-Earth Current ...	6	143	148	172	190	151	173	193	174	157	161	149	155	110	122	113	182	152		
Ratio of Charges	7	2.8	3.4	4.3	4.3	3.0	2.5	2.3	2.7	2.0	1.7	1.9	1.9	2.4	1.7	2.2	2.2	2.3		

Observations at Eskdalemuir during August 1914.

As Kew Observatory is only a few miles from London and practically at sea-level, it will be advantageous to compare the electrical variation with that for Eskdalemuir (237 metres above M.S.L.) in Dumfriesshire, removed from disturbing artificial conditions. The importance of the electrical investigation was increased by the fact that it was possible to include a day of partial solar eclipse, and thus compare the diurnal variation obtained with that for the normal day. Observations of the conductivity variation were made on six days, and of the electric charges on seven days, viz. August 13, 14, 17, 20, 22, 28, and 31. As the eclipse day was August 21, it was considered important for comparison to observe on the preceding and following day, and also to have the other days well balanced with respect to August 21.

Comparing the monthly means at Kew and Eskdalemuir, the conductivity is nearly twice as great at the latter place (see Tables I. and IV.). There is a corresponding early morning minimum about 5 A.M. for both places, and the July depression at Kew about midday is repeated for August at Eskdalemuir. The most remarkable difference in the variation is the sharp increase in conductivity at Eskdalemuir between 8 and 10 A.M., there being only a small rise at Kew. This is certainly to be associated with the rapid disappearance at Eskdalemuir of the morning mist, which was frequently dissipated during these hours. Because of the decreased relative humidity the charged ions in the air would lose a percentage of their water molecules, thus increasing their mobility and, consequently, the conductivity of the air.

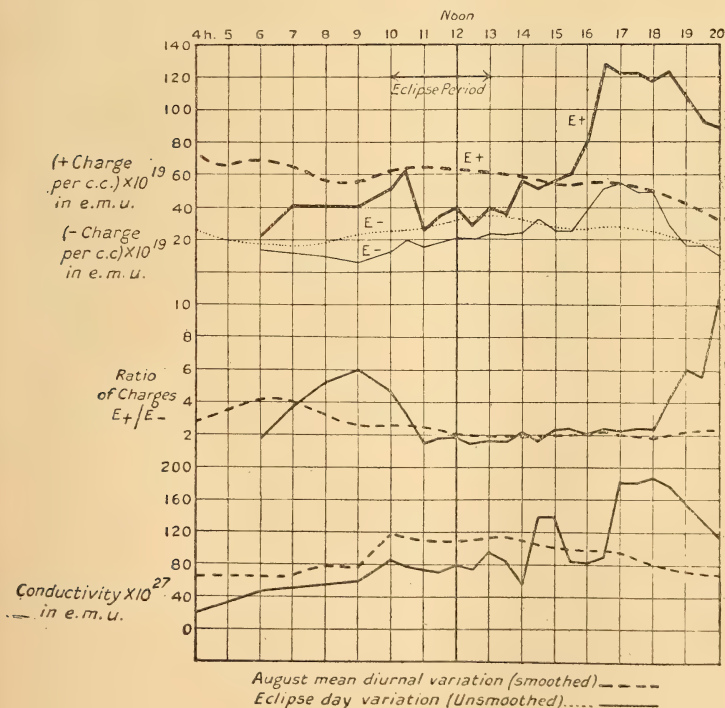
It is noticeable that the charges at Eskdalemuir varied less than at Kew. The mean positive charge is about 6×10^{-18} e.m.u. at both places. For the electric charges at Eskdalemuir there is a maximum about noon, and a minimum during the night. There is also an indication of a secondary maximum and minimum in the early morning.

The ratio of the charges $\left(\frac{E_+}{E_-}\right)$ shows a decrease during the day with a minimum at 1 P.M., which agrees well with the Kew results. The air-earth current is somewhat irregular, and there is no well-marked variation. It should be noted that the conductivity and potential were both high at Eskdalemuir, so that the air-earth current is high, while the mobility shows 100 per cent. increase from 4 to 10 A.M. and maintains an almost steady value for the rest of the day, being about double that obtained at Kew.

Eclipse Observations, August 21, 1914.

Instrumental readings were taken every hour from 4 A.M. to 10 A.M., and then every half-hour to 8 P.M. The duration of the eclipse was from 10 A.M. to 1 P.M., with a maximum phase of 0.7 for Eskdalemuir. The meteorological conditions were not ideal, as in the early morning there was a wet mist, which made observation very difficult. About 8-9 A.M. the air became drier, but it was still dull, the sun being only visible for short periods, and then dimly through thin Stratus and Strato-Cumulus. At 2 P.M. and 2.40 P.M. a little rain fell, but not sufficient to prevent observation. The diurnal variation observed is quite different from that obtained on ordinary days. In order to compare the results more conveniently, they have been represented graphically in figs. 2 and 3.

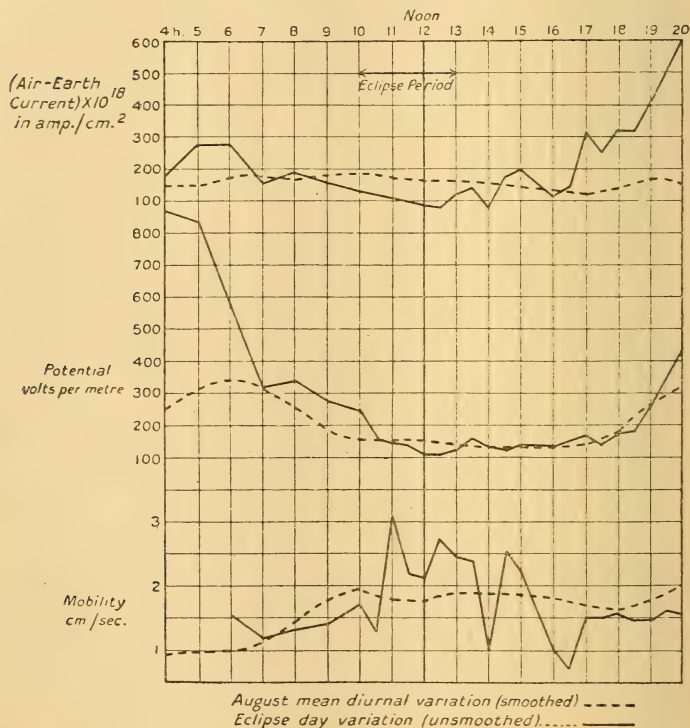
Fig. 2.—*Eskdalemuir, 1914.*



As the normal variation for August is only obtained from 6 or 7 days, it appeared desirable to smooth the values given in Table IV. by means of the formula $\frac{a+2b+c}{4}$. The

eclipse day variation is, however, unsmoothed. There is a considerable decrease in conductivity and electric charges from 10 A.M. to 5 P.M., commencing soon after the primary phase of the eclipse. The maximum for the day, instead of appearing in the early afternoon, is retarded till about 5 P.M. There is a decrease in the mobility which might be explained by the fact that increased relative humidity due to the fall in temperature during the eclipse causes condensation of

Fig. 3.—*Eskdalemuir, 1914* (continued).



water molecules on the ions. There is but little indication of an increase in the ratio of the electric charges, which has been found by some observers, during or near the totality stage, and ascribed to the reduction in the number of the negative ions*.

Superficial observation might connect the depression in the electrical quantities from 10 A.M. to 1 P.M. with the eclipse, but a similar depression is obtained in the August

* Leyst, *Luftelektrischen Beobachtungen*, 1907.

mean, and is more or less distinctly apparent in the individual days except August 31. Thus it seems to indicate a general phenomenon. It was often noticed at Eskdalemuir during this month that an early morning ground-mist was dissipated between 8 and 10 A.M. and was followed by bright sunshine with cloudless sky lasting a short time but becoming overclouded towards noon. Although at 10 A.M. the mist had quite cleared on the slopes near the Observatory, it was often visible in the valleys. It is possible that the dissipation of the valley-mist to higher levels might cause a decrease in the mobility of the ions, and so in the conductivity.

In order to study the eclipse effect more closely, the results for August 20 and 22 may be considered. The general form of the variation for both days was similar except that on August 22 the increase in the conductivity and the electric charges from 7 to 10 A.M. was very marked, and also that instead of a gradual fall from 4 to 8 P.M. there was an increase, this being especially marked in conductivity. The mean for the two days shows a deeper depression in conductivity between 10 A.M. and 1 P.M. than is obtained on August 21. This reduces the probability that any eclipse factors produced the diminution of electrical quantities on the day of eclipse between these hours. But when we consider the retarded maximum in the conductivity and both charges, there is some justification for considering it to be the result of an indirect solar effect. Some observers have considered it sufficient in eclipse work only to investigate the electrical variation during the actual phases, but a lag in producing any electrical effect would make later hours important. Wiechert's reasoning* with respect to solar radiation and the maintenance of atmospheric ionization, indicate the probability of a lag in any eclipse effect which may be due to the suspension of the ionizing influence of ultra-violet light. Although most eclipse observers in the past do not appear to have noticed any striking electrical changes, it is to be hoped that a longer series of observations will be possible in the future at stations situated near the line of totality, with sufficient observations on preceding and following days to obtain the normal form of the diurnal variation.

The few results obtained by G. Dobson † and other observers ‡ show generally a diminution of conductivity and

* "Les Recherches sur l'Electricité Atmosphérique," *Arch. des Sc. Phys. et Nat.* 1912, p. 385.

† Q. J. Roy. Met. Soc. 1913, p. 221.

‡ Ludeling & Nippoldt, *Abhand. d. Kon. Preuss. Met. Inst.* 1908.

number of ions per c.c. during the eclipse period. For the 1912 eclipse at Kew the conductivity and number of ions showed a decided increase during the first phase, but this was followed by a well-marked depression. For Eskdalemuir, no marked change in potential was found, and this agrees with the results from other eclipses.

It is well to emphasize that while small instrumental differences may have existed between the three Ebert electrometers used, as we were at that time dependent on values supplied for the capacity and anemometer readings by the makers, these uncertainties cannot affect the character of the diurnal variation described above, because for any one element only one instrument is involved.

There is some difference of opinion as to the best method of treatment of results from the Wilson electroscope. Some experiments of G. Dobson * at Kew appear to show that the conductivity as measured usually with the Wilson instrument on a tripod-stand 1.3 metres from the ground is smaller than that measured at ground-level, owing to the crowding together of the equipotential surfaces above the test-plate and the production of a saturation current. The correction for the conductivity to bring it to ground-level values was + 20 per cent. The mean for these corrected values is similar to the total conductivity recorded by the Ebert instrument. As the individual results differed so widely, and owing to the insensitiveness of the Ebert apparatus many negative values for mobility and conductivity were obtained with it, the deduction is not convincing that the corrected conductivity obtained from the Wilson apparatus corresponds with the total conductivity measured by the Ebert apparatus. Under existing conditions, it has been thought advisable to continue the old mode of reduction until a more satisfactory method is devised. The results detailed above are, however, quite comparable, as the observational conditions were uniform.

In conclusion, I desire to acknowledge my indebtedness to several members of the Observatory staff at Kew and Eskdalemuir for assistance in carrying out the experimental work involved, especially to Dr. C. Chree, F.R.S., Superintendent of Kew Observatory, under whose supervision the work has been carried out, and also to Lieutenant C. D. Stewart, who performed a number of the earlier observations.

* Proc. Phys. Soc. 1914.

XXXIV. *The Vapour Pressures of Binary Liquid Mixtures: Kinetic Theory based on Dieterici's Equation.* By FRANK TINKER, *M.Sc.**

IN a series of papers published at intervals during the last sixteen years †, Dieterici has advanced a much more fundamental equation of state than the better known Van der Waals' expression. The equation in question, viz.,

$$p = \frac{RT}{v-b} e^{-\frac{a}{vRT}}, \quad [1]$$

in which the constants a and b have a meaning similar to that given them by Van der Waals, is based on the following assumptions ‡ :—

(i.) The pressure π within a fluid is given by the equation

$$\pi(v-b) = RT, \quad [2]$$

i. e., the product pressure \times free space is given by the perfect gas equation; being independent of the size of the molecules, or of the forces between the molecules, and the same for all fluids at the same temperature.

(ii.) The pressure at the surface of the fluid is always less than that in the interior because of the inward pull exerted by the forces of cohesion. Assuming that the boundary pressure is proportional to the number of molecules which have kinetic energy enough to overcome the inward pull, Dieterici showed that this boundary pressure (p) is related to the pressure within the fluid (π) by the exponential equation

$$p = \pi e^{-\frac{A}{RT}}, \quad [3]$$

where A is the work done by the molecule in reaching the surface. Combination with equation [2] gives immediately

$$p = \frac{RT}{v-b} e^{-\frac{A}{RT}}, \quad [4]$$

and making the further assumption that A is proportional to the density of the fluid, and equal to $\frac{a}{v}$ where a is a

* Communicated by Principal Sir O. J. Lodge, F.R.S.

† *Ann. Phys. u. Chem.* xi. p. 700 (1899); *Ann. der Physik* [4] v. p. 51 (1901); *ibid.* xxv. p. 269 (1908); *ibid.* xxxv. p. 220 (1911).

‡ The assumptions are also similar to Van der Waals', but are given a different quantitative expression. The two equations become identical at low and medium pressures.

constant for the substance in question, the equation [4] becomes identical with equation [1].

Dieterici himself has shown that his equation gives excellent results when applied to critical data in the usual way. Recently others also have taken up the development of the subject. In particular McDougall has lately demonstrated* that the pressure of the saturated vapour over a liquid is also given by the equation to a high degree of accuracy.

In view, therefore, of the interest which has been revived in the causes for the deviations of the vapour pressures of binary mixtures from the well-known law of admixture in molecular proportions, and on account of the importance of the question to the subject of osmotic pressure, it seemed that it would be of interest to extend the application of Dieterici's equation to such mixtures also. Van Laar has already studied the subject from the thermodynamic side. It will be seen from the treatment which follows that Dieterici's equation leads to expressions for the partial and total vapour pressures identical in form with those obtained by Van Laar with the aid of the thermodynamic potential; but that the kinetic treatment enables the subject to be developed further, and gives a very simple relationship between the variation of the vapour pressure of the mixture from the theoretical, and the variation of the latent heat of vaporization from the theoretical value calculated by the mixture rule.

(a) *Relation between the Partial and Total Liquid Pressures in a binary mixture and the relative molecular concentrations of the two components*†.

Let X and Y be the two components of the liquid mixture, and let N molecules of X be mixed with n molecules of Y.

* Journ. Amer. Chem. Soc. xxxviii, p. 528 (1916). Mr. McDougall shows that

$$p_s = \frac{RT}{v_1 - b} e^{-\frac{a}{v_1 RT}} = \frac{RT}{v_2 - b} e^{-\frac{a}{v_2 RT}} = \frac{RT}{b} e^{-\frac{a}{2bRT}},$$

where v_1 and v_2 refer to the specific volumes of the liquid and vapour respectively, and where a and b have the values given by the equations

$$\frac{1}{b} = \frac{1}{v_1} + \frac{1}{v_2}; \quad a = \frac{2v_1 v_2}{v_1 + v_2} RT \log e \frac{v_2}{v_1}.$$

† In what follows I use the term liquid pressure (π) to denote the bombardment pressure exerted by the liquid molecules on either side of a plane of unit area placed anywhere within the liquid. It is of course different from the internal or intrinsic pressure due to cohesion. In another place ('Nature,' vol. xcvi, p. 122 (1916)) I have also called it the "diffusion pressure" to distinguish it from the latter.

Also let π_1 = liquid pressure of pure X,
 π_2 = " " " " " Y,
 π_1' and π_2' = partial liquid pressures of X and Y in
the mixture,
 $\pi = \pi_1' + \pi_2'$ = total liquid pressure in mixture,
 V_1 and V_2 = molecular volumes of pure X and Y,
 V_1' and V_2' = " " of X and Y in the
mixture.

In general, V_1' and V_2' will be slightly different from V_1 and V_2 owing to the small volume changes which take place on mixing.

Consider first the partial liquid pressure π_1' of the component X. Before the N molecules of X are introduced into the mixture, the liquid pressure of the pure X is given by the relation

$$\pi_1 = \frac{RT}{N(V_1 - b_1)}, \quad . \quad . \quad . \quad . \quad [5]$$

where the constant R refers to the mass of N molecules. Now add the n molecules of Y to the N molecules of X. We have, volume of free space in mixture

$$= N(V_1' - b) + n(V_2' - b_2).$$

Hence partial pressure π_1' of component will be given by the equation

$$\pi_1' = \frac{RT}{N(V_1' - b_1) + n(V_2' - b_2)} = \frac{RT}{(N + n)(V_1' - b_1)}, \quad [6]$$

since the two "free spaces" $(V_1' - b_1)$ and $(V_2' - b_2)$, being in the same mixture, are equal to one another.

Combining equations [5] and [6] we obtain

$$\frac{\pi_1'}{\pi_1} = \frac{N}{N + n} \cdot \frac{V_1 - b_1}{V_1' - b_1} = \frac{N}{N + n} (1 - \epsilon_1) \text{ approx.}, \quad [7]$$

where ϵ_1 denotes the fractional increase in the "free space" of the molecule of X by the mixing.

Similarly

$$\frac{\pi_2'}{\pi_2} = \frac{n}{N + n} (1 - \epsilon_2). \quad . \quad . \quad . \quad . \quad [8]$$

If instead of $\frac{N}{N + n}$ and $\frac{n}{N + n}$ we substitute the molar fractions x and $(1 - x)$ in the usual way, equations [7] and [8] become

$$\pi_1' = \pi_1 x (1 - \epsilon_1), \quad . \quad . \quad . \quad . \quad [9]$$

$$\pi_2' = \pi_2 (1 - x) (1 - \epsilon_2). \quad . \quad . \quad . \quad [10]$$

We may usually neglect the volume changes of liquids on mixing, so that ϵ_1 and ϵ_2 are practically zero. In this case, the total liquid pressure π becomes

$$\pi = \pi_1' + \pi_2' = \pi_1 x + \pi_2 (1-x), \quad . \quad . \quad . \quad [11]$$

Hence when there are no volume changes on mixing, the total liquid pressure can be calculated from the two partial liquid pressures by the law of admixture in molecular proportions. For the partial liquid pressure of each component is then proportional to its molar fraction. If, however, there is a volume change, the law is only approximately accurate.

But the ratio $\frac{V_1 - b_1}{V_1' - b_1} = (1 - \epsilon_1)$ will in general be so nearly equal to unity that there will be an appreciable deviation from the mixture law only in very abnormal cases.

(b) *Relation between the Partial and Total Vapour Pressures in a binary mixture and the relative molecular concentrations of the two components.*

We can now apply Dieterici's equation proper to the questions in hand.

Let p_1 and p_2 = vapour pressures of pure components X and Y,

p_1' and p_2' = „ „ of X and Y in mixture,
 $p = p_1' + p_2'$ = total vapour pressure of mixture,

A_1 and A_2 = work done when a molecule of X or Y
 is evaporated from the pure liquids,

A_1' and A_2' = work done when a molecule of X or Y
 is evaporated from the mixture.

N, n, x have same values as before.

From equation [3] we have

$$p_1 = \pi_1 e^{-\frac{A_1}{RT}},$$

$$p_1' = \pi_1' e^{-\frac{A_1'}{RT}}.$$

Hence

$$\frac{p_1'}{p_1} = \frac{\pi_1'}{\pi_1} e^{-\frac{(A_1' - A_1)}{RT}}.$$

Combining this equation with equation [9] and neglecting ϵ , we have

$$p_1' = p_1 x e^{-\frac{(A_1' - A_1)}{RT}} \quad . \quad . \quad . \quad . \quad [12]$$

Similarly

$$p_2' = p_2(1-x)e^{-\frac{(A_2'-A_2)}{RT}}$$

and

$$p = p_1' + p_2' = p_1xe^{-\frac{(A_1'-A_1)}{RT}} + p_2(1-x)e^{-\frac{(A_2'-A_2)}{RT}} \quad [13]$$

The last equation is of the same form as the equation Van Laar has obtained for the total vapour pressure of a binary mixture on thermodynamic grounds. Van Laar's expression is

$$p = p_1xe^{-\frac{(\mu_1'-\mu_1)}{RT}} + p_2(1-x)e^{-\frac{(\mu_2'-\mu_2)}{RT}},$$

where μ_1 and μ_2 are the thermodynamic potentials of the two components*.

Equation [13] can also be written in the more convenient form

$$p = p_1xe^{-\frac{\partial A_1}{RT}} + p_2(1-x)e^{-\frac{\partial A_2}{RT}}, \quad [14]$$

where ∂A_1 and ∂A_2 represent the increase in the work done by evaporating a molecule of X or Y from the mixture over the work done when the molecule is evaporated from the pure liquid.

Furthermore, it will be seen that ∂A_1 and ∂A_2 are practically identical with the excess of the molecular latent heats of vaporization ∂L_1 and ∂L_2 of the two components in the mixture over their molecular latent heats in the pure state. Hence we may write†

$$p = p_1xe^{-\frac{\partial L_1}{RT}} + p_2(1-x)e^{-\frac{\partial L_2}{RT}}. \quad [15]$$

We may develop equation [15] still further. Expanding $e^{-\frac{\partial L_1}{RT}}$ and $e^{-\frac{\partial L_2}{RT}}$ we have, since ∂L_1 and ∂L_2 are small,

$$\begin{aligned} p &= p_1x\left(1 - \frac{\partial L_1}{RT}\right) + p_2(1-x)\left(1 - \frac{\partial L_2}{RT}\right) \\ &= p_1x + p_2(1-x) - \frac{1}{RT}\left\{p_1x\partial L_1 + p_2(1-x)\partial L_2\right\}. \end{aligned}$$

Now if p_0 is the vapour pressure of the mixture calculated

* *Zeit. Phys. Chem.* lxxii. p. 723 (1910); *ibid.* lxxxiii. p. 599 (1913); and other papers.

† Equation [15] indicates that $\partial\mu = \partial L$, or that as regards the vaporization of liquids, increase in thermodynamic potential is equivalent to increase in latent heat of vaporization.

according to the law of admixture in molecular proportions, and if L denotes the molecular latent heat of vaporization of the perfect mixture*, we have

$$p_0 = p_1x + p_2(1-x)$$

$$\text{and} \quad p_0L = p_1xL_1 + p_2(1-x)L_2, \quad . \quad . \quad . \quad [16]$$

since the two components evaporate in the ratio of their partial pressures.

Hence also

$$p_1x\partial L_1 + p_2(1-x)\partial L_2 = p_0\partial L.$$

Consequently

$$p = p_1x + p_2(1-x) - p_0 \frac{\partial L}{RT}$$

$$= p_0 - p_0 \frac{\partial L}{RT},$$

i. e.

$$p = p_0 \left(1 - \frac{\partial L}{RT}\right); \quad . \quad . \quad . \quad . \quad . \quad [17]$$

or in words, "the total vapour pressure of a binary mixture is equal to the vapour pressure calculated from the law of admixture in molecular proportions, multiplied by a factor $\left(1 - \frac{\partial L}{RT}\right)$, where ∂L is the excess of the molecular latent heat of vaporization of the mixture over the [theoretical] value calculated from the mixture rule."

This simple relationship shows that the variation of the total vapour pressure from the straight line law is determined almost entirely by abnormalities in the latent heat of vaporization †.

Proceeding from equation [17] we have three possible cases:—

(i.) If the actual molecular heat of vaporization of the

* The *actual* molecular latent heat of the mixture would be the heat of evaporation or condensation of one gm. molecule of *vapour*, i. e. the heat required to produce 22.2 litres of the mixed vapour at N.T.P. The theoretical value L is of course calculated from equation [16].

† Volume changes on mixing will of course have a slight effect on the vapour pressure also. Allowing for these, equation [17] becomes

$$p = p_0 \left(1 - \frac{\partial L}{RT}\right) - (p_1'\epsilon_1 + p_2'\epsilon_2).$$

mixture is equal to the theoretical value ($\partial L=0$), the vapour-pressure curve is a straight line (Diag. I.).

Diagram I.

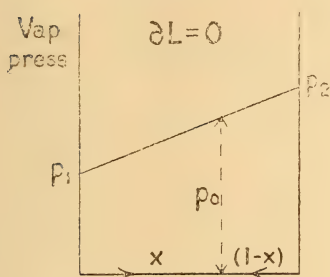
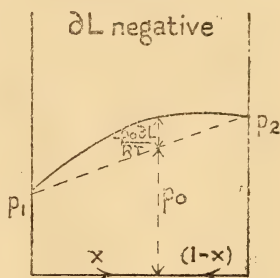


Diagram II.



Molar fractions of two components.

(ii.) If the molecular latent heat of vaporization is less than the theoretical (*i. e.* ∂L is negative) the vapour pressure is greater than the theoretical (Diag. II.).

(iii.) If the molecular heat of vaporization is greater than the theoretical (*i. e.* ∂L is positive) the vapour pressure is less than the theoretical (Diag. III.).

Diagram III.

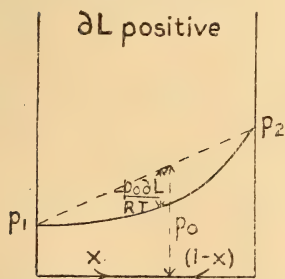
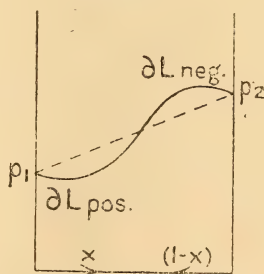


Diagram IV.



If, as the relative molecular concentration of the two components is varied, the value of ∂L changes from positive to negative or *vice versa*, the curve will cross over the diagonal (as in Diag. IV.). In this case ∂L is at first positive, and then changes over to negative.

As yet there have been no measurements made by which the above equation [17] can be subjected to a quantitative test. But its general soundness can be shown indirectly from surface-tension data. It is well known that there is a

very close connexion between the latent heat of vaporization of a liquid and its surface-tension, liquids having a high latent heat, for instance, also having a high surface-tension*. Abnormally high vapour pressures are therefore to be associated also with abnormally low surface-tensions, and *vice versa*. F. P. Worley has already advanced this view on experimental grounds†. Corresponding to the three cases just given for the differences in latent heats, this investigator has given the three following rules for the connexion between the vapour pressures of binary mixtures and their surface-tensions :—

“(i.) If at any given temperature the vapour pressures of mixtures of two liquids agree with the values calculated by the rule of admixture in molecular proportions, the surface-tensions of the mixtures agree with those calculated by the formula

$$S = V_1S_1 + V_2S_2 \ddagger.$$

“(ii.) If the vapour pressures are greater than those calculated, then the surface-tensions are less than those calculated.

“(iii.) If the vapour pressures are less than those calculated, the surface-tensions are greater than those calculated.”

By comparing his own and other surface-tension curves with the vapour-pressure curves of Zawidski and others, Worley has shown that every pair of liquids so far investigated comes under the rules laid down. Mixtures of benzene and ethylene dichloride, for instance, form perfect mixtures and obey the first rule; carbon bisulphide and acetone, benzene and carbon tetrachloride, ether and carbon bisulphide obey the second rule; whilst in the same way, water and the alcohols, and pyridine and acetic acid obey the third.

* So much can be deduced from the Laplace Theory of Capillarity. Also Walden has given the following empirical formula for this relationship, based on a large number of experimental data :

$$L = 3.64 v\gamma,$$

where L = latent heat per gm. in calories; v = sp. vol. in c.c.; γ = surface-tension in dynes/cm.

† Journ. Chem. Soc. Trans. cv. [1] p. 273 (1914).

‡ V_1 and V_2 are the volumes of the two liquids expressed fractionally.

XXXV. *The Hall and Corbino Effects.*

By ALBERT K. CHAPMAN *.

A FEW years ago Professor Corbino observed that when a uniform radial current flows through a circular disk of metal placed in a magnetic field normal to the plane of the disk there is produced a circular current, the density of which is inversely proportional to the radius (*Physikalische Zeitschrift*, xii. pp. 561, 842, 1911). His experiments were confined principally to bismuth and antimony, in which this current is relatively large. Professor Adams (*Phil. Mag.* Feb. 1914, p. 244) has developed an expression for its magnitude on the assumption of the simple electron theory and this, of course, predicts that the effect has the same sign in all metals. In a recent research (*Phil. Mag.* Nov. 1914, p. 692) Professor Adams and the writer quantitatively investigated in this respect a number of metals and found that, in every case, the sign is the same as that of the Hall effect. The method used consisted in measuring the current induced in a coil placed near the disk when the radial current, and hence the circular current, is reversed in sign. This induced current was balanced by the current in the secondary of a known variable inductance, through the primary of which was passed the same current that traversed the disk. Upon reversing the magnetic field the circular current changes sign and may again be balanced by an alteration of the variable inductance. We then have the relation

$$MC = mI,$$

where C is the circular current in the disk, I is the radial current, M is the mutual inductance between the current C and the coil placed near the disk, and m is the reading of the standard mutual inductance. When the magnitude of the circular current permitted, observations for various fields were taken and the values of m/H given. As a result of this investigation there was given also a table comparing the magnitudes of the Hall and Corbino effects, each being referred to copper, which was taken as standard. In general these numbers are of different orders of magnitude; however, if the values of m/H referred to copper are multiplied by the ratio of the specific resistance of the metal in question to that of copper, they are brought into much better agreement. This is to be expected, since, in the Hall effect, differences of potential are measured, while in the Corbino effect actual currents are sought. The numbers for the Hall effect were taken from the *Recueil de Constantes Physiques* (1913), published by the French Physical Society.

* Communicated by Prof. E. P. Adams.

	Field.	Hall Effect.	Corbino Effect.
Copper	ind. H	-1.0	-1.0
Iron	3700	21.9	16.4
Aluminium	ind. H	-0.73	-0.48
Bismuth.....	1650	-19750	-20100
„	3930	-12200	-16300
Antimony	1750	421	557
Platinum	ind. H	-0.46	-0.56
Gold	ind. H	-1.27	-2.34
Silver.....	ind. H	-1.60	-1.61
Nickel	1700	-23.7	-32.0
Zinc	0.63	0.53
Cadmium	ind. H	1.06	1.37
Cobalt	3460	4.61	5.86

The numbers given herewith under "Corbino Effect" are values of m/H referred to copper and multiplied by the ratio of the specific resistance of the particular metal to that of copper. Multiplication by this factor brings the two phenomena to the same order of magnitude, but the agreement is not close except in one or two cases. Without doubt this is due, in part, to the fact that the values were determined for totally different specimens in the two instances; it will be shown later, however, that there remains an outstanding difference between the two effects when measured in the same specimen. Whether this discrepancy is due to the free boundaries necessary in the Hall effect, to some difference in internal behaviour, or to both, seems, in the present state of our knowledge, impossible to say.

Assuming that, in the Corbino effect, the function of the magnetic field is to produce an electromotive force at right angles both to the magnetic force and to the primary current, and that this electromotive force is proportional to the vector product of H , the magnetic force, and E , the primary electromotive force, we have

$$E_1 = cVHE,$$

where V indicates the vector product, E_1 is the electromotive force of the Corbino effect, and c is the proportionality factor (E. P. Adams, Proc. Am. Phil. Soc. vol. liv. no. 216, 1915). Applying this to a disk of the form used in these experiments, we have

$$C = \frac{c}{2\pi} \log \frac{r_2}{r_1} \cdot IH,$$

where C is the total circular current, I is the total radial current, r_2 is the external radius of the disk, and r_1 is the internal radius of the disk. We may then call

$$c = \frac{2\pi C}{HI \log \frac{r_2}{r_1}}$$

the Corbino constant.

The Table below gives the values of this constant for the metals investigated in the previous experiments.

COPPER.		IRON.	
Field.	$c \times 10^7$.	Field.	$c \times 10^7$.
7310	2.38	8450	7.13
6750	2.51	7620	6.60
5920	2.61	6750	6.58
4220	2.40	4890	6.64
2820	3.36	3300	6.55
		2060	8.27
		1390	7.45
ALUMINIUM.		GOLD.	
Field.	$c \times 10^7$.	Field.	$c \times 10^7$.
7970	0.770	6640	1.99
6850	0.69	5250	2.06
5660	0.83		
BISMUTH.		ANTIMONY.	
Field.	$c \times 10^7$.	Field.	$c \times 10^7$.
7720	381.	7210	37.5
6540	415.	6180	38.3
5820	443.	4880	40.7
4690	520.	2590	40.9
2630	613.		
953	666.		
419	771.		
107	739.		
30	711.		
NICKEL.		SILVER.	
Field.	$c \times 10^7$.	Field.	$c \times 10^7$.
6930	11.76	6900	4.78
5970	13.0	6240	5.05
4810	14.6	5510	4.86
3730	14.5	4580	4.84
2570	14.4		
CADMIUM.		COBALT.	
Field.	$c \times 10^7$.	Field.	$c \times 10^7$.
7210	0.715	7570	2.48
		6690	2.61
		5360	2.63
		ZINC.	
Field.	$c \times 10^7$.	Field.	$c \times 10^7$.
		7470	0.343
		PLATINUM.	
Field.	$c \times 10^7$.	Field.	$c \times 10^7$.
		7056	0.199

It is the purpose of the present paper to report on the variations of the Corbino effect with the magnetic field in a number of metals and alloys and, at the same time, to compare these variations with measurements of the Hall effect in a disk exactly similar to the one used for the Corbino effect.

The experimental method used was the same as that described in the former paper. In the earlier experiments some difficulty was experienced with the rotating commutator. It was in series with the galvanometer, and any inequality in the pressure of the brushes or any undue pressure of the brushes always resulted in undesirable electromotive forces which made the galvanometer very unsteady. By lubricating the brushes with a mixture of graphite and oil, and by taking care that each brush had a smooth, broad, but not too heavy contact, a great deal of this trouble was overcome.

The specific resistances of the metals under investigation were determined by measuring the potential difference between the central wire and the circumference of the disk when a known current was flowing radially through it. To take account of any non-uniformity, four points on the circumference, ninety degrees apart, were chosen, the potential difference between each point and the centre measured, and the mean taken. In all cases the disks were of the order of 0.9 mm. thick, so that such measurements by means of a Wolff potentiometer are accurate only for metals of considerable resistance. The values for copper and silver were simply taken from a table. The disks were all immersed in a well-stirred oil-bath during this process.

Circular plates with four integral electrodes, ninety degrees apart, were used for the Hall effect. The electrodes by which the primary current was led into the plates were heavy; the others were much smaller and were connected to the galvanometer circuit. These plates were placed in a hard rubber mounting and introduced into the magnet in the same position as that occupied by the disks used for the Corbino effect. They were made circular in order to include, as nearly as possible, the same flux. In every instance where curves for both effects are given, the two disks were either cut from the same sheet of metal or cast from the same mass of molten metal, as the case may be.

Copper.—It was thought advisable to repeat the measurements on copper under more favourable experimental conditions. The disk, cut from a sheet of commercial copper, was 0.9 mm. thick and 5.0 cm. in diameter. The radii, r_2 and r_1 , were 5.0 cm. and 0.266 cm., as they were for all

the disks investigated. The radial current in every case was about twelve amperes, and the value of m/H for each field is the mean of a number of determinations. One-half the difference between the balancing inductance readings for positive and negative fields is denoted by m .

The Corbino Effect in Copper.

H.	m .	m/H .	$c \times 10^7$.
2590	12.1	0.00469	2.11
4280	19.9	0.00465	2.09
6130	32.0	0.00522	2.35
7590	39.8	0.00524	2.36
8490	37.5	0.00442	1.99
2510	12.1	0.00484	2.18
4060	18.8	0.00462	2.08
5890	28.7	0.00481	2.17
7450	34.2	0.00460	2.07
8530	43.1	0.00505	2.27

Summarizing :—

Field.	m/H .
2500	0.00476
4200	0.00463
6000	0.00501
7500	0.00492
8500	0.00473
Mean	0.00481 = 1/208

$$\text{Mean } c = 2.17 \times 10^{-7}.$$

It will be observed that the values of m/H for the five fields do not depart by more than 4 per cent. from the mean, so we may say with safety that the value $m/H = 1/208$ is accurate to 2 per cent. There is no evidence of any variation of m/H with the field.

Bismuth.—Two disks, from the same body of metal, were cast to exact size in graphite moulds; one for the Corbino effect and one for the Hall effect. The specific resistance, determined by the method above, was found to be 1572×10^{-7} , while the value used for copper is 17×10^{-7} .

The Hall Effect in Bismuth.

H.	$E \times 10^4$.	$K = \frac{Ed}{HI}$.	$(K)_{Cu}$.
208	0.84	7.49	14130
486	1.81	6.90	13020
1050	3.68	6.53	12320
1690	5.55	6.12	11550
3150	8.25	4.85	9150
3600	9.10	4.66	8790
4670	11.0	4.37	8250
5680	12.2	3.99	7530
7060	13.9	3.63	6850
7310	14.6	3.69	6960
8320	15.7	3.49	6580
10400	17.5	3.11	5870
10570	18.1	3.16	5960

The Corbino Effect in Bismuth.

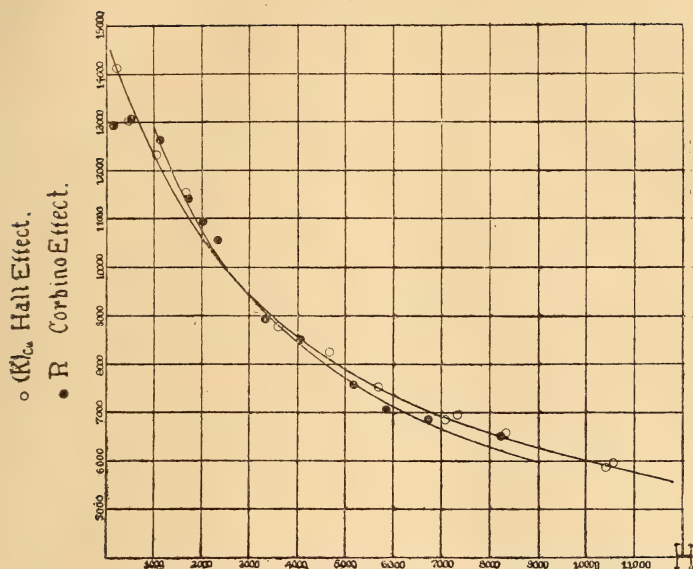
H.	m/H .	$(m/H)_{Cu}$.	R.	$c \times 10^7$.
156	0.673	140.0	12950	303.
510	0.680	141.4	13080	306.
1125	0.657	136.7	12640	296.
1720	0.593	114.0	11410	267.
2020	0.569	118.4	10950	256.
2330	0.549	114.2	10560	247.
3320	0.464	96.5	8930	209.
4050	0.442	91.9	8500	199.
5170	0.394	82.0	7580	177.
5850	0.367	76.3	7060	165.
6720	0.342	71.1	6850	154.
8210	0.338	70.3	6500	152.

In the above tables d is the thickness of the plate (0.0919 cm. for the bismuth plate); E is the potential difference across the plate; I is the primary current flowing through the plate; H is the magnetic field: K is 530×10^{-6} for copper; m/H is $1/208$ for copper; $(m/H)_{Cu}$ is m/H for bismuth divided by m/H for copper; R is $(m/H)_{Cu}$ multiplied by the ratio of the specific resistances of bismuth and copper.

A similar notation is used throughout the remainder of this work.

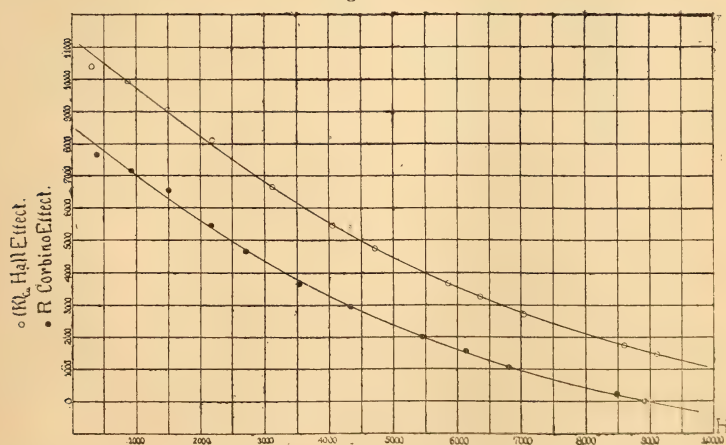
Fig. 1 shows the plot of the two effects against the values of H . It is seen that both decrease very quickly with increasing field, the Corbino effect falling off the more rapidly.

Fig. 1.



Alloys of Bismuth and Tin.—It has been found by von Ettingshausen and Nernst, among others, that the sign of the Hall effect in certain alloys of bismuth and tin is reversed with increasing field. To compare this phenomenon with

Fig. 2.



the Corbino effect, two plates were cast from the same alloy and the two effects measured under like conditions. The

Table gives the numbers, and fig. 2 is a plot of both for an alloy containing 1 per cent. tin; the numbers are all referred to copper, which is taken as unity.

The Hall Effect in a Bismuth-Tin Alloy
(about 1 per cent. tin).

H.	$E \times 10^4$.	$K = \frac{E\bar{d}}{HI}$.	$(K)_{Cu}$.
307	0.91	5.51	10400
869	2.48	5.27	9940
1490	3.88	4.79	9040
2180	5.10	4.29	8100
3120	6.00	3.53	6660
4050	6.35	2.89	5450
4720	6.45	2.51	4740
5850	6.15	1.94	3660
6355	5.90	1.71	3230
7030	5.45	1.43	2700
8600	4.30	0.917	1730
9110	3.88	0.784	1480

$$\bar{d} = 0.0915 \text{ cm.}$$

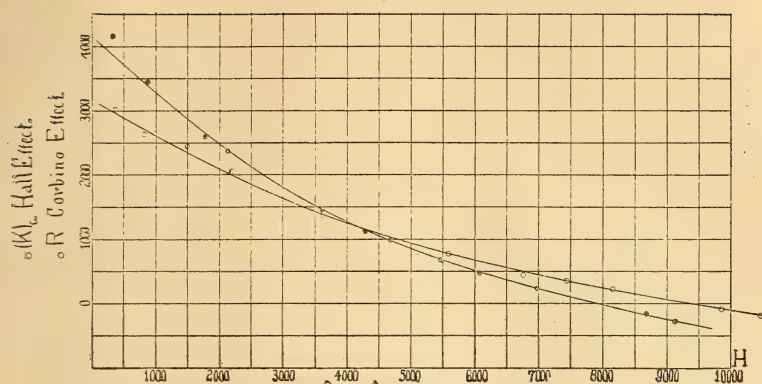
The Corbino Effect in a Bismuth-Tin Alloy
(about 1 per cent. tin).

H.	m/H .	$(m/H)_{Cu}$.	R.	$c \times 10^7$.
390	0.142	29.5	7670	63.8
925	0.132	27.5	7150	59.3
1500	0.121	25.2	6550	54.4
2160	0.101	21.0	5460	45.4
2700	0.0859	17.9	4650	38.8
3540	0.0673	14.0	3640	30.2
4330	0.0544	11.3	2940	24.4
5455	0.0370	7.7	2000	16.66
6130	0.0286	5.95	1550	12.87
6800	0.0195	4.06	1060	8.78
8495	0.0043	0.89	231	1.93
8915	0.0000	0.00

It is seen that both effects decrease rapidly with increasing field, while the sign is that of the effect in bismuth. The two curves are nearly parallel, the Corbino effect being the smaller and falling to zero at 8910, where the Hall coefficient has a value of 1550 referred to copper.

In the case of an alloy containing about 2 per cent. of tin the conditions are somewhat different, as may be seen from the Table and fig. 3.

Fig. 3.



The Hall Effect in a Bismuth-Tin Alloy
(about 2 per cent. tin).

H.	$E \times 10^4$.	$K = \frac{Ed}{HI}$.	$(K)_{Cu}$.
354	0.303	1.594	3010
805	0.600	1.400	2640
1485	1.03	1.294	2440
2185	1.27	1.087	2050
3600	1.39	0.721	1360
4670	1.33	0.521	983
5570	1.21	0.406	766
6750	0.85	0.234	442
7425	0.73	0.183	345
8150	0.485	0.111	209
9845	-0.240	-0.045	-85
10450	-0.545	-0.0976	-184

The Corbino Effect in Bismuth-Tin Alloy
(about 2 per cent. tin).

H.	m/H .	$(m/H)_{Cu}$.	R.	$c \times 10^7$.
317	0.0800	16.64	4170	36.0
855	0.0661	13.75	3450	29.8
1760	0.0497	10.34	2590	22.3
2115	0.0454	9.44	2370	20.4
3600	0.0279	5.80	1450	12.55
4275	0.0212	4.41	1105	9.55
5455	0.0126	2.62	657	5.68
6070	0.00905	1.88	471	4.07
6970	0.00428	0.89	223	1.93
8665	-0.00319	-0.66	-165	-1.43
9110	-0.00560	-1.16	-291	-2.53

Here the curves are no longer parallel; the Corbino effect has a value 4187 and the Hall effect 3000 times that of copper for a field of 250. The former falls off the more rapidly, crosses the Hall effect curve at a field of 4250, becomes zero at a field of 7900, and reaches a value of -291 at 9110. On the other hand, the Hall curve goes to zero at 9325 and is -184 at a field of 10450.

By some physicists it has been suggested that this reversal is the result of the superposition of two or more effects. In the usual method of evaluating the Hall coefficient such a condition might arise through the simultaneous occurrence of the simple Hall effect and the Ettingshausen effect; the latter giving rise to the usual galvanomagnetic temperature difference at the boundaries of the plate, which in turn, because of the Thomson effect, produces a potential difference which appears in the potential reading taken as a measure of the Hall effect. To be sure, under such conditions, the galvanomagnetic potential difference obtained at any given field depends upon the ratio between the Hall and Ettingshausen coefficients, and it is conceivable that the peculiar phenomena in bismuth-tin alloys might be due to this circumstance. It is at once evident, though, that the Corbino method entirely precludes the possibility of the occurrence of the Thomson effect, since there are no free boundaries present. Hence one may say with certainty that the reversal obtained in the Corbino curve is not due to the coexistence of the Hall and Ettingshausen effects.

Because of the junction of dissimilar metals at the centre and circumference of the disks employed, the Peltier effect may enter, producing a radial heat-flow and consequently a circular current. In a recent paper Zahn (*Ann. d. Phys.* xlvii. 1. pp. 49-82, May 11, 1915) has shown that such an error in ordinary experiments on the Hall coefficient amounts actually to a very small fraction of the whole, and it may safely be assumed that such is the case here.

And finally, the best guarantee that such complicating effects do not play any considerable part is the reversal of the radial current, which took place twenty times per second. As is well known, a very definite time is necessary for the Ettingshausen and Ettingshausen and Nernst effects to establish themselves, since each depends upon the rise either of a transverse or of a longitudinal temperature difference. The brief interval between reversals is not adequate for the building up of any great temperature difference.

Nickel.—The disks were cut from a sheet of metal supplied by Messrs. Eimer and Amend.

The Hall Effect in Nickel.

H.	$E \times 10^4$.	$K = \frac{Ed}{HI}$.	$(K)_{Cu}$.
1100	0.0335	0.01491	28.7
2305	0.0682	0.01448	27.8
3420	0.1068	0.01526	29.3
5700	0.1839	0.01581	30.4
7750	0.2060	0.01305	25.1
9150	0.2104	0.01127	21.7
10200	0.2073	0.00998	19.2

The Corbino Effect in Nickel.

H.	m/H .	$(m/H)_{Cu}$.	R.	$c \times 10^7$.
915	0.01689	3.51	19.5	7.60
1830	0.02355	4.90	27.3	10.60
2810	0.02399	4.99	27.8	10.79
4580	0.02434	5.06	28.2	10.97
6550	0.02362	4.91	27.3	10.62
7950	0.02099	4.37	24.3	9.45
9025	0.01873	3.90	21.7	8.44

Fig. 4.

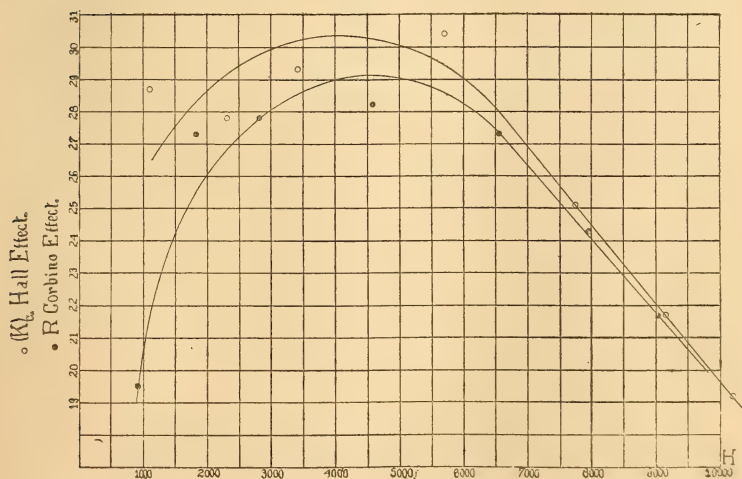


Fig. 4 shows the plots of these numbers. In the case of the Hall effect the points for low fields do not lie well on the curve; this is due to the difficulty of accurately

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measuring the small potential differences obtained in a plate 0.0398 cm. thick. The Corbino effect rises to a maximum at a field of about 4500, and decreases rather rapidly to fields of 9000 lines, which is as far as it was practicable to carry the readings. It is likely that the Hall effect also has a maximum at approximately 4000, and it certainly decreases for increasing field at nearly the same rate as the Corbino effect, the two curves being almost parallel from 6500 to 9500.

The Influence of Thickness.—The theory mentioned above predicts that the Corbino effect is independent of the thickness of the disk employed. This theory has been confirmed in this one respect, within the limits of experimental error, for three different nickel disks of quite different thicknesses. They were supplied by Messrs. Baker & Company, and were of thicknesses 0.0505, 0.1526, and 0.20315 cm. respectively. Here one must take into account the change in mutual inductance between the coils and the disks of varying thickness. To obtain an accurate measure of this change, the method used in the previous paper was employed. The coils were placed against a circular plate of brass, 5 cm. in diameter and having a hole of the same size as the central wire of the other disks drilled in the centre. A sector of small angle was cut out of this disk, to the edges of which heavy leads were soldered. The coil and disk were then placed between the poles of the magnet and their mutual inductance compared with a known standard. A heavy sheet of mica was then interposed between the coil and disk, and the mutual inductance again measured. In the first instance the distance from the centre of the coil to the disk was 0.0438 cm., while the mutual inductance proved to be 21000 e.m.u.; in the second the distance was 0.1317 cm. and the inductance 17500 units. Assuming, then, that the change of inductance with separation is linear for the changes used with the three disks above, we may calculate the inductance between the coils and each disk. Taking the inductance measured with the brass disk of 0.0894 cm. thickness as standard, we find:—

The inductance increase for the thin	disk	3.90	per cent.
„ „ decrease „ medium	„	6.32	„
„ „ „ „ thick	„	11.38	„

Below are the values of δ/F for the three disks, where δ is the change in balancing inductance, measured in degrees

of rotation of the secondary coil. F is the fluxmeter reading on reversing the field.

Thickness of disk.	F .	δ/F .
0.0505 cm.	165	15.3
0.1526 „	166	14.7
0.20315 „	169	15.0
	Mean	15.0

The agreement is very good, especially when it is considered that, in all probability, the disks were cut from different sheets of metal.

The mutual inductance between the coils and the Corbino disk was also determined by a graphical method. This method, which was purely analytical, gave results in good agreement with those obtained experimentally, as is shown below :—

M (calculated) . 11790 e.m.u.

M (experimental) 11860 e.m.u.

This graphical method was of such a nature that one could investigate the most advantageous form of coil for any given conditions.

It was tacitly assumed that the flux calculated for a plane passing through the middle of the coil and parallel to the disk is a mean value of the flux in the surface of the coil next to the disk and the flux in the opposite surface. That this assumption is justified may be inferred from the agreement of the experimental and calculated values of the inductance.

A similar assumption was made in the calculation of the inductances for the nickel disks of varying thickness. That this is legitimate is also borne out by the above theoretical determination, since the change in thickness of the disks was only about half of the actual thickness of the inductance-coil used.

Iron.—This disk was made from a bar of Norway iron, as was the plate used for the determination of the mutual inductance of the two coils and the disk. The inductance was investigated both with and without the external field ; there was no change due to the presence of such a field except for large values beyond 10,000. At this particular point the change was less than 1 per cent. In the study of

this same disk last year, nothing of unusual interest was found. Upon carrying the fields below 1390, which was the lowest used in the previous experiments, a remarkable effect came to light. The value of m/H is zero up to a field of about 760, rises to a sharp maximum at 950, falls to half the maximum value at 1500, and then decreases slowly as the field increases up to 10,000, which was the largest field used.

In order to investigate this region accurately, it was necessary to read small differences in field with a certainty greater than that afforded by the fluxmeter. Accordingly, a standard ohm, immersed in an oil-bath, was put into the magnet circuit and the current flowing determined by taking the potential drop across it; a Wolff potentiometer was employed, together with a standard cell prepared by Professor Hulett of the Department of Chemistry. By taking corresponding current and fluxmeter readings for several fields of very different magnitudes, it was found that, for any given region, the field is proportional to the current. Having determined the factor of proportionality, it was then possible to fix accurately small differences in the field by the corresponding current readings.

During the course of preliminary experiments on this metal, some difficulty was experienced in duplicating the numbers obtained at various times. In view of the fact that the magnetic properties of iron vary rapidly with the temperature, as does the Hall effect, the disk together with the inductance-coils was put into a brass box, so arranged between the poles of the magnet that a constant stream of water could be kept flowing through it. In this way the temperature was, at all times, kept stationary to within one degree, which proved to be sufficiently constant for the work on hand.

Residual magnetism also interfered seriously with the measurements at low fields. This was eliminated by applying an alternating current to the magnet before the desired direct current was passed through it. In every case the initial alternating current employed was much larger than the direct current to be used and was gradually reduced to zero, thus effectively demagnetizing the magnet and disk.

Having taken these two precautions, it was possible to obtain fairly consistent results, as shown by the table.

The Corbino Effect in Iron.

H.	$2\delta/H$.	$c \times 10^7$.
142.3	0.00035	1.05
172.8	0.00033	0.00
221.1	0.0014	4.2
305.3	-0.0003	-0.99
378.0	0.0008	2.4
496.7	-0.00020	-0.60
758.0	-0.00013	-0.39
798.	0.00138	4.16
815.	0.00160	4.82
826.	0.00182	5.48
848.	0.00289	8.7
856.	0.00304	9.2
865.	0.00358	10.8
887	0.00411	12.4
902	0.00432	13.0
911	0.00461	13.9
919	0.00490	14.8
930	0.00516	15.5
944	0.00540	16.2
962	0.00530	16.0
980	0.00510	15.4
996	0.00492	14.8
1014	0.00473	14.2
1053	0.00427	12.9
1081	0.00388	11.7
1117	0.00331	10.0
1162	0.00310	9.3
1180	0.00313	9.4
1197	0.00317	9.5
1211	0.00306	9.2
1227	0.00334	10.1
1304	0.00291	8.8
1392	0.00296	8.9
1425	0.00309	9.3
2125	0.00245	7.4
3530	0.00263	7.9
3580	0.00241	7.3
5500	0.00233	7.0
5680	0.00225	6.8
7120	0.00222	6.7
7290	0.00219	6.6
8270	0.00220	6.6
8640	0.00209	6.3

Fig. 5 gives the plot of m/H against H , the external field; the maximum is very sharp and somewhat resembles a resonance peak. The point at 760, where the effect first appears, also seems to be very definite, while the falling off of the curve beyond 2000 is gradual.

Fig. 5.

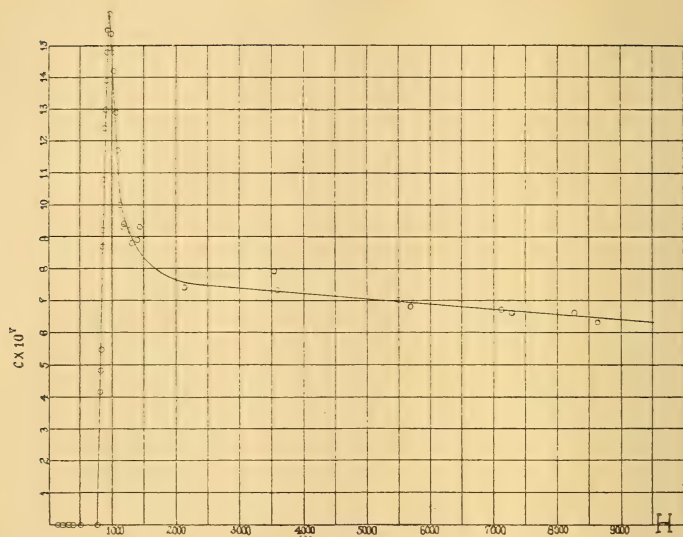


Fig. 6 is a plot of the values of m against H ; m is directly proportional to the circular current. It is seen that, in one region, the circular current actually decreases for an increasing field; it rises quickly to 5.1 and falls to 3.7. Beyond 1179 the curve is a straight line.

It is important to observe that the curve for m/H remains practically of the same form when the internal field is introduced instead of the external field. We have

$$H_i = H - NI,$$

where N represents the demagnetizing factor; H_i the internal field; H the external field; and I the intensity of magnetization. For a disk of the above form N comes out 12.25, which is so near 4π that the external field and the induction are practically the same. We may write the above equation for H_i ,

$$H_i = \frac{H}{1 + \frac{N(\mu - 1)}{4\pi}} = \frac{H}{0.97476\mu} \text{ nearly,}$$

where μ is the permeability.

Since, for this particular specimen of iron, no values of μ were at hand, the numbers given in the *Recueil de Constantes Physiques* par Abraham et Sacerdote (1913) were used. From this H was determined and $2m/H$ plotted against H as in fig. 7; this curve is plotted on a much larger scale than the previous ones.

Fig. 6.

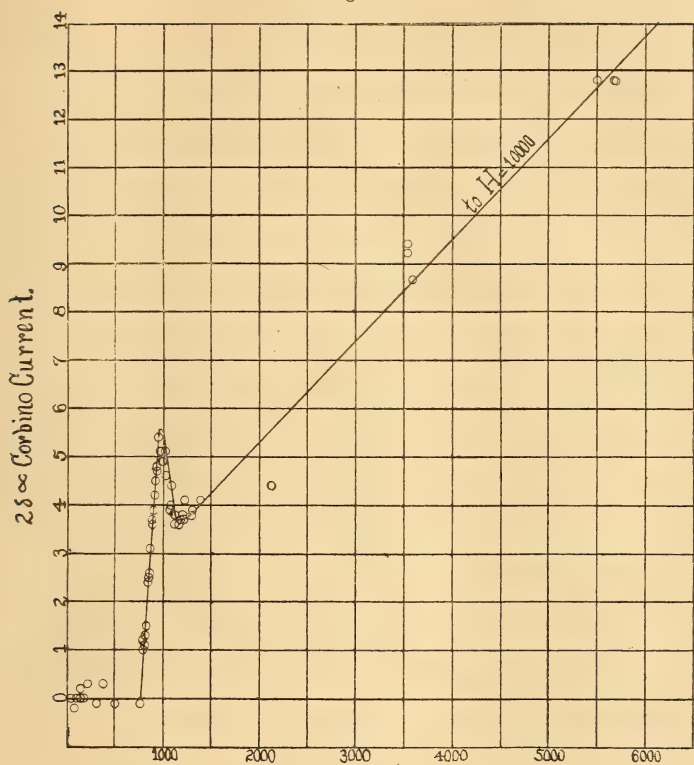
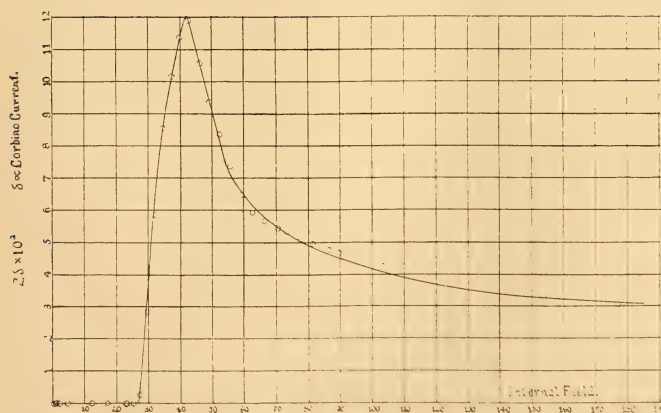


Fig. 7.



That the form of these curves does not depend upon the frequency of reversal of the radial current is demonstrated by the entire agreement of experiments made with twenty reversals per second with those where five or six per second were used.

In view of the possibility that these peculiar effects in the iron disks might arise through the close proximity of the heavy pole-pieces of the magnet, through the fact that one of them is traversed by a one-centimetre hole, or through some other unnoticed circumstance, it was thought that a repetition of the experiments would be of value, with the disk in a solenoid. For this purpose a suitable solenoid, 137 cm. long and 22 cm. in diameter, was constructed. It was wound on a brass tube of 7 cm. internal diameter, and consisted of three sections having respectively 4347, 3450, and 3490 turns of number 10 copper wire, thus giving fields of 40·22, 31·88, and 32·30 lines per square centimetre per ampere. In series with each section was a one-tenth ohm standard resistance immersed in an oil-bath; the potential drop across these standards could be read with a Wolff potentiometer, thus giving an accurate measure of the current, and hence of the field due to each section. At the higher fields the large currents produced a marked heating of the apparatus, so that it was necessary to allow a liberal stream of water to flow through the inner brass tube in which the disk was placed. The disk and inductance-coils were exactly the same as before except for the fact that they were enclosed in a water-tight brass box. In this way the temperature of the disk was kept constant to within a very small range, even though the solenoid itself became warm. As before, readings on the variable inductance were taken with the magnetic field in both directions, and the determinations of the current flowing in the sections were made both before and after adjusting the balancing inductance. The numbers given are the means of series of readings.

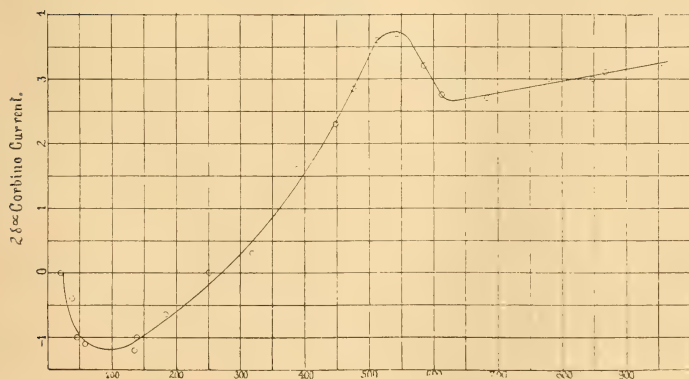
The solenoid has the advantage of affording a uniform field over the whole disk, together with the possibility of dealing accurately with much lower fields and with much smaller variations in higher fields than is possible with the usual type of magnet. It is obvious, from the figures given for the number of lines per square centimetre per ampere, that it is easy to vary the fields to the greatest nicety and also to determine them with an accuracy more than amply sufficient for this purpose. There is another advantage of great importance in the manipulation. In the arrangement used, there are always coils of very high inductance in series with the galvanometer and, at the same time, in the magnetic field, so that any unsteadiness in the magnetizing current,

producing a corresponding change in the field, is immediately apparent in the behaviour of the galvanometer. With the magnet there is a relatively high field produced per ampere, so that any inconstancy in the magnetizing current is very troublesome in producing fluctuations of the galvanometer.

The Corbino Effect in Iron (solenoid).

H.	δ/H .	$c \times 10^7$.
20.2	0.00000
38.5	-0.00519	-31.3
45.9	-0.01089	-65.6
57.6	-0.00977	-58.8
134.6	-0.00446	-26.9
139.0	-0.00360	-21.7
184.3	-0.00176	-10.6
249.5	0.00000	0.0
317.8	0.00094	5.69
389.2	0.00206	12.4
448.5	0.00256	15.4
475.2	0.00300	18.1
511.7	0.00352	21.2
542.7	0.00341	20.6
570.3	0.00307	18.5
585.1	0.00273	16.5
613.5	0.00224	13.5
683.5	0.00197	11.9
780.1	0.00192	11.6
848.0	0.00177	10.7
868.0	0.00179	10.8

Fig. 8.



On the other hand, for the solenoid, the field per ampere is comparatively low and, hence, small variations in the current have little influence on the galvanometer. In fact, while using the solenoid, no such difficulty, on account of small changes in the current, was encountered.

As will readily be seen from fig. 8, in which the balancing

inductance is plotted against the field, the same peculiar maximum occurs, although the field at which it takes place is different and the absolute value of the peak is altered. In all probability the occurrence of the maximum at a distinctly lower field is due to the fact that, when the magnet was used, one of the pole-pieces was traversed by a hole, which so distorted the field that the density of lines was very low over the central part of the plate. It is evident from the formula for the circular current,

$$C = \frac{c}{2\pi} \log \frac{r_2}{r_1} IH,$$

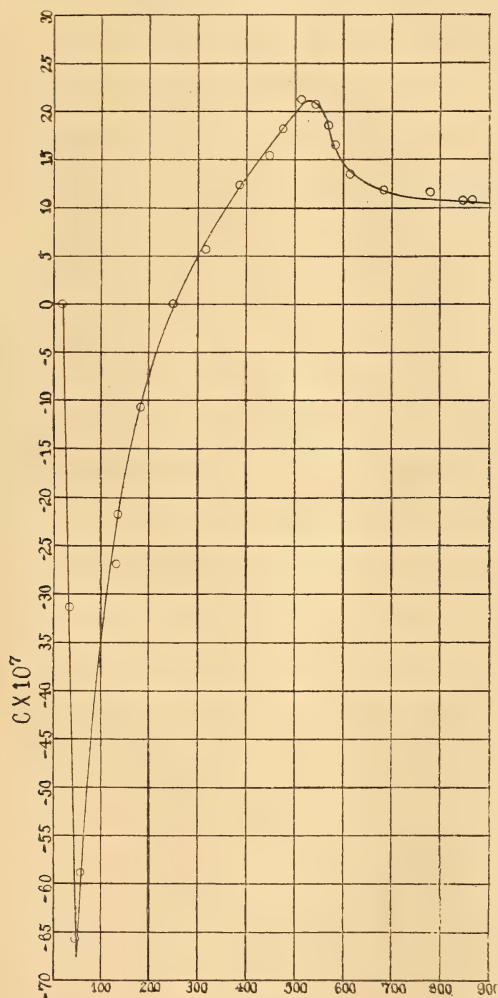
that its magnitude is inversely proportional to $\log I/r_1$, and hence, if the region of small radii is excluded from the field, that a proportionately stronger field is needed to produce a given circular current. Obviously, the field over the whole disk is uniform in the case of the solenoid, and so, since the region of small radii comes under its full influence, it might be expected that a given peculiarity in the curve would take place at a lower field strength.

On pursuing the investigation to values of H below 300, another remarkable effect was brought to light, as is shown by the same curve. Below this point the circular current is negative. So far as the writer is aware, this is the first instance in which such a change in sign in the Corbino effect has been detected in a pure metal. At first sight it seems reasonable to suppose that this reversal might be due to residual magnetism, but such is not the case. A current of 33 amperes was passed through two coils of the solenoid connected in parallel in such a way as to produce a positive flux, then readings of the Corbino effect at a positive field of 100 were obtained and found to be negative. Next, an equally high negative field was applied for a time and a balance again sought at 100 positive. The Corbino effect was negative as before, showing that the reversal is not due to residual magnetism, for the current of 33 amperes was considerably higher than any actually used for the plots given. It should be remarked though, that, after taking a set of determinations for a curve, going from low to high fields, the disk was always carefully demagnetized by reversals before proceeding again to lower fields. This was purely a matter of precaution, as the effect of residual magnetism was actually very small.

In fig. 9 we have a plot of the circular current divided by the field; here the maxima and minima are very sharply

defined. The minimum is made very conspicuous because of the small value of the field and the quite appreciable

Fig. 9.



value of the circular current. This makes m/H very large negatively. The maximum at 525 is much the same as that found in fig. 5.

The argument used in the case of the bismuth-tin alloys

against the possibility of the other two transverse effects seriously interfering with the accurate determination of the Corbino effect applies equally well here. There is the additional assurance that isothermal conditions were approximately realized by the use of the water-bath with both the magnet and solenoid.

Unfortunately it is impossible to attain high values of H with such a solenoid as the one used in this research. For that region one must rest content with the curves obtained with the magnet. At low values of H , for various reasons mentioned above, the results of the experiments with the solenoid must be considered much the more trustworthy.

The Corbino Effect in Bismuth at Low Fields.—In certain metals, notably bismuth and antimony, the Corbino effect is large enough to admit of investigation at low fields. This was done more or less thoroughly for bismuth and roughly for the 1 per cent. alloy of bismuth and tin. It may be observed from fig. 1 that the three points for low fields all correspond approximately to the same value of c . That this is actually the case for this region is shown by the accompanying table. The value of c up to a flux of 800 lines per square centimetre is constant within the limits of error. While the constant c is of the same order of magnitude as that found with the magnet, there is yet, as in the case of iron, a marked difference between them. It is likely that, as before stated, this is due to the distortion of the field occurring when the magnet is made use of.

The Corbino Effect in Bismuth (solenoid).

H.	$2\delta/H$.	$c \times 10^7$.
44.0	0.027	116.4
83.8	0.0276	119.
136.5	0.0258	111.
182.6	0.0268	116.
269.4	0.0267	115.
314.9	0.0267	115.
397.1	0.0277	119.
441.6	0.0272	117.
519.4	0.0271	117.
562.5	0.0270	116.
631.7	0.0269	116.
678.2	0.0267	115.
736.5	0.0272	117.
786.5	0.0266	115.
838.2	0.0264	114.
Mean		... 115.9

Below are the values of c for the bismuth-tin alloy (1 per cent.) ; they must be considered rather unreliable on account of the extremely small effect obtained at these fields.

The Corbino Effect in a Bismuth-Tin Alloy
(about 1 per cent. solenoid).

H.	$2\delta/H$.	$c \times 10^7$.
380	0.8	9.1
499	1.1	9.5
618	1.25	8.7
743	1.6	9.3
876	2.0	9.8
	Mean	... 9.3

The Effect of the Lack of Symmetry of the Field in the Electromagnet.—As was explained before, in taking measurements such as the present ones, the balancing inductance is adjusted so that the galvanometer shows no deflexion. This is done first with the field in the positive direction ; then the field is reversed, the inductance again adjusted for no deflexion, and half the difference between the two readings on the adjustable inductance is taken as a measure of the Corbino current. In every case where the magnet was employed, the two positions of the balancing inductance were asymmetrical with respect to its position for zero field. It was at first supposed that this was due to the lack of exact uniformity of the disks and a consequent circular component of the primary radial current. However, upon experimenting with the solenoid, this lack of symmetry was found to disappear, showing that some other reason must be sought for its presence when the magnet is used. The possibility at once suggests itself that a distortion of the field might be responsible ; it is seen that such a condition could easily arise through the lack of parallelism of the pole-faces, or through a lack of coincidence between the centre of the disk and the centre of the hole traversing one of the pole-pieces, or by a failure to place the disk perpendicular to the flux. In order to test this, a bismuth disk was so arranged as to be movable between the pole-shoes while everything else, including the magnetizing current, remained constant. Approximate settings were then made with the disk in varying positions ; the movements were such as to produce exaggerated representations of changes in position

that might occur during the progress of ordinary experiments. These changes would arise, in practice, through opening the magnet, changing disks, and adjusting the inductance-coils. The figures below show that such changes do alter the aspect of the readings for positive and negative fields as regards their symmetry about the balance-point for zero field.

$P_0 - P_+$	$P_- - P_0$
33	37
31	37
28	33
34.5	32.5
35.5	36.5
36	34
30	36

$P_0 - P_+$ is the difference between inductance balance-points for zero fields and positive fields, and $P_- - P_0$ is the corresponding difference for zero and negative fields.

These rough determinations show that, by a simple change of the position of the disk in a field so distorted, the aspect of the figures as regards symmetry may be totally changed. This is partly taken account of by using half the difference between inductance settings for positive and negative fields as a measure of the Corbino effect. But this does not take into consideration the symmetrical distortion of the magnetic field and, as pointed out above, the experiments with the solenoid are to be regarded as much the more satisfactory in that regard.

The writer is greatly indebted to Professor E. P. Adams, under whose direction this research was undertaken, for his generous help and unflinching interest throughout the progress of the work.

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XXXVI. *Preliminary Note on the Stark Effect of the 4686 Spectrum Line.* By E. J. EVANS, D.Sc. (Lecturer in Physics, Manchester University), and C. CROXSON, B.Sc.*

[Plate V.]

THE effect of an electric field on spectrum lines which was discovered by Stark † is obviously closely connected with the structure of the atom, and it was with the object of further testing existing theories as to the origin of the 4686 spectrum line that this research was undertaken. The Stark effect in the case of hydrogen was examined from the theoretical standpoint by Warburg ‡, and also by Bohr §, who based their work on the view put forward by Sir E. Rutherford that the hydrogen atom consisted of a single electron revolving round a positive nucleus. Bohr deduced that in an electric field each of the hydrogen lines of the Balmer series should consist of two components polarized parallel to the field, and showed that Stark's results for the separation of the two strong outer components polarized parallel to the field for the first five lines of the Balmer series agreed approximately with the values calculated from his theory. Further, in the case of an electron of charge e revolving round a nucleus charge Ne in an electric field of strength E , he deduced the following expression for the frequency difference $\Delta\nu$ between the two outer components

$$\Delta\nu = \frac{3}{4\pi^2} \frac{hE}{Nem} (n_2^2 - n_1^2), \quad . \quad . \quad . \quad (1)$$

where h and m are Planck's constant and mass of electron; and n_2 and n_1 are the numbers representing the position of the spectrum line in a series of the type

$$\nu = N^2K \left[\frac{1}{n_1^2} - \frac{1}{n_2^2} \right],$$

where K is the Rydberg constant.

Stark's experiments had shown that the effect of an electric field was far more complicated than the deductions from the simple theory mentioned above, and this complexity was emphasized in connexion with theories on the structure of the atom. Recently, however, Bohr's theory has been

* Communicated by Sir E. Rutherford, F.R.S.

† *Electrische Spektralanalyse Chemischer Atome*, by J. Stark (Leipzig, S. Hirzel, 1914).

‡ Warburg, *Verhandl. d. Deutsch. Phys. Ges.* xv, p. 1259 (1913).

§ Bohr, *Phil. Mag.* xxvii, p. 506 (1914); xxx, p. 404 (1915).

generalized by Epstein *, who, employing Sir E. Rutherford's model of the hydrogen atom, has obtained results for the positions of the components of the H_{α} , H_{β} , H_{γ} , and H_{δ} lines of the Balmer series in complete agreement with Stark's experiments.

The complete theory of the effect of an electric field on the 4686 line, which on Bohr's theory is due to a helium atom consisting of an electron revolving round a nucleus charge $2e$, has not yet been published, but the ratio of the separation of its components to that of the H_{β} line of hydrogen can be readily deduced from Bohr's simple theory. If, in equation (1), for H_{β} we put $N=1$, $n_2=4$, $n_1=2$, and for the 4686 line $N=2$, $n_2=4$, $n_1=3$, it is found that the ratio of the separation of the outer components of the 4686 line to that of H_{β} is as 7 : 24. This ratio of the separations will also be approximately true for wave-length.

Since the experimental results obtained by us at present are not complete, it is proposed to examine the question further, either with a spectroscope of higher power or by the application of stronger electric fields.

The experiments were carried out by Lo Surdo's † method, in which the spectrum of the dark space in a helium vacuum tube excited by a steady voltage was photographed. The voltage was obtained from a 2000-volt dynamo connected in series with a battery of small cells giving about 3000 volts. The spectrum was photographed by a direct-vision prism instrument which gave a bright spectrum and had such a dispersion that 1 mm. on the photographic plate at λ 4700 represented 10.8 Å.U. Several photographs were taken, one of which is reproduced in Plate V. During this exposure the constant voltage on the tube was 4000 and the pressure of the helium was 1.96 mm.

The strong lines which appeared on the plate consisted of the lines of the ordinary helium spectrum, the H_{β} line (4861), the mercury line (4358.5), and the 4686 line, which has practically the same intensity in the dark space as in the portion of the discharge immediately above. The H_{β} line, the second member (4472) of one of the diffuse series of helium, and the second and third members of the other diffuse series at λ 4922 and λ 4388 showed the well-known effect of the electric field which has been studied in detail by Stark, Lo Surdo, and Brunetti ‡. The maximum distance

* Epstein, *Phys. Zeit.* xvii. pp. 148-150 (1916).

† Lo Surdo, *Estratto da L'Elettrotecnica Giornale ed Atti dell' Associazione Elettrotecnica Italiana*, Anno i. no. 25, Ottobre 5, 1914.

‡ Brunetti, *Rendiconti della R. Accademia dei Lincei*, xxiv. (1915).

between the extreme components of the 4388 line was $15\cdot7$ Å.U., and, according to Stark's measurements in the case of an electric field of 28,500 volts. per cm., this separation would correspond to a field-strength of about 37,000 volts per cm. The 4686 line showed a definite broadening in the dark space, but no definite components could be identified. The line in the strongest photographs appeared to be broadened nearly symmetrically, and the intensity of the outside edges was less than the middle portion of the line. The total width of the line in the dark space was about $3\cdot7$ Å.U., and the corresponding distance between the components of H_{β} was $7\cdot2$ Å.U. Each of the components of H_{β} was double, but the lines were too diffuse for an accurate estimate of the distance between the outside components. However, by making use of Stark's results, it could be calculated that this distance was approximately 10 Å.U. For the sake of comparison the widths of the 4686 and H_{β} lines were measured in the region above the dark space, and found to be in each case about $1\cdot4$ Å.U. Although it is impossible from these results to test quantitatively the simple theory of the Stark effect for the 4686 spectrum line, it seems evident that if the broadening of the line is altogether due to the electric field its magnitude is decidedly less than one-half that of H_{β} . It also follows that it is difficult to explain by means of the Stark effect the great width of the line in comparison with the hydrogen lines when they are generated by a condenser-discharge.

University of Manchester,
July 1916.

XXXVII. *Scattering and Regular Reflexion of Light by an Absorbing Gas.* By R. W. WOOD and M. KIMURA*.

[Plate VI.]

IN previous papers† by one of us, it has been shown that mercury vapour, at room temperature, in an exhausted quartz bulb, when illuminated by the light of a quartz mercury arc, re-emits diffusively a monochromatic radiation of wave-length 2536, which has been named resonance

* Communicated by the Authors.

† R. W. Wood, "Selective Scattering and Absorption by Resonating Gas Molecules," *Phil. Mag.* May 1912.

Phil. Mag. S. 6. Vol. 32. No. 189. Sept. 1916. Z

radiation (Plate VI. fig. 3: lower spectrum, mercury arc; upper, resonance radiation). This radiation appears to be wholly free from polarization, even when the exciting radiation is plane polarized. If the density of the vapour is increased, the diffuse radiation (which is at first a volume radiation) is confined to a shallow layer of molecules lining the front surface of the illuminated bulb, and, with further increment of density, disappears entirely, being replaced by regular reflexion, much as if the inner surface of the bulb was silvered.

In the present investigation an attempt has been made to clear up some of the doubtful points mentioned in the previous papers, and ascertain, if possible, how the diffuse scattering passes over into regular reflexion.

In the earlier work it was found that the resonance radiation had its maximum intensity at the moment of starting the quartz arc, and that it became almost negligible after the lamp had been in operation for eight or ten seconds, as a result of self-reversal of the exciting 2536 line, and consequent absence of the exact frequency necessary for stimulating the mercury molecules in the bulb. In our work, we have used a water-cooled quartz mercury arc of the type described by Kerschbaum*, in which the arc is driven against the front wall of the tube by a weak magnetic field. This reduces the self-reversal to a minimum, for the cooler, non-luminous absorbing layer is "squeezed out," so to speak, the current-carrying vapour being in contact with the quartz wall of the tube. The spectrum of such a lamp is quite unique in appearance; for the 2536 line is so much brighter than any of the other lines, that it is enormously overexposed, appearing much like a photograph of a distant arc light taken at night. Bright diffraction rays radiate from it in all directions, causing it to stand out in the photograph with the conspicuousness of a first-magnitude star in the Milky Way. A photograph of the spectrum is reproduced as a negative on Plate VI. fig. 4. The wavy lines joining the two spectra were caused by the elevation of the plate between the two exposures.

The lamp consists of a straight tube of quartz with the negative electrode (mercury) below, and a positive electrode of tungsten above. It was made to order by the Cooper-Hewitt Co., and operates on 110 volts, with resistance sufficient to hold the current down to about 3.5 amperes. During operation the anode is at a full red heat, but the

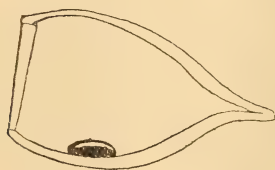
* 'Electrician,' 72. p. 1074 (1914).

rapid circulation of water in the brass jacket keeps the tube quite cold. As a resonance lamp we have used a quartz tube, closed by worked plates of fused quartz, containing a drop of mercury, and highly exhausted. This tube was mounted in front of, and close to, the crystalline quartz plate which formed the window of the water-jacket of the lamp. It is important to have the rays of the arc traverse the mercury vapour as near to the front window of the resonance lamp as possible, since it has been shown in one of the previous papers that the intensity of the resonance radiation is reduced to one-half of its value by traversing a layer of mercury vapour, at room temperature, only 5 millimetres in thickness. A screen of black paper, perforated with a hole, cuts off stray radiation scattered by the walls of the resonance lamp, and it is advantageous to cover the further end of the tube with a small cap of black paper, or provide some other suitable black background.

If the invisible light from the resonance lamp is focussed upon a sheet of uranium glass by means of a large quartz lens, we obtain a bright spot of yellow fluorescent light, and can render visible the vapour rising from a warm drop of mercury by holding it close to the screen in the path of the rays, the shadow of the vapour cast on the uranium glass appearing like a column of black smoke, as shown in one of the photographs published in an earlier paper.

In the present investigation we have used, for the study of the reflexion of the light by the mercury vapour, an exhausted thick-walled bulb of quartz closed by a plate of slightly prismatic form, as shown by fig. 1.

Fig. 1.

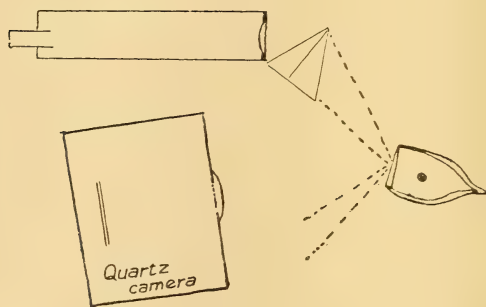


By means of this arrangement the rays reflected from the inner surface could be studied uncontaminated by the reflexion from the outer surface.

In the earlier work, the curved surface of the bulb was used, which made experiments on the polarization of the reflected light impossible.

We first investigated the diffuse resonance radiation as a function of the density of the mercury vapour in the bulb. The light of the quartz arc was passed through a quartz monochromator arranged to give a convergent cone of 2536 monochromatic light. It was simply a roughly constructed quartz spectroscope with a very wide slit and no telescope-tube, shown in diagram in fig. 2. The image of the slit

Fig. 2.



formed by the 2536 rays was located in space by means of a strip of uranium glass, and the bulb mounted in such a position that the image fell upon the centre of the prismatic plate. The dispersion was sufficient to remove the other images of the slit from the bulb, which obviated the use of a second slit and lens for obtaining the monochromatic illuminating beam. The quartz bulb was mounted over a chimney of thin sheet iron, with a Bunsen burner at its base, and the temperature determined by a nitrogen-filled mercury thermometer, the bulb being in contact with the upper surface of the quartz bulb. A camera of very simple construction, furnished with a quartz lens, was focussed upon the bulb, the process consisting in first focussing it with uranium glass upon the image in space of the 2536 line formed by the monochromator, and then measuring the distance between the lens and the image in space. This gives us the proper distance of the bulb from the lens to secure a sharp focus.

The arrangement of the apparatus in this experiment is shown in fig. 2, the rays reflected from the two surfaces of the prismatic plate falling to one side of the lens. In this way we obtain only an image of the scattered resonance radiation from the bulb.

The photographs obtained at different temperatures are

reproduced on Plate VI. fig. 1. The temperatures of the mercury vapour are as follows:—

	Temp.	Pressure of Hg vapour.
<i>a</i>	23° 00168
<i>b</i>	40 00574
<i>c</i>	76 0750
<i>d</i>	100 276
<i>e</i>	175 11.00

The bulb at room temperature (*a*) appears more or less filled with the resonance radiation. This, as has been shown previously, is due to the fact that the radiation from the molecules which lie in the path of the primary beam excite to resonance the entire mass of vapour in the bulb. At 40° (*b*) the pressure has increased about 3.4 times, and the radiation comes chiefly from the front part of the bulb. At 76° (*c*) it is confined chiefly to the inner surface of the plate, though a slight haze to the right of the image indicates that some radiation capable of exciting resonance still penetrates to a depth of a millimetre or so. This radiation is without doubt of wave-length slightly greater and slightly less than that of the centre of the exciting line, in other words the edges of the 2536 line. The size of the patch of resonance radiation contracts rapidly as the temperature goes up, owing to the inability of the radiation to spread out and excite secondary resonance. At 175° (*e*) it has shrunk to the dimensions of the image of the slit thrown upon the bulb by the monochromator. It will be noticed that in case (*d*) where the density is 16 times as great as at room temperature, there is still a slight broadening of the image, due to the secondary resonance. The spreading is of the order of half a millimetre, which is about what we should expect from the known stopping power of the vapour at room temperature. A further increase of temperature causes a rapid diminution in the intensity of the scattered resonance radiation, the energy of the primary beam passing off as a regularly reflected wave. The intensity is a maximum in the vicinity of 100° (*i. e.* at a pressure of about 0.3 mm.). At 150° (pressure about .3 mm.) the intensity has decreased to about half its maximum value; at 200° (pressure 18 mm.) to about one quarter, and at 250° (pressure 76 mm.) to perhaps one tenth. At 270° there is absolutely no trace of the scattered radiation. The estimates were made from a series taken under similar conditions, with a slit somewhat broader, so that a better determination of the relative densities could be made. Equal

exposure times (40 secs.) were given and the images developed simultaneously. It is evident that the scattered resonance radiation decreases (replaced by true absorption probably) long before regular reflexion commences.

Experiments on Selective Reflexion.

In our first experiment on this subject we placed the quartz bulb a little inside of the focus of the monochromator, so that the incident radiations came to a focus after reflexion from the prismatic plate. A plate of uranium glass was mounted in such a position that the two reflected images were focussed on it. The image formed by reflexion from the outer surface was noticeably brighter than the other, owing to absorption by the fused quartz plate, which was twice traversed by the rays reflected from its inner surface. On heating the bulb to a red heat with a Bunsen flame, the latter image brightened up until it appeared to be about three times as bright as the image reflected from the outer surface. In this way it is possible to demonstrate the selective reflexion of the vapour to a small audience at close range. A sheet of heavy plate glass must be used as a protection against a possible explosion of the bulb as the pressure may rise to 15 or 20 atmospheres.

The reflecting power was next determined quantitatively in the following way:—

The total radiation from the water-cooled arc was reflected from the inner surface of the prismatic plate into a small quartz spectrograph, the slit of which was opened rather wide, and shortened to a length of about 1 millimetre. The spectrum lines thus photographed as small rectangular patches, and twenty or thirty exposures could be made on a single plate. The exposures were made by a slow swinging shutter of the pendulum type. We first made a series of exposures of continuously increasing duration by operating the shutter once, twice, three times, &c., with the bulb at room temperature. This gave us a record of the reflecting power of the inner surface of the quartz plate. The bulb was then raised to a red heat, and another series of exposures were made in the same way.

The plate showed that the rectangle representing the 2536 line had the same intensity for an exposure of five seconds for quartz reflexion (bulb cold), and one second for mercury reflexion (bulb hot).

Since the reflecting power of a surface of fused quartz is roughly 5 per cent. in the ultra-violet, this experiment shows

us that the reflecting power of a surface of dense mercury vapour for the light of the 2536 line is not far from 25 per cent., nearly that of most metals in the same region of the spectrum. The next point to determine was the density at which selective or metallic reflexion commenced. Various methods were tried, the following being the one finally adopted.

It is of course desirable to repress as completely as possible the ordinary, or vitreous, reflexion of the quartz surface. This can be done by polarizing the incident beam with its electric vector horizontal, and setting the prismatic face of the bulb at the polarizing angle. Under these conditions the intensity of the two beams reflected from the outer and the inner surface is reduced nearly to zero, and a very slight increment in the reflecting power of the inner surface due to the mercury vapour becomes at once apparent provided the reflecting power is increased regardless of the direction of the plane of polarization, as proved to be the case.

The divergence of the two reflected beams was so great that both were not received by the quartz lens of the camera, consequently the one reflected from the outer surface was made nearly parallel to the other by reflexion from a piece of platinized glass at nearly grazing incidence. Two small images of equal intensity were thus recorded simultaneously on the photographic plate, one above the other. The upper, due to reflexion from the outer surface of the prismatic plate; the lower, representing the reflexion from the inner surface. The temperature of the bulb was now gradually raised, and a number of exposures of equal duration made at different temperatures, the plate being moved slightly between the exposures. Fig. 5, Plate VI. shows the result of the final experiment. The first three exposures, "*a*," "*b*," and "*c*," corresponding to temperatures of 180°, 210°, and 235°, showed both images of equal intensity; in other words, no increment of reflecting power resulted from the presence of mercury vapour up to a pressure of 50 mm. (at 235°). Exposure "*d*," taken at a temperature of 279°, showed the lower image considerably brighter than the upper one. The pressure in this case was about 120 mm.; and since we can infer that the effect would be noticeable at a slightly lower pressure, we are safe in saying that the first appearance of specular reflexion by the vapour takes place at a pressure not very far from 10 cm. Exposures "*e*" and "*f*" were made at temperatures of 300° (pressure 25 cm.), and one other still higher temperature which was beyond the reach of the thermometer. On the print these have practically the same intensity, but

on the original negative the density of "*f*" is certainly double that of "*e*."

As subsequent experiments showed that the dense vapour reflects polarized light in much the same way as a film of metal, it was of some interest to see whether the reflecting power of the quartz surface in contact with the vapour passed through a minimum before beginning to increase with increasing vapour pressure. In the case of metallic deposits on glass, the reflecting power of the glass is considerably diminished by very thin layers of metal, when the reflexion is from the glass side; in fact it is reduced nearly to zero if the thickness of the metallic film is just right. If a number of strips are silvered cathodically on the face of a prism of small angle (five to ten degrees) with exposures to the discharge of from say one minute to ten minutes, one or more strips will be found which appear quite black in reflected light when viewed through the glass, though all reflect much more powerfully than the glass when viewed from the silver side.

The mercury vapour was examined for a similar phenomenon in the following way:—The convergent 2536 beam from the monochromator was reflected from the prismatic plate of the bulb, and the two reflected beams received on a plate of uranium glass. The temperature of the bulb was then gradually raised and the fluorescent images on the uranium plate watched. If the vapour behaved exactly as a metal film of increasing thickness, the image formed by reflexion from the inner surface ought to fade away gradually and then rapidly brighten. No trace of such a phenomenon was observed. The two images remained of the same intensity until a temperature of 250° was reached; above this point the image due to the inner reflexion rapidly brightened as the temperature rose, reaching its maximum brilliancy in the neighbourhood of 300° , at which temperature it appeared to be four or five times as bright as the other image.

This specular or metallic reflexion of the light by the vapour occurs only when there is exact synchronism between the luminous vibration and the free period of the system which causes the 2536 line. This fact is emphasized because there is another type of selective reflexion which occurs when the synchronism is not exact, and which is the result of the refractive index of the vapour. This will be discussed presently, after the polarization experiments have been treated.

Polarization Experiments.

Renewed attempts have been made to detect traces of polarization in the scattered resonance radiation, but without success. Even when the vapour is illuminated with plane polarized light, and the spot of surface luminosity (which we have at say 100°) is photographed through a Savart plate or Fresnel double prism of R. and L. quartz with a polarizing analyser of quartz and Iceland spar, no trace of the fringes appear. The same thing occurs in the case of sodium vapour illuminated with a sodium flame. This is very remarkable, since strong polarization has been found associated with the stimulation of the frequencies corresponding to the band or channeled spectra of sodium and iodine vapour, the polarization showing not only in the line directly excited by the monochromatic light (resonance radiation), but also in all of the other lines (lines of the resonance spectrum *).

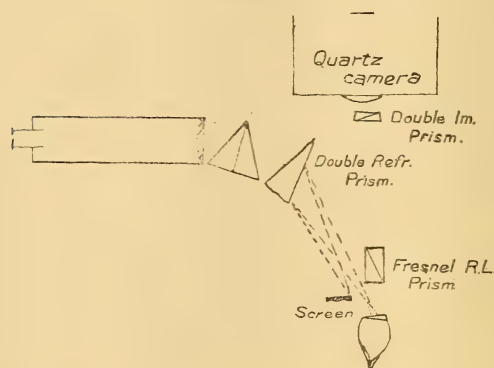
In our present search for traces of polarization we employed the Fresnel double-image prism of right- and left-handed quartz. This gave, with polarized green mercury light, on analysis, six black horizontal bands, while with the light of the 2536 line about thirty bands could be counted. They were very distinct, however, in the photographs made with the quartz camera. As an analyser we used a double-image prism of Iceland spar and quartz, which we made by grinding and polishing a prism of about 8° from a small piece of spar, securing direct vision and fair achromatization by compensating it with a small quartz prism of about the same angle. Both prisms were ground and polished in less than an hour.

The double-image prism was mounted in front of the lens of the camera, and the aperture considerably reduced by a diaphragm to secure sharp definition of the fringes. In some of our experiments we employed a small Foucault prism as a polarizer, but the loss of light was considerable, and we accordingly cut a 60° prism of quartz perpendicular to the axis; this, when mounted about 10 cm. behind the monochromator, in the converging beam of 2536 light, gave two brilliant polarized images on the uranium glass plate, separated by a distance of about 3 cm. One of these was cut off by a screen, and the rays diverging from the other illuminated the prismatic plate of the bulb.

* R. W. Wood, *Phil. Mag.* July 1908, and Oct. 1911, p. 480.

The arrangement of the apparatus is shown in fig. 3, which explains itself. The camera is of course focussed upon

Fig. 3.



the Fresnel prism, by adjusting the distance as described previously. An exposure of four or five minutes was sufficient to give the two images of the spot of resonance radiation formed by the double-image prism. The presence of two or three per cent. of polarization would be indicated by faint traces of the horizontal dark bands in the images.

At first we obtained distinct traces of the bands, but their appearance and intensity were very variable, and we finally found that they were due either to a slight deposit of mercury globules on the inner surface in the high temperature experiments, or to a slight cloudiness of the inner surface of the prismatic plate near one edge. We finally got things so adjusted that no trace of them appeared at any temperature between 20° and 200° , above which the scattered resonance radiation practically disappears; and we feel quite certain that there is no trace of polarization in the scattered radiation, even when the incident light is plane polarized, and the density of the vapour is so great that we are approaching the stage at which selective metallic reflexion begins. This seems very remarkable, since, as we shall see presently, if the incident light is polarized, the metallically reflected wave is polarized also.

This makes it appear probable that we shall have to reject the idea that the reflected wave can be accounted for by the application of the Huygens principle to the waves emitted by the resonators. Moreover, as the vapour density increases, the scattered resonance radiation practically disappears some

time before the appearance of the reflected wave, which is additional evidence against such a view. We had hoped to find that, as the density increased, an increasing percentage of polarization would be found in the scattered radiation, but such does not seem to be the case.

The complete absence of polarization appears to be rather remarkable, and not easy to explain. A rapid rotation of the molecule would probably act as a depolarizing factor; and if the vibrations of the resonator were constrained to take place along fixed lines or planes in the molecule, these would be oriented in all possible positions, and we should expect, from the most elementary considerations, less than 50 per cent of polarization, even with the incident light plane polarized. This was found to be the case with the resonance spectra of sodium and iodine vapour; but in the present case, where we are apparently dealing with a vibration of much simpler type, there is no trace of polarization whatever.

If we are to regard the metallically reflected wave as the resultant effect of vibrations emitted by a closely packed system of resonators in perfect synchronism, we must show how the polarization results, for the reflected wave is plane polarized as we shall now see.

In the case of the reflexion of light from metal surfaces, if the plane of polarization is parallel or perpendicular to the plane of incidence, the reflected light is also plane polarized. If, however, the incident light is polarized in an azimuth of 45° , the reflected light is usually more or less elliptically polarized, due to the phase difference between the two reflected components.

We examined all three cases, but need mention in detail only the one in which the incident light is polarized in azimuth 45° . The light of the water-cooled arc was passed through a small Foucault prism arranged to transmit vibrations inclined at 45° to the vertical. It was then reflected at the polarizing angle from the inner surface of the prismatic plate of the bulb into a quartz spectrograph. Between the slit and the collimating lens, and close to the latter, we mounted the double-image prism already referred to, arranged so as to transmit horizontal and vertical vibrations, the images formed by it lying one above the other. It may be worthy of mention that in working with a quartz spectrograph it is important to analyse the polarized light before it enters the lenses, on account of the natural rotation of the latter.

The rays which entered the spectroscope were of course

plane polarized by reflexion from the inner surface of the prismatic plate, or, in other words, the incident polarized light was split into two components, one of which was wholly transmitted, while the other was in part reflected, with its vibrations parallel to the reflecting surface. The light thus reflected gave a single image after passage through the double-image analyser. The slit of the spectrograph was opened somewhat and its length contracted to such a degree that the spectrum-lines of the mercury are appeared as small squares on the plate. Exposures were made with the bulb at room temperature (fig. 6, Plate VI., upper spectrum), and at a red heat (lower spectrum). In the latter case, owing to the metallic reflexion of the 2536 line, both components are reflected with equal intensity, and unite into a plane polarized resultant in azimuth 45° . This is doubly refracted by the analyser, yielding *two* images of the square for the wave-length in question. It will be noticed that all of the other spectrum-lines (squares) are represented by single images only.

This experiment does not prove, however, that the reflected light is plane polarized, for we should have a similar appearance if it was depolarized or circularly polarized by the reflexion.

To prove that it is plane polarized we must rotate the double-image prism into such a position that one of the two images disappears. It was assumed that this would occur for a rotation of 45° , and we accordingly turned the prism through an angle of about 40° , and then made a number of successive exposures, turning the prism through a small additional angle each time. One of these exposures showed the second image completely absent, proving that elliptical polarization was not present, or at all events that it was too small to be detected.

Selective Reflexion and Refractive Index.

Selective reflexion of another type occurs at the boundary surface separating quartz from dense mercury vapour. This occurs in the case of frequencies slightly higher than that of the 2536 line. The mercury resonators in this case emit no scattered radiation, and there is practically no loss by absorption. In some earlier work *, in studying the reflexion of the light of the iron arc by mercury vapour, it was found

* R. W. Wood, "Selective Reflexion of Monochromatic Light by Mercury Vapour," *Phil. Mag.* xviii. p. 187 (1909).

that an iron line one Ångström unit on the short wave-length side of the 2536 mercury line, was much more powerfully reflected than a pair of iron lines on the long wave-length side situated at 0.1 and 0.4 Å.U. from the mercury line. No explanation of this was given in the paper, but the suggestion was made later in Wood's 'Physical Optics' (second edition), page 432, that it undoubtedly resulted from the sudden change in the refractive index of the vapour in the vicinity of the absorption-line.

"The 2536 line shows powerful selective dispersion and the refractive index, in its immediate vicinity on the short wave-length side, is much below unity, probably as low as 0.5, or even much less close to the line. In the case of light going from a rare to a dense medium, a *high* value of the refractive index for the latter is accompanied by strong reflexion. When, however, the ray goes from dense to rare (quartz-mercury vapour) as in the present case, a low value of the index for the latter is accompanied by strong reflexion.

"On the long wave-length side, for a region very close to the line, the index of the dense mercury vapour may rise to a value as high as that of quartz, in which case there will be no reflexion at all."

It would appear then that, if we could employ light of two frequencies, one slightly higher and the other slightly lower than the frequency of the 2536 line, the former would be powerfully reflected and the latter not at all. This condition was realized by employing as our source of light a quartz-mercury arc operated at a potential just sufficient to distinctly double the 2536 line by self-reversal. On Plate VI. fig. 7 we have four views of the 2536 line taken with a small Fuess quartz spectrograph, very accurately focussed. This line has a faint companion on the short wave-length side, indicated by an arrow on the photographs. If the light is first passed through mercury vapour in a heated quartz tube, the main line is weakened or removed by absorption, and the faint companion remains, as shown by fig. 7 (*a*), in which the upper and lower figures represent the line without and with mercury absorption. Fig. 7 (*b*) shows the appearance of the line when the quartz arc, designed to operate at a potential drop of 170 volts, is run at 30 volts, while (*c*) and (*d*) show it reversed at 60 and 80 volts.

We made our experiment as follows:—The light of the lamp running at 80 volts was reflected from the inner surface of the prismatic plate of the quartz bulb into the quartz spectrograph, the slit of which was reduced to a length of

1 mm. by a diaphragm which could be raised by a micro-meter-screw. An exposure of one minute was given: the slit diaphragm was then raised 1 mm., the quartz bulb raised to a red heat by a Bunsen burner, and a second exposure of fifteen seconds made. Fig. 2 (Plate VI.) shows the result of the experiment. The reversed 2536 line appears as a doublet and is indicated by an arrow, the faint companion on the short wave-length side appearing to its left. This was the exposure made by light reflected from the cold bulb. Above it we have the exposure made with the hot bulb. The light reflected from the hot bulb is seen to consist solely of the short wave-length component of the doublet (widened and reversed 2536 line), for which the reflecting power of the quartz-mercury vapour surface is very slight. The long wave-length component has disappeared entirely, owing to the very low value of the reflecting power for this frequency. The width of the doublet is about 0.8 \AA.U. It is perhaps worthy of mention that we have here a rather efficient method of isolating from the total radiation of a quartz-mercury arc running at a moderately high temperature, a single line of wave-length about 0.4 \AA.U. less than that of the 2536 line of a similar lamp running at a low temperature.

This might be very useful in certain special investigations. One could of course make the difference even less than 0.4 \AA.U. by operating the lamp at a lower voltage.

The reflecting power of the dense vapour for the light forming the continuous background of the spectrum of the cadmium spark was also investigated. It was found that the spectrum of the reflected light showed a bright line in coincidence with the 2536 line of mercury. No cadmium line appears at this point, and the continuous background is absent or very faint on the long wave-length side of the bright line, due to the low value of the reflecting power for this region. An enlargement of the photograph is reproduced on Plate VI. fig. 8. The quartz bulb was at a full red heat during this experiment. It is evident from the photograph that the spectroscope lacked sufficient resolving power to show the minimum to advantage. The bright line with its dark border on the right, due to the powerful and feeble reflexion of these wave-lengths in the light of the continuous background of the spark, is indicated by an arrow. Above is the 2536 line of mercury for comparison.

The Structure of the 2536 Line.

The interesting work of Malinowski (*Ann. der Physik*, xliv. p. 947, 1914) has made it appear extremely probable that the 2536 line is, in reality, not a single line, but a very close multiple line. On this account we have investigated the structure of the resonance line with a quartz Fabry and Perot interferometer.

Malinowski found that, if the resonance lamp was placed in a magnetic field, and excited by a lamp outside of the field, the intensity of the resonance radiation increased rapidly with an increase of field strength, reaching a maximum at 1000 gauss, and then passing through successive minima and maxima.

The first maximum is easily explained as follows :—

Assuming the 2536 line of the exciting lamp to be more or less reversed, as is sure to be the case with a lamp of the ordinary type, it is clear that the magnetic components of the resonating vapour in the magnetic field will respond to the frequencies to the right and left of the centre of the reversed line where the intensity is greater.

The fact that successive maxima and minima occur indicates that we are not dealing with a single line, and a normal Zeeman effect. Malinowski obtained a curve of similar type by placing the resonance lamp in a magnetic field and passing its radiation through a quartz cell containing mercury vapour

at room temperature. As ordinates he took the ratio $\frac{J}{J_0}$, in which J is the intensity after passage through the absorption cell, and J_0 the intensity (for a given field) without the cell. This compensates for the variations of J_0 with the field, and the curve shows the transmitting power of unmagnetized mercury vapour for the resonance radiation of mercury vapour in a field of varying strength. A similar curve was also obtained when the absorption cell was placed in the magnetic field and the resonance lamp outside of the field.

These results indicated that the absorption line and emission (resonance) line had a similar complicated structure, and behaved in a similar manner in a magnetic field.

Malinowski showed also that the secondary resonance radiation, excited in a bulb outside of the field by the light of a resonance lamp in the field, gave a curve, the maxima of which occupied the positions of the minima of the curve obtained with the absorption cell, showing that the energy, which was refused passage by the cell, was re-emitted. It was not possible to draw any very definite conclusions

regarding the exact structure of the line from these results, though Malinowski was able to make some interesting calculations as to the probable widths of the component lines.

We have made a preliminary investigation with the interferometer of the structure of the 2536 line of the water-cooled mercury arc and the resonance radiation of the vapour at room temperature. The results are not as satisfactory as we hoped for, owing to the low reflecting power of the metals in this region of the spectrum. Mirrors of silicon were tried, but if thick enough to exhibit the high reflecting power found by Hurlbut, their absorption was too great. With mirrors of cathodically deposited cadmium we obtained fair results, however.

The interference fringes disappeared completely when the Fabry and Perot plates were separated by a distance of 6 mm., and re-appeared with a separation of 12 mm., both for the water-cooled arc and the resonance radiation.

If we consider that the resolving power of the plates is no greater than that of a Michelson interferometer, our inference from this result would be that we were dealing with a doublet with a separation of about 0.03 Å.

As a matter of fact, the resolving power was somewhat greater, as the fringes obtained with small path difference were distinctly narrower than those of a system formed by two interfering beams. The finite widths of the components of the assumed doublet may, however, be sufficient to account for their disappearance with a 6 mm. path difference, and their diminished visibility at 12 mm.; so that, for the present at least, we are justified in considering the line a doublet.

It is obvious that a further study of the line with a higher resolving power will be necessary before any positive statement can be made. This we intend to do in the autumn with a plane grating of 90,000 lines in the 10th-order spectrum, which should give us a resolving power of about .0028.

On looking over the plates made in this region with the very powerful quartz spectrograph of Professor Weiss with a resolving power equivalent to that of 13 large 60° quartz prisms, it seems as if the study of the line with this instrument would scarcely be worth while, for the resolving power is only about .033. On this account it will be necessary to use a large grating unless we succeed in getting something better in the way of a metal film for the interferometer.

In this investigation we have been greatly aided by a substantial grant from the Solvay Institute of Physics of Brussels.

XXXVIII. *Notices respecting New Books.*

Studies in Terrestrial Magnetism. By C. CHREE, M.A., Sc.D. LL.D., Superintendent of Kew Observatory, etc., etc. Macmillan's Science Monographs. Macmillan & Co., Ltd.: London, 1912. Pp. xii + 206.

THIS volume is "intended to give a connected account" of the author's "own original work" in terrestrial magnetism, without aiming at being a text-book or even at summarizing the results of other investigators in those branches of terrestrial magnetism with which it actually deals. The author gives us "almost entirely facts, or supposed facts." Absence of theory is by no means due to lack of curiosity as to "the causes of things," but to the author's well-founded belief that for the time being theorising is less promising than the extension of "positive knowledge."

The book being mainly concerned with results derived from *magnetograms*, the author gives in Chapter I. a useful explanation how such records are obtained and interpreted. The next chapter is dedicated to Secular Changes of the magnetic elements; the subject is illustrated by tables and a diagram extending from 1860 up to 1910. Chapter III. deals with Non-Cyclic Changes, and Chapter IV. with Diurnal Inequalities, obtained by taking in succession the difference of each hourly value from the mean of the twenty-four; here, as in the remaining chapters, many numerical data and several diagrams are given. Chapters V. and VI. are dedicated to Diurnal Inequality on Ordinary, and on Disturbed Days, respectively. The next chapter treats of the use of Fourier series for the representation of diurnal inequalities. Although this method is to be employed with discernment, the author agrees with the well-established opinion of investigators of other branches of terrestrial physics that "even if there should be no natural force answering to each term of a Fourier expansion, a study of individual terms of the series may prove of marked utility." Chapter VIII. deals with Annual Variation, and Chapter IX. with the Absolute Daily Range, *i. e.* the excess of the largest over the smallest value during the twenty-four hours. Comparatively much space is dedicated to Antarctic Magnetic Results (Chap. X.). Chapters XI. and XII., dealing with Magnetic Storms and their "sudden commencements," contain a number of very interesting curves which are likely to attract the attention even of non-specialists. Of equal interest will be the Comparison of Arctic and Antarctic Disturbances, treated in Chapter XIII., in which the author gives us numerous Kew-curves juxtaposed with Birkeland's arctic, and corresponding antarctic

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curves. The remaining chapters of this valuable Monograph are : XIV. Sunspots and Terrestrial Magnetism ; XV. Wolf's Sunspot Formula ; XVI. Nature of Sunspot Relationship ; XVII. General Conclusions,—in which the contents of the preceding chapters are shortly resumed. It is to be regretted that the author has taken no notice of Zeeman and Winawer's researches in solar magneto-optics in connexion with the spectroscopic properties of sunspots.

Edinburgh Mathematical Tracts. London : G. Bell & Sons, Ltd., 1915 :—

No. 1. *A Course in Descriptive Geometry and Photogrammetry for the mathematical laboratory*, by E. LINDSAY INCE, M.A., B.Sc., Pp. viii+79. 2s. 6d.

No. 2. *A Course in Interpolation and Numerical Integration for the mathematical laboratory*, by DAVID GIBB, M.A., B.Sc. Pp. viii+90. 3s. 6d.

No. 3. *Relativity*, by A. W. CONWAY, D.Sc., F.R.S. Pp. 43. 2s.

No. 4. *A Course in Fourier Analysis and Periodogram Analysis for the mathematical laboratory*, by G. A. CARSE, M.A., D.Sc. and G. SHEARER, M.A., B.Sc. Pp. viii+66. 3s. 6d.

No. 5. *A Course in the Solution of Spherical Triangles for the mathematical laboratory*, by HERBERT BELL, M.A., B.Sc. Pp. viii+66. 2s. 6d.

No. 6. *An Introduction to the Theory of Automorphic Functions*, by LESTER R. FORD, M.A. Pp. viii+96. 3s. 6d.

1. The first Tract of this carefully edited and beautifully published series begins with a very clear and interesting exposition of the purpose and the methods of Descriptive Geometry. The subject is introduced by a plain description of Monge's method, of the fundamental properties of projections, and of the methods of "Contours" and of "Perspective." Some useful hints concerning Laboratory Methods are given. The introductory chapter closes with an account of the evolution of descriptive geometry, from Vitruvius up to Lambert. Chapter II. treats of the straight line and the plane in orthogonal projection, and contains a good number of fundamental problems with their solutions, explained on clear diagrams. A few numerical examples accompany each of these problems. The chapter closes with a collection of appropriate exercises. Chapter III. is dedicated to curved surfaces and space-curves, in orthogonal projection, the subject being again developed chiefly upon a series of interesting problems concerning cylindrical surfaces and solids of revolution. In the two concluding sections, the method of contours is applied to topographical surfaces. Chapter IV. initiates the reader in the perspective representation of regular solids, of plane and of twisted

curves, and of curved surfaces, and closes with an exposition of the relation between the method of perspective and that of Monge. Chapters III. and IV. are each provided with appropriate exercises. The last chapter of this exceedingly useful and fascinating Tract is dedicated to the so-called Photogrammetry or Metrophotography, whose aim is to obtain from photographs a correct metrical representation of the object photographed.

2. Contains a collection of useful hints for the practical processes of Interpolation and Numerical Integration. The first two chapters give some theorems in the Calculus of Finite Differences, and formulæ of Interpolation, *e. g.* those of Lagrange, Newton, Stirling, Gauss, and Bessel. The subject is illustrated by numerous concrete examples. Chapter III. treats of the construction and use of Mathematical Tables. Many mathematical and physical students will be particularly grateful to the author for that chapter, in which they will find details on the subject, otherwise not very accessible. The last chapter of this very useful "Course" is dedicated to Numerical Integration.

3. Professor Conway gives in the four chapters of "Relativity" the substance of his four lectures delivered before the Edinburgh Mathematical Colloquium: I. Einstein's Deduction of Fundamental Relations. II. Transformation of Electromagnetic Equations. III. Applications to Radiation and Electron Theory,—with a short investigation concerning relativistic dynamics (pp. 28–34). IV. Minkowski's Transformation,—containing a very short exposition of Minkowski's "geometrical" representation of the Lorentz transformation, and of the meaning and use of Minkowski's four- and six-vectors. This little "Tract" can be recommended as a rapid but easily intelligible introduction to the theory of relativity.

4. The object aimed at by the authors is "not to discuss the series of Fourier from the theoretical standpoint, but rather to provide what is necessary for the practical application of the subject." Chapter I. contains generalities on Fourier's Theorem, illustrated by a number of useful and elegant examples. From Chapter II. the reader learns the art of the practical evaluation of the coefficients of a Fourier expression; here the 12-ordinate and the 24-ordinate Arithmetical Methods are fully explained, a concise computing form, devised by Prof. Whittaker, is annexed, and an account of Perry's, Wedmore's, Harrison-Ashworth's, and Beattie's graphical methods is given. Chapter III. is dedicated to Periodogram Analysis, a subject which "has not received adequate treatment hitherto in any English text-book." This chapter will be very helpful to the physicist, as well as to the astronomer and meteorologist. The last chapter is a brief introduction to Spherical Harmonic Analysis. The exposition of the subject is based on Whittaker's general solution of Laplace's

equation $\nabla^2 V = 0$,

$$V = \int_0^{2\pi} f(z + ix \cos u + iy \sin u, u) du,$$

f being an arbitrary function of its arguments. Thence spherical (and, more especially, zonal) harmonics are deduced. Further, their integral properties are demonstrated, which lead at once to the expansion of a function as a series of surface harmonics. Stress is laid (pp. 57-66) on the practical evaluation of the coefficients, Bauschinger's scheme for the calculation of the auxiliary coefficients being fully explained by means of two numerical tables.

5. The aim of this Tract is to give an account of various methods, both numerical and graphical, of solution of Spherical Triangles. Every teacher of mathematics will gladly share the author's opinion that this subject is of importance not only in view of its applications, but "has also considerable value from the educational point of view." After a few remarks on the use of Logarithmic Tables (Chap. I.), the fundamental formulæ of spherical trigonometry are developed in Chapter II. in an easily accessible and attractive manner. Chapter III. treats of the numerical solution of the Right-angled, and Chapter IV. of that of the General spherical triangle. The following chapter contains special applications of the most important and instructive kind: Great Circle Sailing, Finding the Longitude at Sea, Reducing an Angle to the Horizon, and Conversion of Star Coordinates, with numerical examples. The last chapter is dedicated to Graphical Methods of Solution of spherical triangles.

6. The last of the Edinburgh Tracts hitherto published contains a most fascinating and easily intelligible introduction to the theory of Automorphic Functions which will, doubtless, greatly contribute to the diffusion of this beautiful but by no means very accessible branch of mathematics. The reviewer, tempted to enter upon the numerous attractive details of this beautiful work, regrets the necessity, dictated by reasons of "space," of limiting himself to a mere quotation of the chapters. These are: I. Linear Transformations. II. Groups of Linear Transformations. III. Automorphic Functions. IV. The Riemann-Schwartz Triangle Functions. V. Non-Euclidean Geometry. VI. Uniformisation. —A very full Bibliography of automorphic functions (pp. 88-96), up to 1913, will be helpful to those who desire to pursue the subject further.

Bulletin of the Bureau of Standards.

(Washington: Government Printing Office, 1915.)

AMONGST the publications of scientific interest during 1915 we note the following:—

In the March number is described a determination by the bomb method of the heats of combustion of Cane Sugar, Benzoic acid,

and Naphthalene by H. C. Dickinson. The object of these determinations is to provide standard values to aid in the calibration of the various forms of bomb calorimeters in use in commercial laboratories as well as by scientific investigators. The following table shows the degree of agreement with other recent observers. The results are expressed in 15° calories.

Authority.	Naphthalene.	Benzoic acid.	Sucrose.
Fries (1910)	6318	
Wrede (1910)	9633	6323	3952
Roth (1910)	9643		
Laroux (1910)	9631		
Dickinson (1910-1912)...	9612	6323	3945

In the same number appears a paper by D. R. Harper on the specific heat of copper in the interval 0° to 50°, with a note on vacuum-jacketed calorimeters. The value obtained (correct to third significant figure with temperature coefficient correct to 10 per cent.) is

$$c = 0.383_4 + 0.00020(t - 25) \frac{\text{joules}}{\text{gr. deg.}},$$

or

$$c = 0.0917 + 0.000048(t - 25) \frac{\text{cal.}_{20}}{\text{gr. deg.}}.$$

In the May number appears an investigation on the insulating properties of dielectrics, dealing with the volume resistivity and surface leakage of a number of insulating materials with the view of ascertaining their suitability for replacing hard rubber. The volume resistivity (in ohm-centimetres) is over 5000×10^{15} for special paraffin, ceresin, and fused quartz; it is about 1000×10^{15} for hard rubber. Mica comes next with a value 200×10^{15} , and then sulphur with a value 100×10^{15} . The most insulating variety of bakelite has a value of only 20×10^{15} , while many varieties of bakelite lie very low (*e. g.* No. 140, 20×10^6). A large number of determinations are made on surface resistivity as affected by light and moisture. These are represented by curves. This is due to a surface film, usually of water or oil, on the insulator, and is generally the important factor in determining the leakage between two conductors. However, for insulators having a vol. resistivity less than 10^4 ohm cms. placed in an atmosphere having a humidity less than 25 per cent., the greater part of the current may flow through the insulator.

The same number contains a determination by W. W. Coblentz of the absorption, reflexion, and dispersion constants of quartz, an accurate knowledge of which is so necessary in order to determine spectral energy curves of a black body when a quartz prism is used. Data are given extending from the ultra-violet to

3μ in the infra-red. Quartz is practically transparent from the ultra-violet to 1.8μ . Tabulated data are given for allowing for the effect of the absorption which enters beyond this point. The results show that (within the errors of observation) in unpolarized light, the transmission is not affected by the direction in which the radiations pass through the material with respect to the optic axis.

The Science of Musical Sounds. By Prof. DAYTON C. MILLER, D.Sc. 10s. 6d. net. New York: The Macmillan Co. London: Macmillan & Co. Ltd., 1916. Pp. viii + 286.

THIS work will be heartily welcomed by those who are fascinated by the various problems and methods of harmonic analysis and its applications in connexion with musical sounds. It presents, substantially as given, a course of eight experimental lectures delivered by the author at the Lowell Institute during January and February 1914. As befits a course of lectures intended for a general audience, the subject matter consists in large part of elementary and familiar material, but selected and arranged to develop the main purpose. But, in accordance with the design of all such lectures at the Lowell Institute, however elementary their foundation, they meet the legitimate expectation that they shall include the most recent progress of the science under review and experimentally illustrate this to the fullest possible extent. It is somewhat difficult to adequately present in book form this demonstration aspect of a course of lectures. But in this case the lavish use of diagrams and photographs of special apparatus and their records, or of arrays of forks, pipes, &c., leaves nothing to be desired.

After two preliminary lectures on waves, vibrations, and tones, we have a lecture devoted to methods of recording sound waves. After passing in review the time-honoured methods, the author introduces his own device the *phonodeik* or *sound-demonstrator*. This consists essentially of a horn and glass diaphragm for receiving the sound, and a connected mirror for demonstrating its amplitude, frequency, and quality. The mirror has a few silk fibres from the diaphragm passed once round its axle and then held tight by a spring. The light reflected from the mirror may be used for demonstration purposes or for yielding permanent photographic records.

The next lecture deals with the analysis and synthesis of harmonic curves, the fifth with the influence of the horn and diaphragm on the sound waves. The sixth and seventh lectures are occupied respectively with the tone qualities of musical instruments and of vowels.

The history of theories of vowel sounds is given clearly and compactly. It is interesting to note that the results of the author's work here described are in entire agreement with the fixed-pitch theory of vowels due to Helmholtz.

After a concluding chapter, there occurs a valuable bibliography of over a hundred references, general and special.

To mathematical treatment this work makes no pretence, scarcely a couple of pages of equations occurring throughout the text. Its strength lies in its experimental and graphical treatment, and in the photographic records giving analyses of sounds from the simplest vibrations of forks up to vowels and the vibrato notes of operatic artists.

Both for showing what has been accomplished up to date and for the stimulation of further research on this subject, this work may be heartily recommended.

A Course of Modern Analysis: an Introduction to the General Theory of Infinite Series and of Analytic Functions; with an account of the Principal Transcendental Functions. By E. T. WHITTAKER, F.R.S., and G. N. WATSON, M.A. Cambridge: At the University Press. 1915. Second Edition, completely revised.

THE first edition of this well-known work was reviewed in these pages in 1904 (Phil. Mag. vii. p. 605). In the preparation of the new edition Professor Whittaker has had the assistance of his friend and former pupil, Mr. G. N. Watson, to whom are due the new chapters on Riemann Integration, on Integral Equations, and on the Riemann Zeta-Function. In addition to these, however, important changes have been made throughout the work. New paragraphs have been added, certain discussions formerly included in one chapter are now rearranged and distributed in two, and in many cases the demonstrations have been altered so as to be in line with the growing demands for greater rigour. The main character of the treatise remains the same, Part I. treating of the Processes of Analysis, and Part II. of the Transcendental Functions. In the Second Part there are introduced, in addition to the Zeta Function already named, special discussions of the Mathieu Function and the Theta Functions. There is no doubt that all these changes are of the nature of improvements, and have led naturally to a marked increase in the size of the volume. Nor is this increase merely to be measured by the greater number of pages, practically one half more than the number in the first edition. An important feature of the new work is the use of large and small type to distinguish what might be termed the main stream of discussion from what is more subsidiary; and by this means more matter is packed into the page. Numerous examples are appended to the chapters, and many of these are quite as important as the illustrations given in the text. By such methods the volume is kept within reasonable dimensions, and yet the ground covered is amazingly extensive. The work abounds in historic references and indications to the student for more extended reading in both original memoirs and treatises on special branches. It is interesting to note that recent work done in Professor

Whittaker's own mathematical institute in the University of Edinburgh is explicitly recognized as adding to the sum of mathematical knowledge. Through his inspiration the 'Proceedings of the Edinburgh Mathematical Society,' which for many years were mainly devoted to geometrical developments, now contain research papers on transcendental functions, certain results of which find their place in this valuable treatise on Modern Analysis. The centre of gravity of mathematical development in the United Kingdom has appreciably shifted since Professor Whittaker flitted from Cambridge to Edinburgh by way of Dublin.

It should be said in conclusion that the book is beautifully printed, and that its usefulness is greatly enhanced by a very full general subject index as well as a complete list of authors quoted.

Diophantine Analysis. By ROBERT D. CARMICHAEL. New York: John Wiley & Sons. London: Chapman & Hall.

THIS is No. 16 of the mathematical monographs edited by Mansfield Merriman and Robert S. Woodward. The most familiar Diophantine problem is that connected with the property of the right-angled or Pythagorean triangle, and may be stated in these words: to find two integers the sum of whose squares is a square. Theorems of this nature have had, and no doubt will always have, a great fascination for all interested in the theory of numbers; and the fascination is probably the greater on account of the lack of a general method of investigation. Fermat is universally recognized as the great master of Diophantine analysis. He left behind him many theorems without proof; and some of these still await a demonstration. Fermat's Last Theorem that, if n is an integer greater than 2, there do not exist integers x, y, z , all different from zero, such that $x^n + y^n = z^n$, is one of those for which no general solution has yet been found. The Academy of Sciences of Göttingen holds a sum of a hundred thousand marks to be presented as a prize to the person who first gives a rigorous proof of the theorem. Professor Carmichael devotes the fifth chapter of his book to a discussion of this equation, the earlier chapters being taken up with the consideration of equations of the second, third, and fourth degrees. The sixth and last chapter deals with what is called the method of functional equations, a method of considerable use in the investigation of Diophantine problems. In particular, it is applied to the solution of another of Fermat's problems, namely: To find three squares such that the product of any two of them, added to the sum of those two, gives a square. The book is clearly written and fully carries out the aim of the author, which is "to systematize, as far as possible, a large number of isolated investigations and to organize the fragmentary results into a connected body of doctrine. The principal single organizing idea here used and not previously developed systematically in the literature is that connected with the notion of a multiplicative domain."

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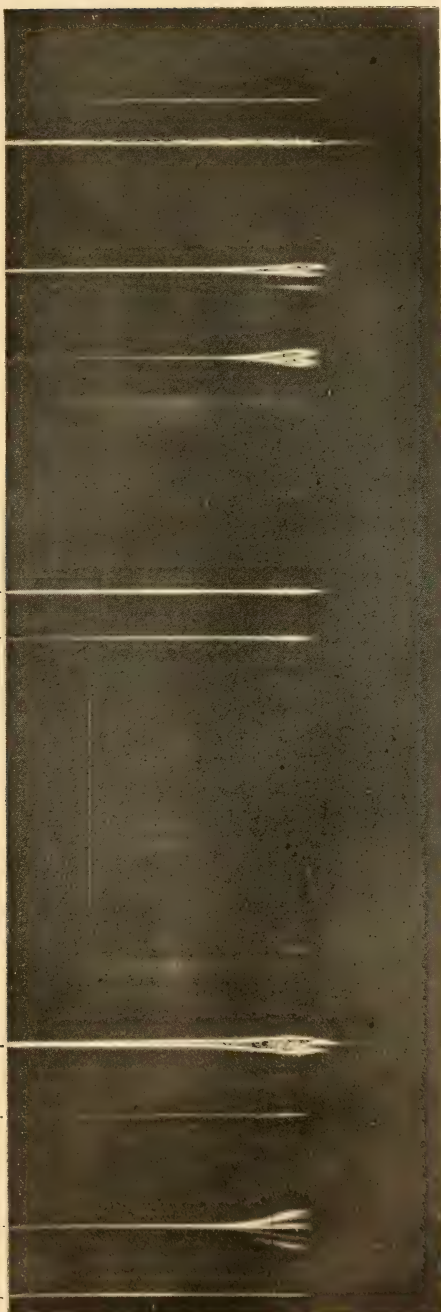
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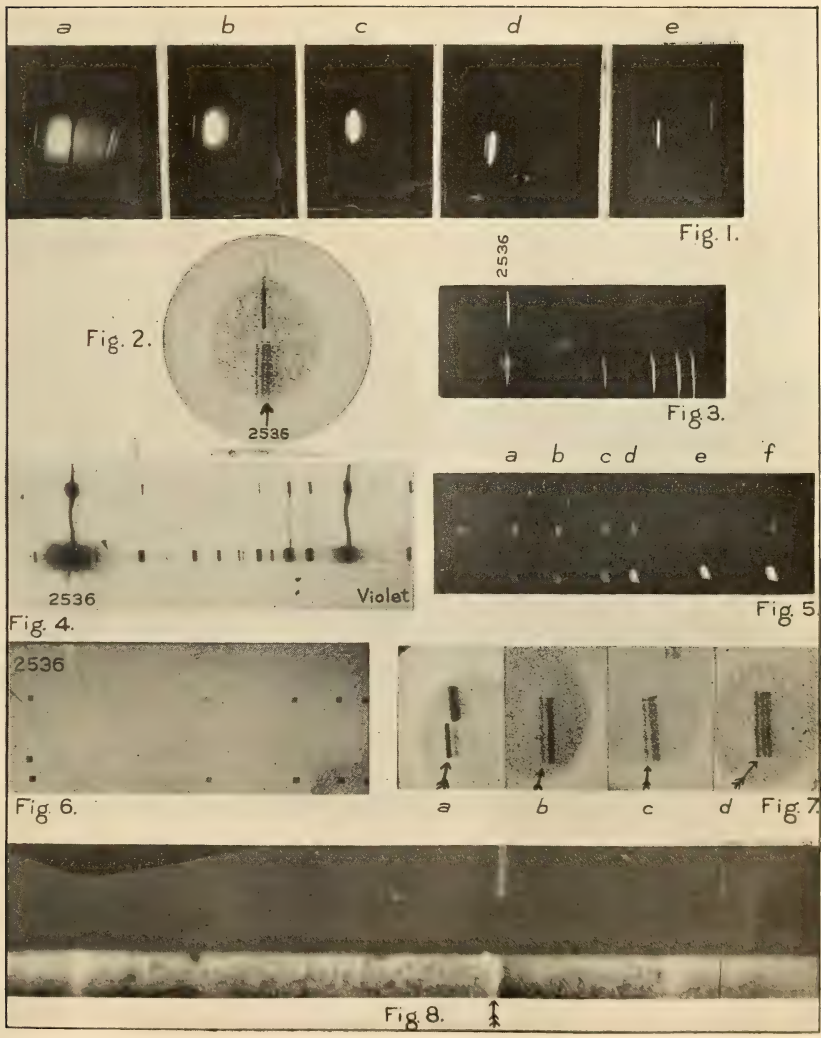
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THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

OCTOBER 1916.



XXXIX. *On Vibrations and Deflexions of Membranes, Bars, and Plates.* By Lord RAYLEIGH, O.M., F.R.S.*

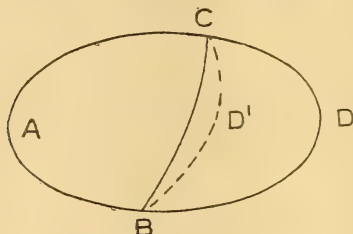
IN 'Theory of Sound,' § 211, it was shown that "any contraction of the fixed boundary of a vibrating membrane must cause an elevation of pitch, because the new state of things may be conceived to differ from the old merely by the introduction of an additional constraint. Springs, without inertia, are supposed to urge the line of the proposed boundary towards its equilibrium position, and gradually to become stiffer. At each step the vibrations become more rapid, until they approach a limit corresponding to infinite stiffness of the springs and absolute fixity of their points of application. It is not necessary that the part cut off should have the same density as the rest, or even any density at all."

From this principle we may infer that the gravest mode of vibration for a membrane of any shape and of any variable density is devoid of internal nodal lines. For suppose that ACDB (fig. 1) vibrating in its longest period (τ) has an internal nodal line CB. This requires that a membrane with the fixed boundary ACB shall also be capable of vibration in period τ . The impossibility is easily seen. As ACDB gradually contracts through ACD'B to ACB, the longest period diminishes, so that the longest period of ACB is less than τ . No period possible to ACB can be equal to τ .

* Communicated by the Author.

If we replace the reactions against acceleration by external forces, we may obtain the solution of a statical problem.

Fig. 1.



When a membrane of any shape is submitted to transverse forces, all in one direction, the displacement is *everywhere* in the direction of the forces.

Similar conclusions may be formulated for the conduction of heat in two dimensions, which depends upon the same fundamental differential equation. Here the boundary is maintained at a constant temperature taken as zero, and "persistances" replace the periods of vibration. Any closing in of the boundary reduces the principal persistence. In this mode there can be no internal place of zero temperature. In the steady state under positive sources of heat, however distributed, the temperature is above zero everywhere. In the application to the theory of heat, extension may evidently be made to three dimensions.

Arguments of a like nature may be used when we consider a bar vibrating transversely in virtue of rigidity, instead of a stretched membrane. In 'Theory of Sound,' § 184, it is shown that whatever may be the constitution of the bar in respect of stiffness and mass, a curtailment at either end is associated with a rise of pitch, and this whether the end in question be free, clamped, or merely "supported."

In the statical problem of the deflexion of a bar by a transverse force locally applied, the question may be raised whether the linear deflexion must everywhere be in the same direction as the force. It can be shown that the answer is in the affirmative. The equation governing the deflexion (w) is

$$\frac{d^2}{dx^2} \left(B \frac{d^2 w}{dx^2} \right) = Z, \quad . \quad . \quad . \quad . \quad (1)$$

where $Z dx$ is the transverse force applied at dx , and B is a coefficient of stiffness. In the case of a uniform bar B is constant and w may be found by simple integration. It suffices to suppose that Z is localised at one point, say at

$x=b$; and the solution shows that whether the ends be clamped or supported, or if one end be clamped and the other free or supported, w is everywhere of the same sign as Z . The conclusion may evidently be extended to a force variable in any manner along the length of the bar, provided that it be of the same sign throughout.

But there is no need to lay stress upon the case of a uniform bar, since the proposition is of more general application. The first integration of (1) gives

$$\frac{d}{dx} \left(B \frac{d^2 w}{dx^2} \right) = \int_0^x Z dx + C, \quad . \quad . \quad . \quad (2)$$

and $\int Z dx = 0$ from $x=0$ at one end to $x=b$, and takes another constant value (Z_1) from $x=b$ to the other end at $x=l$. A second integration now shows that $B d^2 w / dx^2$ is a linear function of x between 0 and b , and again a linear function between b and l , the two linear functions assuming the same value at $x=b$. Since B is everywhere positive, it follows that the curvature cannot vanish more than twice in the whole range from 0 to l , ends included, unless indeed it vanish everywhere over one of the parts. If one end be supported, the curvature vanishes there. If the other end also be supported, the curvature is of one sign throughout, and the curve of deflexion can nowhere cross the axis. If the second end be clamped, there is but one internal point of inflexion, and again the axis cannot be crossed. If both ends are clamped, the two points of inflexion are internal, but the axis cannot be crossed, since a crossing would involve three points of inflexion. If one end be free, the curvature vanishes there, and not only the curvature but also the rate of change of curvature. The part of the rod from this end up to the point of application of the force remains unbent and one of the linear functions spoken of is zero throughout. Thus the curvature never changes sign, and the axis cannot be crossed. In this case equilibrium requires that the other end be clamped. We conclude that in no case can there be a deflexion anywhere of opposite sign to that of the force applied at $x=b$, and the conclusion may be extended to a force, however distributed, provided that it be one-signed throughout.

Leaving the problems presented by the membrane and the bar, we may pass on to consider whether similar propositions are applicable in the case of a flat plate, whose stiffness and density may be variable from point to point. An argument similar to that employed for the membrane shows that when the boundary is clamped any contraction of it is attended

by a rise of pitch. But ('Theory of Sound,' § 230) the statement does *not* hold good when the boundary is free.

When a localised transverse force acts upon the plate, we may inquire whether the displacement is at all points in the same direction as the force. This question was considered in a former paper * in connexion with a hydrodynamical analogue, and it may be convenient to repeat the argument. Suppose that the plate (fig. 2), clamped at a distant boundary, is almost divided into two independent parts by a straight partition CD extending across, but perforated by a narrow aperture AB; and that the force is applied at a distance from CD on the left. If the partition were complete, w and dw/dn would be zero over the whole (in virtue of the clamping), and the displacement in the neighbourhood on the left would be simple one-dimensional bending, with w positive throughout. On the right w would vanish. In order to maintain this condition of things a certain couple acts upon the plate in virtue of the supposed constraints along CD.

Fig. 2.



Along the perforated portion AB the couple required to produce the one-dimensional bending fails. The actual deformation accordingly differs from the one-dimensional bending by the deformation that would be produced by a couple over AB acting upon the plate, as clamped along CA, BD, but otherwise free from force. This deformation is evidently symmetrical with change of sign upon the two sides of CD, w being positive on the left, *negative* on the right, and vanishing on AB itself. Thus upon the whole a downward force acting on the left gives rise to an upward motion on the right, in opposition to the general rule proposed for examination.

If we suppose a load attached at the place where the force acted, but that otherwise the plate was devoid of mass, we see that a clamped plate vibrating freely in its gravest mode may have internal nodes in the sense that w is there evanescent, but of course not in the full sense of places which behave as if they were clamped.

In the case of a plate whose boundary is merely supported, *i. e.* acted upon by a force (without couple) constraining w to remain zero†, it is still easier to recognize that a part of

* Phil. Mag. vol. xxxvi. p. 354 (1893); Scientific Papers, vol. iv. p. 88.

† It may be remarked that the substitution of a supported for a clamped boundary is equivalent to the abolition of a constraint, and is in consequence attended by a fall in the frequency of free vibrations.

the plate may move in the direction opposite to that of an applied force. We may contemplate the arrangement of fig. 2, where, however, the partition CD is now merely supported and not clamped. Along the unperforated parts CA, BD the plate must be supposed cut through so that no couple is transmitted. And in the same way we infer that internal nodes are possible when a supported plate vibrates freely in its gravest mode.

But although a movement opposite to that of the impressed force may be possible in a plate whose boundary is clamped or supported, it would seem that this occurs only in rather extreme cases when the boundary is strongly re-entrant. One may suspect that such a contrary movement is excluded when the boundary forms an oval curve, *i. e.* a curve whose curvature never changes sign. A rectangular plate comes under this description; but according to M. Mesnager*, “M. J. Résal a montré qu’en appliquant une charge au centre d’une plaque rectangulaire de proportions convenables, on produit très probablement le soulèvement de certaines régions de la plaque.” I understand that the boundary is supposed to be “supported” and that suitable proportions are attained when one side of the rectangle is relatively long. It seems therefore desirable to inquire more closely into this question.

The general differential equation for the equilibrium of a uniform elastic plate under an impressed transverse force proportional to Z is†

$$\nabla^4 w = (d^2/dx^2 + d^2/dy^2)^2 w = Z. \quad (3)$$

We will apply this equation to the plate bounded by the lines $y=0$, $y=\pi$, and extending to infinity in both directions along x , and we suppose that external transverse forces act only along the line $x=0$. Under the operation of these forces the plate deflects symmetrically, so that w is the same on both sides of $x=0$ and along this line $dw/dx=0$. Having formulated this condition, we may now confine our attention to the positive side, regarding the plate as bounded at $x=0$.

The conditions for a supported edge parallel to x are

$$w=0, \quad d^2 w/dy^2=0; \quad (4)$$

and they are satisfied at $y=0$ and $y=\pi$ if we assume that w as a function of y is proportional to $\sin ny$, n being an

* *C. R.* t. 162, p. 826 (1916).

† ‘Theory of Sound,’ §§ 215, 225; Love’s ‘Mathematical Theory of Elasticity,’ Chapter xxii.

integer. The same assumption introduced into (3) with $Z=0$ gives

$$(d^2/dx^2 - n^2)^2 w = 0, \quad . \quad . \quad . \quad . \quad (5)$$

of which the general solution is

$$w = \{(A + Bx)e^{-nx} + (C + Dx)e^{nx}\} \sin ny, \quad . \quad . \quad (6)$$

where A, B, C, D , are constants. Since $w=0$ when $x=+\infty$, C and D must here vanish; and by the condition to be satisfied when $x=0$, $B=nA$. The solution applicable for the present purpose is thus

$$w = A \sin ny \cdot (1 + nx)e^{-nx}. \quad . \quad . \quad . \quad . \quad (7)$$

The force acting at the edge $x=0$ necessary to maintain this displacement is proportional to

$$\frac{d\nabla^2 w}{dx} + (1-\mu) \frac{d^2}{dy^2} \frac{dw}{dx}, \text{ that is } \frac{d^3 w}{dx^3} \text{ simply,} \quad . \quad (8)$$

in virtue of the condition there imposed. Introducing the value of w from (7), we find that

$$d^3 w/dx^3 = 2n^3 A \sin ny, \quad . \quad . \quad . \quad . \quad (9)$$

which represents the force in question. When $n=1$,

$$w = A \sin y \cdot (1 + x)e^{-x}; \quad . \quad . \quad . \quad . \quad (10)$$

and it is evident that w retains the same sign over the whole plate from $x=0$ to $x=\infty$. On the negative side (10) is not applicable as it stands, but we know that w has identical values at $\pm x$.

The solution expressed in (10) suggests strongly that Résal's expectation is not fulfilled, but two objections may perhaps be taken. In the first place the force expressed in (9) with $n=1$, though preponderant at the centre $y=\frac{1}{2}\pi$, is not entirely concentrated there. And secondly, it may be noticed that we have introduced no special boundary condition at $x=\infty$. It might be argued that although w tends to vanish when x is very great, the manner of its evanescence may not exclude a reversal of sign.

We proceed then to examine the solution for a plate definitely terminated at distances l , and there *supported*. For this purpose we resume the general solution (6),

$$w = \sin ny \{(A + Bx)e^{-nx} + (C + Dx)e^{nx}\}, \quad . \quad (11)$$

which already satisfies the conditions of a supported edge at $y=0, y=\pi$. At $x=0$, the condition is as before $dw/dx=0$.

At $x=l$ the conditions for a supported edge give first $w=0$, and therefore $d^2w/dy^2=0$. The second condition then reduces to $d^2w/dx^2=0$. Applying these conditions to (11) we find

$$D = B e^{-2nl}, \quad C = -e^{-2nl}(A + 2lB). \quad . \quad . \quad (12)$$

It remains to introduce the condition to be satisfied at $x=0$. In general

$$\begin{aligned} \frac{dw}{dx} = \sin ny [e^{-nx} \{ -n(A + Bx) + B \} \\ + e^{nx} \{ n(C + Dx) + D \}]; \quad . \quad . \quad (13) \end{aligned}$$

and since this is to vanish when $x=0$,

$$-nA + B + nC + D = 0. \quad . \quad . \quad . \quad (14)$$

By means of (12), (14) A , C , D may be expressed in terms of B , and we find

$$\begin{aligned} \frac{dw}{dx} = \frac{nB \sin ny}{1 + e^{-2nl}} [e^{-nx} \{ -x + (2l-x)e^{-2nl} \} \\ + e^{-n(2l-x)} \{ -(2l-x) + xe^{-2nl} \}]. \quad (15) \end{aligned}$$

In (15) the square bracket is negative for any value of x between 0 and l , for it may be written in the form

$$-xe^{-nx} \{ 1 - e^{-2n(2l-x)} \} - (2l-x)e^{-2nl} \{ e^{nx} - e^{-nx} \}. \quad (16)$$

When $x=0$ it vanishes, and when $x=l$ it becomes

$$-2le^{-2nl}(e^{nl} - e^{-nl}).$$

It appears then that for any fixed value of y there is no change in the sign of dw/dx over the whole range from $x=0$ to $x=l$. And when $n=1$, this sign does not alter with y . As to the sign of w when $x=0$, we have then from (11)

$$w = \sin ny(A + C) = B \sin ny \frac{e^{2nl} - e^{-2nl} - 4nl}{n(e^{nl} + e^{-nl})},$$

so that dw/dx in (15) has throughout the opposite sign to that of the initial value of w . And since $w=0$ when $x=l$, it follows that for every value of y the sign of w remains unchanged from $x=0$ to $x=l$. Further, if $n=1$, this sign is the same whatever be the value of y . Every point in the plate is deflected in the same direction.

Let us now suppose that the plate is clamped at $x=\pm l$,

instead of merely supported. The conditions are of course $w=0$, $dw/dx=0$. They give

$$D = e^{-2nl} \{ 2nA + B(2nl - 1) \} \quad . \quad . \quad . \quad (17)$$

$$C = -e^{-2nl} \{ A(1 + 2nl) + 2nl^2 B \} \quad . \quad . \quad . \quad (18)$$

The condition at $x=0$ is that already expressed in (14).

As before, A, C, D may be expressed in terms of B. For shortness we may set $B=1$, and write

$$H = 1 + e^{-2nl}(2nl - 1) \quad . \quad . \quad . \quad (19)$$

We find

$$-nA + 1 = 2n^2 l^2 e^{-2nl} / H,$$

$$D = (2nl + 1 - e^{-2nl}) e^{-2nl} / H,$$

$$nC + D = -e^{-2nl} \cdot 2n^2 l^2 / H.$$

Thus

$$\begin{aligned} \frac{dw}{dx} &= \sin ny [e^{-nx} (-nA + 1 - nx) + e^{nx} (nC + D + nDx)] \\ &= H^{-1} \sin ny \cdot e^{-nx} [2n^2 l^2 e^{-2nl} - nx \{ 1 + e^{-2nl}(2nl - 1) \}] \\ &\quad + H^{-1} \sin ny \cdot e^{n(x-2l)} [-2n^2 l^2 + nx \{ 2nl + 1 - e^{-2nl} \}], \end{aligned}$$

vanishing when $x=0$, and when $x=l$.

This may be put into the form

$$\begin{aligned} \frac{dw}{dx} &= -H^{-1} \sin ny [2n^2 l(l-x) e^{-2nl} (e^{nx} - e^{-nx}) \\ &\quad + nx e^{-nl} (1 - e^{-2nl}) (e^{n(l-x)} - e^{-n(l-x)})], \quad . \quad (20) \end{aligned}$$

in which the square bracket is positive from $x=0$ to $x=l$.

It is easy to see that H also is positive. When nl is small, (19) is positive, and it cannot vanish, since

$$e^{2nl} > 1 > 1 - 2nl.$$

It remains to show that the sign of w follows that of $\sin ny$ when $x=0$. In this case

$$w = (A + C) \sin ny; \quad . \quad . \quad . \quad (21)$$

and

$$\begin{aligned} \frac{n(A + C)}{H} &= 1 - e^{-2nl} (2 - 2nl + 4n^2 l^2) + e^{-4nl} \\ &= e^{-2nl} (e^{2nl} + e^{-2nl} - 2 + 2nl - 4n^2 l^2). \quad . \quad (22) \end{aligned}$$

The bracket on the right of (22) is positive, since

$$e^{2nl} + e^{-2nl} = 2 \left(1 + \frac{4n^2 l^2}{2} + \frac{16n^4 l^4}{4!} + \dots \right).$$

We see then that for any value of y , the sign of dw/dx

over the whole range from $x=0$ to $x=l$ is the opposite of the sign of w when $x=0$ *; and since $w=0$ when $x=l$, it follows that it cannot vanish anywhere between. When $n=1$, w retains the same sign at $x=0$ whatever be the value of y , and therefore also at every point of the whole plate. No more in this case than when the edges at $x=\pm l$ are merely supported, can there be anywhere a deflexion in the reverse direction.

In both the cases just discussed the force operative at $x=0$ to which the deflexion is due is, as in (8), proportional simply to d^3w/dx^3 , and therefore to $\sin ny$, and is of course in the same direction as the displacement along the same line. When $n=1$, both forces and displacements are in a fixed direction. It will be of interest to examine what happens when the force is concentrated at a single point on the line $x=0$, instead of being distributed over the whole of it between $y=0$ and $y=\pi$. But for this purpose it may be well to simplify the problem by supposing l infinite.

On the analogy of (7) we take

$$w = \Sigma A_n (1 + nx) e^{-nx} \sin ny, \quad . \quad . \quad . \quad (23)$$

making, when $x=0$,

$$d^3w/dx^3 = 2 \Sigma n^3 A_n \sin ny. \quad . \quad . \quad . \quad (24)$$

If, then, Z represent the force operative upon dy , analysable by Fourier's theorem into

$$Z = Z_1 \sin y + Z_2 \sin 2y + Z_3 \sin 3y + \dots, \quad . \quad . \quad (25)$$

we have

$$Z_n = \frac{2}{\pi} \int_0^\pi Z \sin ny dy = \frac{2}{\pi} Z_{11} \sin n\eta, \quad . \quad . \quad (26)$$

if the force is concentrated at $y=\eta$. Hence by (24)

$$A_n = \frac{Z_{11}}{\pi} \frac{\sin n\eta}{n^3}, \quad . \quad . \quad . \quad (27)$$

so that

$$w = \frac{Z_{11}}{2\pi} \Sigma \frac{\cos n(y-\eta) - \cos n(y+\eta)}{n^3} e^{-nx} (1 + nx), \quad (28)$$

where $n=1, 2, 3$, &c. It will be understood that a constant factor, depending upon the elastic constants and the thickness of the plate, but not upon n , has been omitted.

The series in (28) becomes more tractable when differentiated. We have

$$\frac{dw}{dx} = - \frac{x Z_{11}}{2\pi} \Sigma \frac{\cos n(y-\eta) - \cos n(y+\eta)}{n} e^{-nx}; \quad (29)$$

* This follows at once if we start from $x=l$ where $w=0$.

and the summations to be considered are of the form

$$\Sigma n^{-1} \cos n\beta e^{-nx} (30)$$

This may be considered as the real part of

$$\Sigma n^{-1} e^{-n(x-i\beta)}, (31)$$

that is, of

$$- \log (1 - e^{-(x-i\beta)}). (32)$$

Thus, if we take

$$\Sigma n^{-1} e^{-n(x-i\beta)} = X + iY, (33)$$

$$e^{-X-iY} = 1 - e^{-(x-i\beta)}, \text{ and } e^{-X+iY} = 1 - e^{-(x+i\beta)},$$

so that

$$e^{-2X} = 1 + e^{-2x} - 2e^{-x} \cos \beta. (34)$$

Accordingly

$$\Sigma n^{-1} \cos n\beta e^{-nx} = -\frac{1}{2} \log (1 + e^{-2x} - 2e^{-x} \cos \beta); . (35)$$

and

$$\frac{dw}{dx} = \frac{x Z_{11}}{4\pi} \log \frac{1 + e^{-2x} - 2e^{-x} \cos (y-\eta)}{1 + e^{-2x} - 2e^{-x} \cos (y+\eta)}. . . (36)$$

From the above it appears that

$$W = x \log \{1 + e^{-2x} - 2e^{-x} \cos (y+\eta)\} = x \log h$$

must satisfy $\nabla^4 W = 0$. This may readily be verified by means of

$$\nabla^2 \log h = 0, \text{ and } \nabla^2 W = x \nabla^2 \log h + 2d \log h/dx.$$

We have now to consider the sign of the logarithm in (36), or, as it may be written,

$$\log \frac{e^x + e^{-x} - 2 \cos (y-\eta)}{e^x + e^{-x} - 2 \cos (y+\eta)}. (37)$$

Since the cosines are less than unity, both numerator and denominator are positive. Also the numerator is less than the denominator, for

$$\cos (y-\eta) - \cos (y+\eta) = 2 \sin y \sin \eta = +,$$

so that $\cos (y-\eta) > \cos (y+\eta)$. The logarithm is therefore negative, and dw/dx has everywhere the opposite sign to that of Z_{11} . If this be supposed positive, w on every line $y = \text{const.}$ increases as we pass inwards from $x = \infty$ where $w = 0$ to $x = 0$. Over the whole plate the displacement is positive, and this whatever the point of application (η) of the force. Obviously extension may be made to any distributed one-signed force.

It may be remarked that since the logarithm in (37) is unaltered by a reversal of x , (36) is applicable on the negative as well as on the positive side of $x = 0$. If $y = \eta$,

$x=0$, the logarithm becomes infinite, but dw/dx is still zero in virtue of the factor x .

I suppose that w cannot be expressed in finite terms by integration of (36), but there would be no difficulty in dealing arithmetically with particular cases by direct use of the series (28). If, for example, $\eta=\frac{1}{2}\pi$, so that the force is applied at the centre, we have to consider

$$\Sigma n^{-3} \sin \frac{1}{2}n\pi \cdot \sin ny \cdot e^{-nx}(1+nx), \quad . \quad . \quad (38)$$

and only odd values of n enter. Further, (38) is symmetrical on the two sides of $y=\frac{1}{2}\pi$. Two special cases present themselves when $x=0$ and when $y=\frac{1}{2}\pi$. In the former w is proportional to

$$\sin y - \frac{1}{3^3} \sin 3y + \frac{1}{5^3} \sin 5y - \dots, \quad . \quad . \quad (39)$$

and in the latter to

$$e^{-x}(1+x) + \frac{1}{3^3}e^{-3x}(1+3x) + \frac{1}{5^3}e^{-5x}(1+5x) + \dots \quad (40)$$

August 2, 1916.

Added August 21.

The accompanying tables show the form of the curves of deflexion defined by (39), (40).

y .	(39).	y .	(39).
0°	·0000	50	·7416
10	·1594	60	·8574
20	·3162	70	·9530
30	·4675	80	1·0217
40	·6104	90	1·0518

x .	(40).	x .	(40).
0·0	1·0518	3·0	·1992
0·5	·9333	4·0	·0916
1·0	·7435	5·0	·0404
2·0	·4066	10·0	·0005

In a second communication * Mesnager returns to the question and shows by very simple reasoning that all points

* *C. R.* July 24, 1916, p. 84.

of a rectangular plate supported at the boundary move in the direction of the applied transverse forces.

If z denote $\nabla^2 w$, then $\nabla^2 z = \nabla^4 w$, is positive over the plate if the applied forces are everywhere positive. At a straight portion of the boundary of a supported plate $z=0$, and this is regarded as applicable to the whole boundary of the rectangular plate, though perhaps the corners may require further consideration. But if $\nabla^2 z$ is everywhere positive within a contour and z vanish on the contour itself, z must be negative over the interior, as is physically obvious in the theory of the conduction of heat. Again, since $\nabla^2 w$ is negative throughout the interior, and w vanishes at the boundary, it follows in like manner that w is positive throughout the interior.

It does not appear that an argument on these lines can be applied to a rectangular plate whose boundary is clamped, or to a supported plate whose boundary is in part curved.

P.S. In connexion with a recent paper on the "Flow of Compressible Fluid past an Obstacle" (Phil. Mag. July 1916), I have become aware that the subject had been treated with considerable generality by Prof. Cisotti of Milan, under the title "Sul Paradosso di D'Alembert" (*Atti R. Istituto Veneto*, t. lxxv. 1906). There was, however, no reference to the limitation necessary when the velocity exceeds that of sound in the medium. I understand that this matter is now engaging Prof. Cisotti's attention.

XL. *The Condensation and Reflexion of Gas Molecules.* By R. W. WOOD, *Professor of Experimental Physics, Johns Hopkins University* *.

[Plate VII.]

IN a previous communication† I have shown that a jet of mercury vapour, in which the molecular motion is restricted to one dimension, is reflected from a flat plate of glass approximately according to the cosine law. The mercury vapour after reflexion was condensed as a metallic film on the wall of the bulb, which contained the reflector at its centre and was immersed in liquid air.

Removal of the bulb from the low-temperature bath caused the film of solid mercury to melt and collect into small drops, which made measurements of its thickness in different gases impossible.

* Communicated by the Author.

† Phil. Mag. August 1915, "Experimental. Determination of the Law of Reflexion of Gas Molecules."

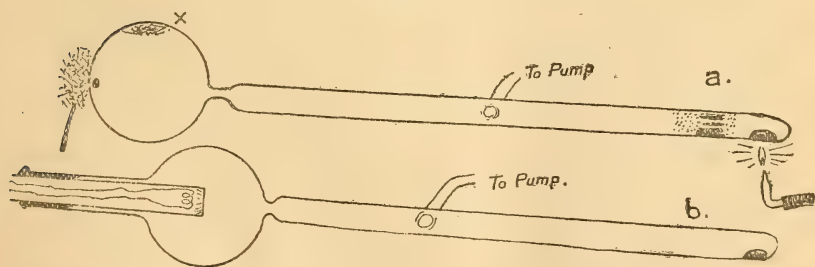
During the past winter the work has been continued, cadmium being used in place of the mercury. Permanent films can be obtained in this way, and measurements made with the photometer. The chief point of interest in connexion with the work, however, has been the discovery of the fact that there is a sort of critical temperature for each substance, below which condensation appears to take place at the first collision, in other words the chance that reflexion takes place is zero. This is true, however, only in case the wall is of some substance other than the metal in question. The critical temperature is surprisingly low. For mercury it is in the vicinity of -140° , for cadmium about -90° , and for iodine about -60° .

We will take up first, however, the law of reflexion.

In my first communication, I drew attention to the supposed circumstance that there was practically no reflexion for directions which made small angles with the reflecting surface; in other words, there was a zone of clear glass just above the circle cut by the plane of the reflecting surface in the wall of the bulb. Above this clear zone the gradually increasing thickness of the deposit indicated that the law of reflexion approximated the cosine law, and was independent of the angle of incidence.

As soon as it was found that permanent films of cadmium could be obtained, quantitative experiments were commenced.

Fig. 1.



The form of tube used is indicated by fig. 1 (a) and (b). (a) is the type used for demonstrating the one-dimensional motion of the cadmium molecules, to which attention was first drawn by L. Dunoyer in the case of sodium. The tube must be kept in communication with a Gaede pump during the experiment as an extremely high vacuum is necessary, and the cadmium is heated by a gas-flame 3 or 4 mm. high burning at the tip of a glass tube drawn down to a fine capillary.

If the bulb is kept at room temperature no trace of a deposit appears, for reasons which will be given presently; but if the wall opposite the constriction in the tube is cooled with a small pad of cotton wet with liquid air, a small circular deposit of the metal at once forms, showing that the molecular stream shoots across the vacuum-bulb, without spreading out laterally. If the vacuum is sufficiently high, prolonging the experiment for fifteen or twenty minutes causes no appreciable increase in the diameter of the circular spot, which becomes distinctly visible in ten seconds, after the cadmium has reached the proper temperature (which is only a little above the melting-point). If the vacuum is inferior, the diameter of the spot may be much greater, owing to the deflexion of the cadmium molecules by the residual gas.

A photograph of the bulb with the circular deposit is reproduced on Plate VII. fig. 1.

The most instructive way of performing the experiment is as follows.

The bulb and tube are kept at room temperature until the cadmium melts. If now the *side* of the bulb is touched at X with a pellet of cotton wet with liquid air, a large deposit of irregular shape at once forms, showing that the bulb is filled with cadmium vapour (with 3-dimensional motion). It seems rather paradoxical that we can have a bulb at room temperature filled with cadmium vapour, which apparently can be condensed only by the application of liquid air. If the bulb is touched by the cotton pellet for only a second or two the deposit forms, and continues to increase in thickness indefinitely, after the removal of the cotton, for the chance of reflexion at room temperature from a cadmium surface is zero, *i. e.* condensation occurs at the first collision.

If now the wall of the bulb opposite the constriction is touched with liquid air, the small circular spot at once forms, and goes on building up indefinitely after the removal of the cotton. No further deposit occurs on the side of the bulb, for the circular deposit, once formed, serves as a trap for the one-dimensional molecular stream which impinges on it.

These phenomena will be more fully discussed later on.

Tube (b) was designed for obtaining a deposit after reflexion of the molecular stream from a flat polished surface of glass. In order to obtain films thick enough for quantitative measurements, it is necessary to keep the reflecting surface at a moderately high temperature for at least 30 minutes. This was accomplished by a small electrical heating-coil introduced into the tube which carried the reflecting surface. The tube was closed in a flame and the closed end ground flat and

highly polished. It was cemented in position with sealing-wax, with the reflecting surface at the centre of the bulb.

The bulb was wrapped in cotton, which was kept soaked with liquid air throughout the experiment, which lasted about thirty minutes.

A photograph of the deposit is shown on Plate VII. fig. 2; and it is evident that the clear zone is absent, the deposit coming quite down to the plane of the reflecting surface.

Measurements of the thickness of the film for different angular distances from the line normal to the centre of the reflecting surface, made by observing photometrically the transmission of red light by the film, showed that the cosine law was obeyed to within the probable error of the experiment.

During the progress of the investigation, however, the exceeding ingenious method devised by Knudsen was communicated by him to me in a letter, and as this method was so obviously superior to that above described, the quantitative investigation of the law of reflexion was abandoned in its early stages.

Knudsen's method, which was published in the *Annalen der Physik* in the early part of 1916, is based upon the fact that for reflexion from a *spherical* surface under certain conditions a film of uniform thickness will be formed *if the reflexion from each element of the spherical surface follows the cosine law*. This he found to be the case.

The general subject of condensation and reflexion as a function of the temperature was next investigated.

It is apparent, from the experiment with the bulb which I have described, that cadmium vapour, at least when highly attenuated, will not condense as a homogeneous film on a glass wall at room temperature, whereas it will on a surface of metallic cadmium.

In the vicinity of the heated bead of the metal we may have a metallic film produced, but here the nature of the deposit is very variable, depending on the condition of the glass surface. Black splotches sometimes form, granular deposits of white metal and mirror surfaces may alternate, and in fact no two tubes act alike.

In the bulb, however, the conditions are different, and the phenomena can be reproduced at will.

I first attempted to get some idea of how many collisions with the wall were possible without condensation, by substituting for the bulb a glass tube bent at a number of right angles, as shown on Plate VII. fig. 3. The end of the tube

was blown out into a small thin-walled bulb, the surface of which was touched with a pellet of cotton wet with liquid air. A film formed on the cooled spot, as shown in the photograph, proving that the vapour could traverse the bent tube without condensation. Tubes with over a dozen bends were used, proving that a large number of reflexions were possible, the number of bends being the minimum number of reflexions sufficient to carry the molecule to the condensation bulb.

It was found that a much longer time was required to form a film in the bulb when a large number of bends were introduced, which indicated that a portion of the molecules had been condensed on the way down the tube. In the case of the tube shown on Plate VII. fig. 4 the deposit in the bulb was produced in fifteen minutes. The cotton pellet was then applied at the bend marked A, and a deposit of equal density was obtained in two minutes. Transferring the cotton to the point B gave an equal deposit in about 10 seconds.

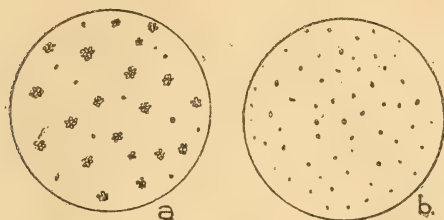
It is evident that the density of the cadmium vapour is considerably decreased by its passage down the bent tube, but as no visible deposit could be seen on the walls, it was not at once evident what became of it.

A glass stopcock was introduced between the bulb and the straight tube containing the cadmium, which was then heated above its melting-point. The stopcock was then closed and at the same instant the cotton pellet wet with liquid air was applied to the bulb, but no film was produced, even after many trials. This seemed to show that the vapour was incapable of existing as vapour in a bulb at room temperature for any appreciable time.

A very thin-walled bulb of exceptionally clean glass was then blown on a tube, and after forty minutes heating of the cadmium, it was removed from the pump and examined in a strong light against a black background. The wall of the bulb diffused a small amount of light, as if a light smoke had deposited on it, and examination under the microscope with $\frac{1}{12}$ -inch oil-immersion objective showed widely separated aggregates of the metal which appeared to be clusters of crystals (fig. 2 a). More frequently, however, deposits such as shown by fig. b were obtained. When we consider the excessive thinness of metal films which are partially transparent to light, it is clear that an amount of cadmium sufficient to form a film of moderate opacity over the entire surface of the bulb, if collected into a single crystal, would probably not be much larger than one of these small aggregates.

Experiments were next made to determine the temperature necessary for the formation of a homogeneous film of cadmium.

Fig. 2.



The bulb was cooled by the application of alcohol a little above its solidifying point (with a consistency of Canada balsam), but no trace of a film could be detected at the end of five minutes. Solid alcohol, however, caused a film to appear in ten or fifteen seconds. As the alcohol contained some water, its freezing-point was in the neighbourhood of -100° , which can be considered as the approximate critical temperature for cadmium. Mercury gave no film in a bulb cooled to -130° , but a film immediately appeared if the temperature was reduced to -150° . The critical temperature for mercury is therefore in the neighbourhood of -140° .

I am not yet certain how sharply these temperatures can be determined, or what the nature of the deposit will be for prolonged action at a temperature a few degrees above the critical temperature; but it seems probable that in this case a granular deposit will be formed made up of exceedingly small grains very close together.

As I have said before, if any portion of the inner surface of the bulb has received a low-temperature film deposit, a similar type of condensation continues even at room temperature, the rest of the bulb remaining clean.

These experiments show pretty clearly how condensation takes place on metal and glass surfaces. In the case of surfaces composed of the same metal as the vapour, the molecules stick at the first impact, the probability of reflexion being zero, even at room temperature. The same is true for a surface, such as glass, if it is cooled below a certain critical temperature which is characteristic for each substance investigated. If the temperature is above this, the probability of reflexion is large, and only a few molecules

are fixed at the first impact. These, however, act as traps and gather others about them, just as does a small area of film. A granular deposit therefore forms, the grains increasing in size until they are finally visible under the microscope.

In working with cadmium it was found that a film formed in the bulb before the cadmium had reached the melting-point. This film was black or neutral grey by transmitted light. If, however, a piece of cadmium which had been heated above the melting-point *in vacuo* for several minutes was used over again in a fresh tube, the deposit was blue by transmitted light, resembling a film of silver. Apparently the cadmium contained some impurity which distilled off at a comparatively low temperature.

To determine the nature of this substance the deposit was condensed in the bulb of a small end-on vacuum-tube with external electrodes of tinfoil.

The tube was sealed off from the cadmium tube, heated, and the spectrum of the discharge photographed. The cadmium and mercury lines were found on the plate. The experiment was repeated with a long U-tube immersed in liquid air between the pump and the tube, to prevent diffusion of mercury vapour; but again the black deposit was obtained which showed the mercury lines. Evidently the cadmium contains a trace of mercury, which distils off first, carrying some cadmium with it.

The critical temperature for the alloy of cadmium and mercury has not yet been determined. It is probable that we are dealing with molecules of the alloy, for it has been found that mercury vapour is capable of dissolving less volatile metals, in the same way that the dense vapours of certain organic compounds are capable of dissolving non-volatile compounds, such as potassium iodide.

Experiments were also made with iodine, for which the critical temperature is in the neighbourhood of -60° . Above this temperature a black granular deposit forms, below it a deep red film. The iodine crystals at one end of a tube bent at two right angles were kept at 0° . The other end was then plunged into a bath of known temperature, and the nature of the deposit observed. When working in the vicinity of the critical temperature, deposits were sometimes obtained which were bright green by transmitted light. These probably were made up of exceedingly minute granules, analogous to the coloured films and fogs obtained with sodium and potassium.

The whole subject is one that merits an extended investigation. I have not yet examined the impact of the vapour of one metal against a film composed of another metal, nor have I studied the behaviour of non-metallic surfaces other than glass.

Some very extraordinary results were obtained in the case of sodium condensed on surfaces cooled with liquid air, the colour of the film changing as soon as the liquid air was removed. I believe that there is a change in the structure of the film with rising temperature, but the phenomena were too complicated and not sufficiently studied to make their discussion worth while at the present time.

It is my plan to continue the study of the subject with improved apparatus; for it is obvious that we must be able to control the temperature of the metal, the distillation-tube, and the bulb. It will also be necessary to arrange matters so that the behaviour of superheated vapours can be studied.

XLI. *On the Boiling-Points of Homologous Compounds.*

By H. C. PLUMMER*.

1. **I**N discussing the relations which subsist among the boiling-points of series of homologous compounds, without the guidance of any physical theory, the difficulty is to avoid the representation of a large mass of data by a number of unconnected formulæ on the one hand, and on the other the too hasty generalization of some particular type which happens to hold for some special class of substances. The first difficulty is the greater because the observations relate to very short series in many cases, and these are naturally affected by more or less considerable errors. The second difficulty is well illustrated by the early experience of Kopp.

As an example of the way in which a short series may be represented by a very simple formula, the isoparaffins may be chosen. The formula in this case is

$$T = 430 \cdot 5 \log n,$$

where n is the number of carbon atoms. The comparison with observation is shown in Table I. This is satisfactory

* Communicated by the Author.

TABLE I.

Isoparaffins : C_nH_{2n+2} .

<i>n</i> .	Abs.B.P.*	Calc.	O—C.
4.	259·7	259·2	+0·5
5.	300·95	300·91	+0·04
6.	335·0	335·0	0·0
7.	363·3	363·8	—0·5

enough as far as it goes. But obviously it does not go very far. The more general formula of this type, however,

$$T = a \log (bn + c)$$

or its equivalent

$$\Delta \cdot 10^{T/a} = b, \text{ const.},$$

may prove more widely useful in connexion with similar series. This is suggested by some experience to be described later.

As another example the small class of hydrosulphides available may be quoted. Here the simple formula suggested is

$$T = 30^{\circ} \cdot 93 (n + 8),$$

and the comparison with observation is given in Table II.

TABLE II.

Hydrosulphides : $C_nH_{2n+1}-SH$.

<i>n</i> .	Abs. B.P.†	Calc.	O—C.
1.	279·0	278·4	+0·6
2.	309·0	309·3	—0·3
3.	340·5	340·2	+0·3
4.	370·5	371·2	—0·7

Again the range is too small to make any wide deduction possible.

2. Nevertheless, the linear formula, though pushed beyond its legitimate limits by Kopp, certainly has its area of useful application. This is to be seen in the case of the alcohols, the boiling-points of which are fairly represented by the formulæ :

$$\text{Normal primary : } T = 18^{\circ} \cdot 64 (n + 17),$$

$$\text{Normal secondary : } 18^{\circ} \cdot 64 (n + 16),$$

$$\text{Primary iso : } 18^{\circ} \cdot 64 (n + 16 \cdot 5).$$

* Young, Scientific Proc. R.D.S., xv. p. 97 (1916).

† Young, Phil. Mag. ix. p. 6 (1905); *J. de Chimie Physique*, iii. p. 245.

TABLE III.

Alcohols : $C H_{2n+1}-OH$.

n.	Normal Primary.			Normal Secondary.			Primary Iso.		
	T.	Calc.	O-C.	T.	Calc.	O-C.	T.	Calc.	O-C.
1.	337.7	335.5	+2.2						
2.	351.3	354.2	-2.9						
3.	370.2	372.8	-2.6	355.45	354.16	+1.3			
4.	389.9	391.4	-1.5	373.0	372.8	+0.2	381.05	382.12	-1.1
5.	411.0	410.1	+0.9	392.0	391.4	+0.6	405.05	400.76	+4.3
6.	431.0	428.7	+2.3	409.0	410.1	-1.1	423.0	419.4	+3.6
7.	449.0	447.4	+1.6	437.5	428.7	+8.8	437.0	438.0	-1.0
8.	464.0	466.0	-2.0	452.0	447.4	+4.6			
9.	486.5	484.6	+1.9						
10.	504.0	503.3	+0.7						

The calculated are compared with the observed values in Table III. A serious error in one of the latter is plainly evident in the secondary series, recognized by Young. The run of the residuals in the iso-group may point to curvature. But as a whole the experimental results appear to be in better accord with a linear than any other formula. The constant $18^{\circ}64$ agrees with Kopp's original ideas, and it may be noticed—though this may be accidental—that, as in the case of the hydrosulphides, the constants additive to n are simple. This point is perhaps further illustrated by the aldehydes and amines, which can be represented by :

Aldehydes : $T = 26.76 (n + 10)$,Amines : $26.76 (n + 9)$,

as shown in Table IV. The lowest member of the aldehyde series is quite anomalous, and the amines may show a slight curvature. But the simple linear formula is clearly not without its uses in representing the facts.

TABLE IV.

Aldehydes : $C_n H_{2n+1} - CHO$. Amines : $C_n H_{2n+1} - NH_2$.

n.	T.	Calc.	O-C.	T.	Calc.	O-C.
0.	252.0	267.6	-15.6			
1.	293.8	294.4	-0.6	267.0	267.6	-0.6
2.	322.0	321.1	+0.9	291.7	294.4	-2.7
3.	347.0	347.9	-0.9	322.7	321.1	+1.6
4.	376.0	374.6	+1.4	348.5	347.9	+0.6
5.	400.9	401.4	-0.5	376.0	374.6	+1.4
6.	428.0	428.2	-0.2	402.0	401.4	+0.6
7.				427.0	428.2	-1.2

3. The ethers form a double series, $C_sH_{2s+1}-O-C_rH_{2r+1}$, for which a considerable number of boiling-points have been determined, and it seems desirable to connect them if possible by a single formula. Since the composition of these substances is symmetrical the absolute boiling-points $T_{rs}=T_{sr}$ are symmetrical functions of r and s . Hence a function of the form

$$T_{rs}=a+b(r+s)+c(r+s)^2+d(r-s)^2$$

is suggested, and a simple calculation gives

$$T_{rs}=189\cdot75+32\cdot42(r+s)-0\cdot557(r+s)^2+0\cdot138(r-s)^2,$$

the last term allowing for some variation among isomeric compounds. The numbers resulting from this formula are given in Table V.(a), and the result of subtracting these

TABLE V. (a).

Ethers : $C_sH_{2s+1}-O-C_rH_{2r+1}$. T_{rs} calc.							
r .	$s=1$.	2.	3.	4.	...	7.	8.
1.	252 ⁰ ·4	282 ⁰ ·1	311 ⁰ ·1	339 ⁰ ·2	...	418 ⁰ ·5	443 ⁰ ·2
2.	282·1	310·5	338·1	364·8	...	439·9	463·2
3.	311·1	338·1	364·2	389·5	...	460·5	482·4
4.	39·2	364·8	389·5	413·5	...	480·2	500·8
...
7.	418·5	439·9	460·5	480·2	...	534·5	550·9
8.	443·2	463·2	482·4	500·8	...	550·9	565·9

TABLE V. (b).

Ethers : O-C.

r .	$s=1$.	2.	3.	4.	...	7.	8.
1.	-3 ⁰ ·0	+1 ⁰ ·7	+0 ⁰ ·8	+4 ⁰ ·1	...	+4 ⁰ ·3	+2 ⁰ ·8
2.	+1·7	-2·9	-1·5	-0·4	...	-0·3	-1·0
3.	+0·8	-1·5	-0·5	+0·6	...	+0·1	-2·4
4.	+4·1	-0·4	+0·6	+0·4	...	-1·5	-2·1
...
7.	+4·3	-0·3	+0·1	-1·5	...	+0·4	+0·9
8.	+2·8	-1·0	-2·4	-2·1	...	+0·9	-1·2

from the observed boiling-points, which it is unnecessary to reproduce, is shown in Table V.(b). The comparison may be considered satisfactory on the whole, though when r (or s) = 1 the discrepancies are not altogether negligible.

4. This tolerable success in representing the boiling-points of the ethers by a single formula suggests a similar treatment for the esters. Here the composition, $C_sH_{2s+1}-COOC_rH_{2r+1}$, is no longer symmetrical in r and s , but a formula of the same type may be tried. Numerically this takes the form

$$T_{rs} = 277^{\circ}68 + 25^{\circ}55(r+s) - 0^{\circ}368(r+s)^2 + 0^{\circ}1153(r-s)^2,$$

which leads to the numbers given in Table VI. (a). When these are subtracted from the corresponding observed values

TABLE VI. (a).

Esters: $C_sH_{2s+1}-COOC_rH_{2r+1}$. T_{rs} calc.

r .	$s=0$.	1.	2.	3.	4.	5.	6.	7.	8.
1.	303 ⁵	327 ³	351 ¹	374 ⁴	396 ³	419 ⁶	441 ⁴	462 ⁷	483 ⁵
2.	327 ⁸	351 ¹	374 ⁰	396 ⁴	418 ²	439 ⁵	460 ⁴	480 ⁷	500 ⁵
3.	352 ¹	374 ⁴	396 ⁴	417 ⁷	438 ⁶	459 ⁰	478 ⁹	498 ²	517 ¹
4.	375 ⁸	396 ³	418 ²	438 ⁶	458 ⁵	477 ⁹	496 ⁸	515 ²	533 ¹
5.	399 ¹	419 ⁶	439 ⁵	459 ⁰	477 ⁹	496 ³	514 ³	531 ⁷	548 ⁷
6.	421 ⁹	441 ⁴	460 ⁴	478 ⁹	496 ⁸	514 ³	531 ³	547 ⁸	563 ⁷
7.	444 ²	462 ⁷	480 ⁷	498 ²	515 ²	531 ⁷	547 ⁸	563 ²	578 ²
8.	465 ⁹	483 ⁵	500 ⁵	517 ¹	533 ¹	548 ⁷	563 ⁷	578 ²	592 ³

TABLE VI. (b).

Esters: O—C.

r .	$s=0$.	1.	2.	3.	4.	5.	6.	7.	8.
1.	+1 ⁴	+2 ⁹	+1 ⁶	+1 ³	+4 ⁰	+3 ⁰	+3 ⁷	+3 ²	+3 ⁰
2.	-0 ⁵	-1 ⁰	-2 ²	-3 ⁵	-0 ⁵	+0 ¹	-0 ³	-1 ⁹	0 ⁰
3.	+1 ⁸	+0 ¹	-1 ¹	-2 ⁰	+1 ⁹	-0 ⁵	+0 ⁵	-0 ⁵	...
4.	+4 ¹	+1 ²	+0 ²	+0 ¹	+0 ³	-0 ⁶	+1 ³	-1 ⁷	...
5.	+4 ³	+1 ⁰	...	-1 ²	-1 ²
6.	+4 ⁷	+0 ⁸	...	-0 ⁸	0 ⁰
7.	+5 ⁵	+1 ⁶	+0 ³	0 ⁰	+1 ⁴	+0 ⁷	-0 ²	-0 ⁴	...
8.	+5 ²	-0 ⁵	-1 ¹	-1 ⁹	+0 ¹	-0 ⁵	-0 ³	+0 ⁷	...

the differences shown in Table VI. (b) result. On the whole they may be regarded as satisfactorily small. But a sensible and systematic discrepancy again appears in the first line and column, of the same kind as that suggested in Table V. (b). The boiling-points of the initial series are rather higher than the general law indicates. This is not surprising, however, because these series are known to be otherwise abnormal in their physical properties. The convenient assumption of symmetry in the formula does not seem to have had any serious detrimental effect on the representation.

5. But by far the most important single series is presented by the normal paraffins, both on account of the number of consecutive members and the special care with which the boiling-points have been determined. It has also received the greatest amount of discussion. Table VII. conveys the results of considering this series from more than one point of view. In the first place, the boiling-points in the third column are calculated by the formula

$$T = -69.0 + 184.65\sqrt{n} - 6.89n.$$

The comparison given in the next column shows that this formula successfully represents T as a function of n . In

TABLE VII.

Normal Paraffins: C_nH_{2n+2} .

n .	T .	Calc.	O-C.	ΔT .	Calc.	O-C.	Cum.sum.	(Young.)
1.	108.3	108.7	-0.4	71.7	70.5	+1.2	-0.2	+1.2
2.	180.0	178.3	+1.7	49.0	51.4	-2.4	+1.0	+1.9
3.	229.0	230.1	-1.1	43.8	42.7	+1.1	-1.4	-1.2
4.	272.8	272.7	+0.1	36.4	36.4	0.0	-0.3	-0.2
5.	309.2	309.4	-0.2	32.8	32.5	+0.3	-0.3	-0.6
6.	341.95	342.0	0.0	29.4	29.3	+0.1	0.0	-0.4
7.	371.4	371.3	+0.1	27.2	26.8	+0.4	+0.1	-0.3
8.	398.6	398.2	+0.4	24.9	24.8	+0.1	+0.5	+0.1
9.	423.5	423.0	+0.5	22.5	23.0	-0.5	+0.6	+0.2
10.	446.0	446.0	0.0	21.0	21.6	-0.6	+0.1	-0.2
11.	467.0	467.6	-0.6	20.5	20.4	+0.1	-0.5	-0.8
12.	487.5	487.9	-0.4	19.5	19.3	+0.2	-0.4	-0.6
13.	507.0	507.2	-0.2	18.5	18.3	+0.2	-0.2	-0.2
14.	525.5	525.6	-0.1	18.0	17.3	+0.7	0.0	+0.1
15.	543.5	542.8	+0.7	17.0	16.5	+0.5	+0.7	+0.8
16.	560.5	559.4	+1.1	15.5	15.8	-0.3	+1.2	+1.2
17.	576.0	575.2	+0.8	14.0	15.2	-1.2	+0.9	+0.9
18.	590.0	590.4	-0.4	13.0	14.6	-1.6	-0.3	-0.3
19.	603.0	605.0	-2.0				-1.9	-1.9

the second place, the observed boiling-points have been differenced, following the ideas of Prof. Young, and represented by the formula

$$\Delta T = \frac{20648}{T + 136.6} - 13.82.$$

The numbers calculated in the sixth column by this formula are based on the observed values of T in the second column. The residuals are shown in the seventh column, and the cumulative sum of these in the eighth column, the first term being chosen so that the sum of the column is nearly zero. This column fairly shows the result of building up the series by successive differences and should be comparable with the

fourth column. Finally, for the sake of comparison, are added in the last column the residuals which follow similarly from Young's own formula

$$\Delta T = 144.86 T^{-0.0148} \sqrt{T}.$$

The author's differences (being C—O) have been reversed in sign and also altered by the addition of a small constant (0°·4) in order to make their mean value zero, an adjustment practically equivalent to a small change in the initial point assumed.

A comparison of the fourth, eighth, and ninth columns of Table VII. shows that the representation by three distinct methods is not only equally good, but that after the first term or two it is almost identical in detail. Strictly within the range considered the three formulæ are virtually the same, and it would appear that the common residuals do represent approximately the actual errors of observation reduced to a homogeneous system. On the other hand, the facility with which the series of numbers can be represented in ways mathematically different, makes it impossible to lay stress on any particular functional form or to pass beyond the precise limits covered by the experiments.

6. It is useful to consider the locus of the points $x=n$, $y=T$, as a curve. According to Kopp this curve is a straight line, but the truth of this is at best limited. According to the formulæ found in this paper, the curve is more generally parabolic in shape. Again, the locus of the points $x=T$, $y=\Delta T$ may be considered as a curve. For the normal paraffins it has been shown that this curve may be treated as hyperbolic. Now, according to Young, this curve is approximately the same for all substances which are not associated, so that ΔT is a function of T only, independent of the particular substance. Let the curves be $T=f(n)$ and $\Delta T=\phi(T)$, and consider the f curves of two different series. By a displacement parallel to the axis of x (or n) they may be brought into coincidence at some point T . But by Young's law ΔT , being a function of T only, is the same for both curves. Hence the curves will coincide at the next point, and the next, and so on. Thus the two curves will coincide altogether, and it follows that if the equation of one in its original position is $T=f(n)$, the equation of the other must be $T=f(n+a)$, where a is a constant. This principle should be very useful, but it proves not to be so. The reason for this is that comparatively small errors in ΔT , if systematic, are fatal to the satisfactory representation of $f(n)$. The law, though confessedly only

approximate, is good and useful in its own sphere. A residual of 10 per cent. in a difference is often innocuous and smaller than the effect of experimental error. But a residual of 1 per cent. in the absolute temperature is generally unsatisfactory. Hence the determination of $f(n)$ and $\phi(T)$ are in practice quite different problems, and the former is the more exacting.

Thus we have been led to treat differently three classes of compounds. The first, typified by the alcohols, with marked chemical association, is represented by a linear formula. The second, an intermediate type containing the ethers and esters, has slight molecular association and the boiling-points show a slight degree of curvature which is fairly represented by a parabolic formula. Finally, the very complete series of normal paraffins has been taken as the type of compounds free from association, and the well-developed curve formed by their boiling-points has also been represented by a parabolic formula of another kind. In one or another of these three classes probably all series of boiling-points can be more or less well represented. Thus to the first class may be attributed :

Cyanides ($C_nH_{2n+1}-CN$) : $T=19^{\circ}14 (n+17.5)$,

Nitro-compounds ($C_nH_{2n+1}-NO_2$) : $T=16^{\circ}9 (n+21)$,

Ketones ($CH_3-CO-C_nH_{2n+1}$) : $T=23^{\circ}52 (n+13)$,

and to the second class :

Acids ($C_nH_{2n+1}-COOH$) : $T=368^{\circ}83 + 23^{\circ}34n - 0^{\circ}643n^2$.

The comparison, which suggests some irregularities in the experimental data, need not be given.

So far as the parabolic formula found to represent the normal paraffins can be regarded as strictly typical, it applies to series of hydrocarbons which are for the most part very fragmentary. Two examples are given in Table VIII. In the first series the constant additive to n in the formula is 0, and the calculated values are taken without change from the third column of Table VII. In the second case the constant added to n in the formula is 6.4.

TABLE VIII.

$HC_nH_{2n}C=CHC_nH_{2n}$				$C_6H_5C_nH_{2n+1}$			
n .	T.	Calc.	O - C.	n .	T.	Calc.	O - C.
4.	274.0	272.7	+1.3	1.	383.6	382.4	+1.2
5.	309.5	309.4	+0.1	2.	409.0	408.4	+0.6
6.	341.0	342.0	-1.0	3.	431.5	432.4	-0.9
7.	371.0	371.3	-0.3	4.	453.0	454.9	-1.9
				5.	474.5	476.0	-1.5

7. More extensive material is provided by the halogen compounds, and since the differences between the boiling-points are fairly represented by Young's law, it follows that the boiling-points themselves can be represented approximately by the normal paraffin curve. The necessary constants to be added to n are about :

$$\begin{aligned} \text{C}_n\text{H}_{2n+1}\text{Cl} : a &= +2.4; \text{C}_n\text{H}_{2n+1}\text{Br} : a = +3.2; \\ \text{C}_n\text{H}_{2n+1}\text{I} : a &= +4.2. \end{aligned}$$

But the residuals, though fairly small, show a marked systematic tendency, which may otherwise be expressed by an increase of the constant a with n . Hence it may be opportune to illustrate the utility of another formula already suggested, namely

$$T = a \log (bn + c),$$

which may be taken to have the numerical forms :

$$\begin{aligned} \text{C}_n\text{H}_{2n+1}\text{Cl} : T &= 1000^\circ \log (0.155n + 1.6209), \\ \text{C}_n\text{H}_{2n+1}\text{Br} : &1000 \log (0.155n + 1.7446), \\ \text{C}_n\text{H}_{2n+1}\text{I} : &1000 \log (0.155n + 1.9082), \end{aligned}$$

where something of individual accuracy is sacrificed to uniformity and simplicity. This is not serious, however, as

TABLE IX.

$\text{C}_n\text{H}_{2n+1}\text{Cl.}$				$\text{C}_n\text{H}_{2n+1}\text{Br.}$				$\text{C}_n\text{H}_{2n+1}\text{I.}$			
$n.$	T.	Calc.	O—C.	T.	Calc.	O—C.		T.	Calc.	O—C.	
1.	249.3	249.4	—0.1	277.5	278.7	—1.2		315.8	314.5	+1.3	
2.	285.5	285.8	—0.3	311.4	312.7	—1.3		345.5	346.0	—0.5	
3.	319.0	319.3	—0.3	343.8	344.3	—0.5		375.5	375.3	+0.2	
4.	351.0	350.2	+0.8	374.0	373.8	+0.2		403.0	402.8	+0.2	
5.	379.6	379.5	+0.1	402.5	401.3	+1.2		429.0	428.7	+0.3	
6.	406.0	406.7	—0.7	429.0	427.3	+1.7		452.0	453.0	—1.0	
7.	433.0	432.3	+0.7	452.0	451.7	+0.3					
8.	457.0	456.5	+0.5	474.0	474.9	—0.9					

is shown by the comparison given in Table IX. The three constants c may be expressed in the forms :

$$\begin{aligned} \text{C}_n\text{H}_{2n+1}\text{Cl} : c &= 1.509 + 35.46 \div 318, \\ \text{C}_n\text{H}_{2n+1}\text{Br} : &1.509 + 79.92 \div 339, \\ \text{C}_n\text{H}_{2n+1}\text{I} : &1.509 + 126.92 \div 318; \end{aligned}$$

and though the divisors are not quite the same this suggests at least some relation between the constant and the atomic weight of the halogen element entering into the compound.

8. It may be noticed that the logarithmic form of $f(n)$ is capable of a considerable range of adaptability. Thus when b is small in comparison with c the differences become nearly constant, and the linear formula is practically reproduced. Again, it satisfies the halogen compounds, as shown in Table IX., and in another form it represents the fragmentary series of iso-paraffins, as shown in Table I. The corresponding difference equation is

$$\Delta T = \phi(T) = a \log(1 + b \cdot 10^{-T/a}).$$

If ΔT is strictly a function of T only, a and b must be absolute constants. But in practice the merely approximate truth of this law is probably consistent with a certain range of variation in these numbers. The values used for the halogen groups were $a=1000^\circ$, $b=0.155$. It is more difficult to compare the logarithmic with the hyperbolic difference equation for the normal paraffins. But the values $a=815$, $b=0.223$ make the two nearly coincide from 400° to 600° , and below 400° they diverge more and more.

Hence it is impossible to represent the normal paraffin series as a whole by the logarithmic formula. And yet the relation is worth examination. Experience suggests that it is more important to represent the higher members than to imitate the whole series by an indifferent compromise, for the greatest difficulties are always met with in the lower terms. When this is done the lower members show marked residuals which can, however, be easily represented by an empirical correcting term. The result is to give the formula :

$$T = 800^\circ \log(0.2323n + 1.290) - 70^\circ/2^n,$$

which is compared with experiment in Table X. The formula is certainly artificial, but its type is suggested by

TABLE X.

Normal Paraffins: C_nH_{2n+2} .

<i>n</i> .	T.	Calc.	O - C.	<i>n</i> .	T.	Calc.	O - C.
0.	20.4	18.5	+1.9	10.	446.0	446.2	-0.2
1.	108.3	111.0	-2.7	11.	467.0	467.9	-0.9
2.	180.0	177.8	+2.2	12.	487.5	488.3	-0.8
3.	229.0	229.7	-0.7	13.	507.0	507.6	-0.6
4.	272.8	272.6	+0.2	14.	525.5	525.8	-0.3
5.	309.2	309.3	-0.1	15.	543.5	543.1	+0.4
6.	342.0	341.9	+0.1	16.	560.5	559.6	+0.9
7.	371.4	371.2	+0.2	17.	576.0	575.4	+0.6
8.	398.6	398.1	+0.5	18.	590.0	590.5	-0.5
9.	423.5	423.1	+0.4	19.	603.0	604.9	-1.9

the analogous formula of Ramage and may perhaps be justified on similar grounds. The interesting point, however, is that this formula does what no other equation suggested has succeeded in doing. It leads the series right down so as to include the boiling-point of hydrogen as the zero paraffin. The wish to do this constituted the original incentive to the work contained in this paper. It was suggested by reading the two papers by my colleague, Prof. Sydney Young, to which reference has been made and from which all the experimental material used above has been taken. To his kindly interest this paper is entirely due.

Dunsink Observatory,
July 31, 1916.

XLII. *A Wehnelt Cathode-Ray Tube Magnetometer.* By CHAS. T. KNIPP, M.A., Ph.D., Associate Professor of Experimental Electricity, and L. A. WELO, M.S., Research Assistant in Astronomy, University of Illinois, U.S.A.*

[Plate VIII.]

I. Introduction.

IN a recent paper † the authors described an apparatus for determining the horizontal intensity (H) of the earth-magnetic field, depending on the measurement of the magnetic and electrostatic deflexions of a beam of cathode rays and on the assumption of the ratio of the charge to the mass of the electron. It has since been suggested to the authors that it might be well to deduce H by comparing the deflexion produced by the earth's magnetic field with the deflexion produced by a known calculated field, thus avoiding the assumption of the ratio e/m . It was for the purpose of carrying out this suggestion, and also in the hope of obtaining a more compact and convenient instrument, that the work now to be described was undertaken.

II. The Apparatus and the Manipulation.

The discharge-tube is a glass jar about 40 cm. in height and of 12 cm. diameter, clamped at the bottom to a brass plate, which is in turn fastened to the widened part of a brass pipe about 5 cm. in diameter. The last fastening is

* Communicated by the Authors.

† Knipp and Welo, *Terrestrial Magnetism and Atmospheric Electricity*, vol. xx. 1915, pp. 53-68.

adjustable so that the axis of the discharge-tube can be made to coincide exactly with that of the pipe, which turns in holes cut in a wooden frame. The pipe thus serves as a bearing, permitting the discharge-tube to be turned about a vertical axis. The rotation about the vertical axis is a necessary feature of the instrument, for the earth's magnetic field cannot very well be eliminated for a zero reading of the deflexions, but they must be observed for various orientations of the tube.

The tube is exhausted through a glass stem inside of the brass pipe. One end of the stem is waxed into a hole at the bottom of the jar, and the other is fitted by a long and well-ground joint to another glass stem which connects to a Gaede capsule-pump, to a bulb containing the drier P_2O_5 , and to the charcoal-liquid-air bulb for further improving the vacuum.

The cathode shown at C, fig. 1, is of a form previously described by the authors * but has, in this case, been somewhat simplified and made more compact. The object of using the Wehnelt cathode rather than the cold aluminium cathode is to secure electrons having a low velocity with consequent measurable deflexions for the relatively weak earth-field, and at the same time get a definite beam of sufficient range without the use of diaphragms, as was found possible when the spot of lime (Bank of England wax) does not exceed 0.02 cm. in diameter. The direction that the beam takes is uncertain, and two adjustments of the cathode are necessary, one in the plane of the drawing and one perpendicular to it. If the beam should take some such direction as that indicated in the drawing, fig. 1, it can be made to take a vertical direction by turning at the joint J_2 , and a similar adjustment in a plane perpendicular to the plane of the drawing can be made by turning at J_3 .

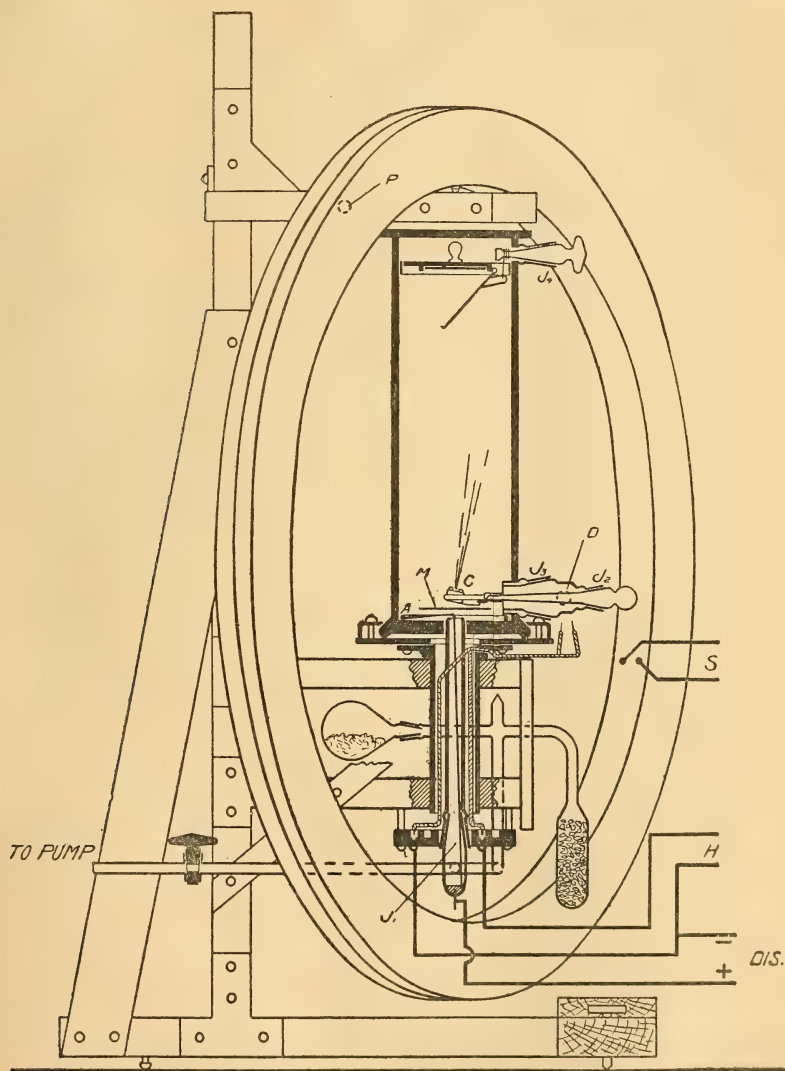
The anode is an aluminium ring of nearly the internal diameter of the jar, lying at its bottom, and connected by a wire to mercury at the bottom of the glass stem below the joint J_1 . The mercury is in contact with a sealed-in platinum wire which is connected to the positive terminal of a high potential battery. Since the cathode and the wire leading to the anode from the inside of the exhaust-tube are very close together, a mica screen is inserted at M to prevent a direct discharge which might take place if the vacuum were at a favourable stage, with possible injury to the battery.

The deflexions of the beam of particles are recorded on a

* *Loc. cit.*

photographic plate held near the top of the tube in a simple form of plateholder and shutter. The holder, which rests on a brass ring waxed to the inside of the tube, is a disk of

Fig. 1.



- S—Standard Coil, to Battery.
H—Heating Circuit, to Battery.
DIS—Discharge Circuit, to High Potential Battery.

aluminium with a small segment cut away. On the under side of this disk a shallow box is built. One end of the box is slightly undercut on the inside so that the photographic plate can be wedged in, and a wedge-shaped clamp actuated by a spring in the wall at the other end secures the plate in position, not only from falling out but presses it up against the disk, so that its position is never in doubt. The cover is a thin aluminium plate, hinged at a side, bent over at the edges to prevent light from entering edgewise, and is backed by a spring so that it will stay closed. The lid is opened by the winch shown at J_4 , which winds a string hooked to the handle on the lid. The discharge-tube is closed at the top by a piece of plate-glass, sealed on with half-and-half (beeswax and resin) wax.

The platinum strip on which the lime is placed is heated electrically. To do away with troublesome winding of the wires as the tube is turned, a hard rubber plate containing two rings of mercury was arranged, immediately beneath the brass pipe bearing. The rings are connected to the terminals of a battery by binding-posts leading up under the rings through the rubber. From the rings, wires lead up the inside of the brass pipe but outside the exhaust stem, through the brass pipe above the wooden frame, and thence to sealed-in platinum terminals set in tubes perpendicular to the plane of the drawing and indicated by the dotted circle at D. Wires on the inside of the cathode-mounting lead to the cathode itself. One terminal of the heating circuit leads to the negative end of a high potential battery, while the positive end of the high potential battery is connected to the sealed-in platinum wire below J_1 , as has already been stated.

The coil of six turns, wound so close that the current may be considered a circular one, is wound on a built-up and accurately turned wooden core. It is pivoted on the horizontal support, shown above the discharge-tube, so that it hangs vertically, but is prevented from turning about a vertical axis by an adjustable clamp (not shown) on the support. A pointer on the coil which moves over a graduated circle on the support gives the orientation. A pointer on the brass supporting-pipe and a graduated circle on the wooden frame gives, it might be added, the orientation of the tube itself. When it is desired to remove the cover from the discharge-tube, the coil is lifted off and the support swung back about the pivot P.

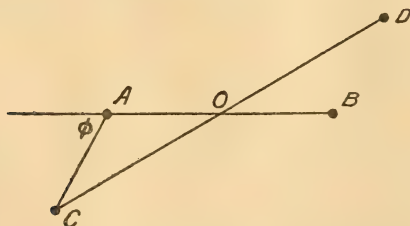
When making a determination, the pump is generally run for over an hour, the charcoal bulb being gently heated during the while to drive off gas absorbed at ordinary temperatures.

When the bulb has cooled it is immersed in liquid air. The heating circuit may soon be closed, the resistance adjusted until the platinum glows with a red heat, and 1000 volts now applied between the cathode and anode. It is well to leave the glowing cathode thus for, say, five minutes. If the heating current is then slowly increased a diffused spot will show on the plateholder-lid which is covered with willemite. On further heating, the diffused spot becomes either a well-defined spot or a sharp line in a plane containing the platinum strip. After the cathode has been adjusted so that the spot is east of the centre of the plateholder by an amount equal to the estimated deflexion (and in the plane of the coil), and the orientation of the tube noted, the plateholder is opened and an exposure made for 5 to 15 seconds, depending on the intensity of the beam. The switch of the coil circuit is then closed so that the beam is deflected still more, and the exposure at the new spot made. Then the plateholder is closed, the tube turned to 180° , the plateholder again opened and the process repeated. We thus get four spots or lines on the plate, which should be arranged along a straight line. Half of the distance between the two inner spots is the deflexion due to the earth's field alone, while half the distance between the outer two is the deflexion with the earth's field increased by the standard one.

III. The Declination.

It is unnecessary to determine the declination by another instrument in order to set the coil in a plane perpendicular to the meridian. The instrument itself is indeed a sufficiently accurate declinometer for the purpose of setting the coil.

Fig. 2.



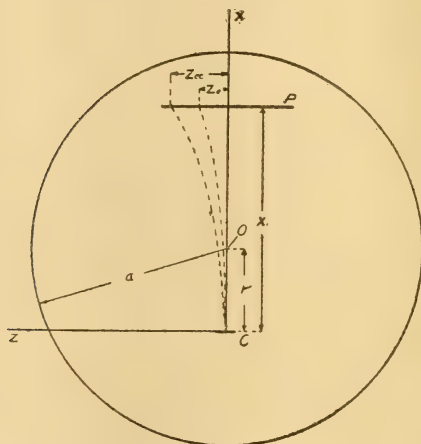
A test photographic plate is first taken with the coil in any position, which is noted on the graduated circle, and four spots obtained. Fig. 2 represents the general case. OA is the deflexion due to the earth's field alone, and OC that due to both fields. But OD is the resultant of OA

and the component AC of the coil-field making an angle ϕ with BA produced. On this test-plate, then, the line BA is drawn and produced; draw the line AC and measure the angle ϕ . The coil must be turned through the angle ϕ in order that it be perpendicular to the magnetic meridian.

IV. The Formula.

Fig. 3 shows the relative positions of the cathode C which is taken as the origin, the photographic plate P at a distance x_1 , the centre of the coil O at a distance r , the undeflected

Fig. 3.



path of the beam along the diameter taken as the x axis and the deflected paths, with and without a current in the coil.

Calling z_e the deflexion produced by the earth's magnetic field alone, the equation *

$$z_e = \frac{e}{mv} \int_0^{x_1} H(x_1 - x) dx \quad . \quad . \quad . \quad (1)$$

holds when the deflexions are so small that $\left(\frac{dz_e}{dx}\right)^2$ may be neglected. When a current flows in the coil so that the two fields have the same direction the resultant field is

$$H + IH_c.$$

* J. J. Thomson, Phil. Mag. vol. xviii. pp. 844-845 (1909).

Letting z_{ec} be the deflexion then produced we have, similarly,

$$z_{ec} = \frac{e}{mv} \int_0^{x_1} (H + IH_c)(x_1 - x) dx. \quad (2)$$

H is constant so that, integrating (1) and the first term of the right-hand member of (2), we have

$$z_e = \frac{e}{mv} H \frac{x_1^2}{2} \quad (3)$$

and

$$z_{ec} = \frac{e}{mv} \left[\frac{H x_1^2}{2} + I \int_0^{x_1} H_c (x_1 - x) dx \right]. \quad (4)$$

Because of the low velocity of the electrons and the high vacuum used, the factor $\frac{e}{mv}$ is constant and is eliminated by dividing (4) by (3). Solving for H the formula appears

$$H = \frac{2I}{x_1^2} \cdot \frac{z_e}{z_{ec} - z_e} \cdot \int_0^{x_1} H_c (x_1 - x) dx.$$

In the circular coil, used because of ease of construction and to secure compactness, the field H_c is not constant along the path of the beam, but must be expressed as a function of the distance from the cathode.

Calling x' the distance from the centre of the coil to the point in question, we have, per unit current*,

$$H_c = \frac{4an}{a^2 - x'^2} E,$$

where

$$E = \frac{\pi}{2} \left[1 - \left(\frac{1}{2} \right)^2 \frac{x'^2}{a^2} - \left(\frac{1.3}{2.4} \right)^2 \frac{1}{3} \frac{x'^4}{a^4} - \dots \right],$$

but $x' = x - r$, so that we have for the field, in terms of the distance from the cathode, the centre of coordinates,

$$H_c = \frac{2\pi an}{a^2 - (x-r)^2} \left[1 - \left(\frac{1}{2} \right)^2 \frac{(x-r)^2}{a^2} - \left(\frac{1.3}{2.4} \right)^2 \frac{1}{3} \frac{(x-r)^4}{a^4} - \dots \right] \quad (5)$$

Let

$$M = \int_0^{x_1} H_c (x_1 - x) dx$$

and substitute the value of H_c from (5), we have, using four

* A. Russell, Phil. Mag. vol. xiii. pp. 420-446 (1907).

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terms of the series,

$$\begin{aligned} M = 2\pi an \int_0^{x_1} & \left[\frac{dx}{a^2 - (x-r)^2} - \left(\frac{1}{2}\right)^2 \frac{1}{a^2} \frac{(x-r)^2 dx}{a^2 - (x-r)^2} \right. \\ & - \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} \frac{1}{a^4} \frac{(x-r)^4 dx}{a^2 - (x-r)^2} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \frac{1}{a^6} \frac{(x-r)^6 dx}{a^2 - (x-r)^2} \Big] \\ & - 2\pi an \int_0^{x_1} \left[\frac{x dx}{a^2 - (x-r)^2} - \left(\frac{1}{2}\right)^2 \frac{1}{a^2} \frac{x(x-r)^2 dx}{a^2 - (x-r)^2} \right. \\ & - \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} \frac{1}{a^4} \frac{x(x-r)^4 dx}{a^2 - (x-r)^2} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \frac{1}{a^6} \frac{x(x-r)^6 dx}{a^2 - (x-r)^2} \Big]. \end{aligned}$$

Introducing a new variable $y = x - r$, we have, after collecting terms wherever possible,

$$\begin{aligned} M = 2\pi an \int & \left[(x_1 - r) \frac{dy}{a^2 - y^2} - \frac{y}{a^2 - y^2} \frac{dy}{dy} - \left(\frac{1}{2}\right)^2 \frac{1}{a^2} (x_1 - r) \frac{y^2}{a^2 - y^2} \frac{dy}{dy} \right. \\ & + \left(\frac{1}{2}\right)^2 \frac{1}{a^2} \frac{y^3}{a^2 - y^2} \frac{dy}{dy} - \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} \frac{1}{a^4} (x_1 - r) \frac{y^4}{a^2 - y^2} \frac{dy}{dy} \\ & + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} \frac{1}{a^4} \frac{y^5}{a^2 - y^2} \frac{dy}{dy} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \frac{1}{a^6} (x_1 - r) \frac{y^6}{a^2 - y^2} \frac{dy}{dy} \\ & \left. + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \frac{1}{a^6} \frac{y^7}{a^2 - y^2} \frac{dy}{dy} \right]. \end{aligned}$$

Integrating, collecting terms, restoring the variable x and putting in the limits, we have, finally, from the four terms of the series with which we started,

$$\begin{aligned} M = 2\pi an \Big\{ & \frac{x_1 - r}{2a} \left[1 - \left(\frac{1}{2}\right)^2 - \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \right] \log_e \frac{(a + x_1 - r)(a + r)}{(a - x_1 + r)(a - r)} \\ & - \frac{1}{2} \left[1 - \left(\frac{1}{2}\right)^2 - \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \right] \log_e \frac{a^2 - r^2}{a^2 - (x_1 - r)^2} \\ & + \frac{x_1 - r}{a^2} \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \right] x_1 \\ & - \frac{1}{a^2} \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \right] \frac{x_1(x_1 - 2r)}{2} \\ & + \frac{x_1 - r}{a^4} \left[\left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \right] \frac{(x_1 - r)^3 + r^3}{3} \\ & - \frac{1}{a^4} \left[\left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \right] \frac{(x_1 - r)^4 - r^4}{4} \\ & + \frac{x_1 - r}{a^6} \left[\left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \right] \frac{(x_1 - r)^5 + r^5}{5} \\ & \left. - \frac{1}{a^6} \left[\left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} \right] \frac{(x - r)^6 - r^6}{6} \right\}. \quad \dots \dots \dots (6) \end{aligned}$$

The law of the series is apparent. The convergence is slow, but the coefficients are easily evaluated if it is noted that the series of the first two terms in (6) when multiplied by $\frac{\pi}{2}$ is the E series of the elliptic functions when the modulus is one. Its value is then unity, so that

$$1 - \left(\frac{1}{2}\right)^2 - \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} - \dots = \frac{2}{\pi} = 0.63662,$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{3} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{5} + \dots = 1.00000 - 0.63662 = 0.36338,$$

and so forth.

Then, neglecting terms of the power five and higher because the fourth significant place in M is not affected, there results

$$M = 2\pi a n \left\{ \frac{x_1 - r}{2a} \cdot 0.63662 \cdot \log_e \frac{(a + x_1 - r)(a + r)}{(a - x_1 + r)(a - r)} \right. \\
- \frac{1}{2} \cdot 0.63662 \cdot \log_e \frac{a^2 - r^2}{a^2 - (x_1 - r)^2} \\
+ \frac{x_1 - r}{a^2} \cdot 0.36338 \cdot x_1 - \frac{1}{a^2} \cdot 0.36338 \cdot \frac{x_1(x_1 - 2r)}{2} \\
+ \frac{x_1 - r}{a^4} \cdot 0.11338 \cdot \frac{(x_1 - r)^3 + r^3}{3} \\
\left. - \frac{1}{a^4} \cdot 0.11338 \cdot \frac{(x_1 - r)^4 - r^4}{4} \right\}.$$

It should be remarked that in evaluating

$$M = \int_0^{x_1} H_c(x_1 - x) dx$$

we have assumed the same distribution of the field along the actual path of the beam as along the diameter of the coil. It was the plan at first to use opposed fields so that the beam would follow the diameter very closely, but it was noted on Plate No. 26 and two previous ones where deflexions were taken in both directions, that, so far as measurements indicated, there was no difference in the distances from either of the spots to the middle one corresponding to no current in the coil. The assumption as to the distribution is thus justified.

V. *The Results.*

Measurements were made on seven photographic plates, numbered 27-34, number 29 being unexposed, for the platinum strip was accidentally burned out while the heating current was adjusted. Photographs 28, 30, 33, with number 26, are shown in Plate VIII., and the data from the measured photographs are collected in Table I.

TABLE I.

Plate No.	M.	x_1 cm.	I amperes.	z_e cm.	z_{ec} cm.	$z_{ec}-z_e$ cm.	H c.g s.	H-H _m .
27	48.61	33.13	1.402	1.059	1.886	0.827	0.1590	+0.0007
28	48.61	33.13	1.463	0.993	1.812	0.819	0.1571	-0.0012
30	48.61	33.13	1.432	1.024	1.863	0.839	0.1548	-0.0035
31	48.61	33.13	1.423	1.032	1.846	0.814	0.1598	+0.0015
32	48.61	33.13	1.417	0.980	1.759	0.779	0.1579	-0.0004
33	48.61	33.13	1.323	0.952	1.652	0.700	0.1594	+0.0011
34	48.61	33.13	1.473	1.009	1.831	0.822	0.1602	+0.0019
							0.1583	± 0.0005

The constant M was calculated by the formula of the preceding section, and agrees satisfactorily with the value determined graphically. The diameter of the coil, the distance between the cathode and the photographic plate and the vertical distance of the latter from the edge of the coil, were measured with a cathetometer, and it was deduced that $a=43.46$ cm., $x_1=33.13$ cm., and $r=5.01$ cm. It will be remembered that the coil has six turns.

The current was measured with a Siemens & Halske precision ammeter, removed from the magnetometer, to 1 part in 1500, with no observed fluctuations during the time of exposure. The deflexions tabulated are the half means of five measurements of the distances between the outer two and the inner two spots on each plate. The precision attained in measuring deflexions is indicated by taking Plate no. 33 as an example, where the maximum deviation of a single measurement of z_e from the mean of five measurements is 1 part in 316, and for z_{ec} it is 1 part in 235.

The deviations from the mean of the values of H are tabulated and the relative probable error of ± 0.00046 or $\frac{1}{344} H$ calculated, but the performance of the instrument is not

fairly indicated because of the presence of electrical circuits in the neighbouring rooms of the laboratory and a changing distribution of iron about the building.

VI. *Summary.*

1. A magnetometer has been designed and built, depending on the property of the cathode rays being deflected by a magnetic field, the deflexion being compared with that produced by a known standard field.

2. A Wehnelt cathode is used to impart a low velocity to the electrons, resulting in measurable deflexions for the weak magnetic fields. The deflexions are recorded on a photographic plate.

3. The elimination of the earth's magnetic field for a zero reading of the deflexion is avoided by mounting the tube so as to turn about a vertical axis, and deflexions observed for different orientations.

4. The magnetometer is also a declinometer of sufficient precision for setting the coil in a plane perpendicular to the magnetic meridian.

5. Account is taken of the fact that the field of the standard coil is not uniform along the path of the rays, and an analytical solution of the problem is found.

6. Seven determinations of H have been made with a relative probable error in the fourth significant place.

The general design and construction of the apparatus is that of the senior author, who also did a part of the preliminary work. The junior author supplied some of the details of construction, deduced the formula, and made the final observations and the necessary calculations.

Laboratory of Physics,
University of Illinois,
June 9, 1916.

XLIII. *On the "Wolf-note" in Bowed Stringed Instruments.* *By C. V. RAMAN, M.A.**

[Plate IX.]

IT has long been known that on all musical instruments belonging to the violin family there is a particular note which it is difficult to elicit in a satisfactory manner by bowing. This is called the "wolf-note," and when it is sounded the body of the instrument is set in vibration in an

* Communicated by the Author.

unusual degree; and it appears to have been realized that the difficulty of maintaining the note steadily is due in some way to the sympathetic resonance of the instrument*. In a recent paper †, G. W. White has published some interesting experimental work on the subject, confirming this view. The most striking effect noticed is the *cyclical* variation in the intensity of the note when the instrument is forced to speak at this point. White suggests as an explanation of these fluctuations of intensity that they are due to beats which accompany the forced vibration impressed on the resonator when the impressed pitch approaches the natural pitch of the system. The correctness of this suggestion seems open to serious criticism. For, the beats which are produced when a periodic force acts on a resonator are of brief duration, being merely due to the superposition of its forced and free oscillations, and when, as in the present case, the resonator freely communicates its energy to the atmosphere and the force itself is applied in a progressive manner and not suddenly, such beats should be wholly negligible in importance, and should, moreover, vanish entirely when the impressed pitch coincides with the natural pitch.. In the present case the essential feature is the *persistency* of the fluctuations of intensity and their markedness over a not inconsiderable range; and it is evident that an explanation of the effect has to be sought for on lines different from those indicated by White. I had occasion to examine this point when preparing my monograph on the "Mechanical Theory of the Vibrations of Bowed Strings," which will shortly be published, and the conclusions I arrived at have since been confirmed by me experimentally.

From the mechanical theory, it appears that when the pressure with which the bow is applied is less than a certain critical value, proportionate to the rate of dissipation of energy from the vibrating string, the bow is incapable of maintaining the ordinary mode of vibration in which the fundamental is dominant, and the mode of vibration should progressively alter into one in which the octave is the predominant harmonic‡. In the particular case in which the frequency of free oscillation of the string coincides very nearly with that of the bridge of the violin and associated masses, the mode of vibration of the string is *initially* of the well-known type in which the fundamental is dominant.

* Guillemin, "The Application of Physical Forces," 1877.

† G. W. White, Proc. Camb. Phil. Soc. June 1915.

‡ Compare with the observations of Helmholtz, 'Sensations of Tone,' English Translation by Ellis, p. 85.

But the vibrations of the string excite those of the instrument, and, as the vibrations of the latter increase in amplitude, the rate of dissipation of energy increases continually till it outstrips the critical limit, beyond which the bow fails to maintain the usual type of vibration. As a result of this, the mode of vibration of the string progressively alters to a type in which the fundamental is subordinate to the octave in importance. The vibration of the belly then begins to decrease in amplitude, but, as may be expected, this follows the change in the vibrational form of the string by a considerable interval. The decrease in the amplitude of the vibrations of the belly results in a falling off of the rate of dissipation of energy, and, when this is again below the critical limit, the string regains its original form of vibration, passing successively through similar stages, but in the reverse order. This is then followed by an increase in the vibrations of the belly, and the cycle repeats itself indefinitely. The period of each cycle is approximately twice the time in which the vibrations of the belly would decrease from the maximum to the minimum, if the bow were suddenly removed.

The foregoing indications of theory are amply confirmed by the photographs reproduced in Plate IX., which show the simultaneous vibration-curves of the belly and string of a 'cello at the wolf-note pitch. It will be seen that the form of vibration of the string alters cyclically in the manner predicted by theory, and that the corresponding changes in the vibration-curve of the belly *follow* those of the string by an interval of about quarter of a cycle. That the two sets of changes are dynamically interconnected in the manner described is further confirmed by the prominence of the octave in both curves at the epochs of minimum amplitude. The explanation of the cyclical changes given above is also in accordance with the observed fact that they disappear and are replaced by a steady vibration when the ratio of the pressure to the velocity of bowing is either sufficiently increased or sufficiently reduced. In the former case the string vibrates in its normal mode, and in the latter case the fundamental disappears altogether and the string divides up into two segments.

Effect of Muting on the "Wolf-note."

Since the pitch of the wolf-note coincides with that of a point of maximum resonance of the belly, we should expect to find that by loading the bridge or other mobile part of the body of the instrument important effects are produced.

This is readily shown by putting a mute on the bridge. The pitch of the wolf-note then falls immediately by a considerable interval. On the particular 'cello I use, a load of 17 grammes fixed at the highest point of the bridge lowers the wolf-note pitch from 176 to 160 vibrations per second. A larger load of 40.4 grammes depresses it further to 137 vibrations per second, and also causes two new but comparatively feeble resonance-points to appear at 100 and 184 respectively, without any attendant cyclical phenomena. An ordinary brass mute has a very similar effect.

The Formation of Violin-tone and its Alteration by a Mute.

The positions of the frequencies of maximum resonance of the bridge and associated parts of the belly for notes over the whole range of the scale are undoubtedly of the highest importance in determining the character of violin-tone, and the explanation of the effect of a mute on the tone of the instrument is chiefly to be sought for in the effect of the loads applied on the frequencies of the principal free modes of vibration of the bridge and associated parts of the belly. The observations of Dr. P. H. Edwards on the effect of the mute* are evidently capable of explanation on the basis of the lowering of the frequencies of maximum resonance by the loading of the bridge. But a more detailed understanding of the dynamics of the problem requires further theoretical and experimental investigation. Recently, I have secured an extensive series of photographs showing the effect on the motion of the bridge in its own plane produced by fixing a load on it at one or other of a variety of positions. The close parallelism between the effect of loading, as shown by these photographic curves and as observed by the ear, seems to show that the motion of the bridge in its own plane determines the quality of violin-tone to a far greater extent than might be supposed from the work of Giltay and De Haas†. A detailed discussion of this and other problems relating to the physics of bowed instruments is reserved for a separate communication.

This investigation was carried out in the Laboratory of the Indian Association for the Cultivation of Science, Calcutta.

20th May, 1916.

* P. H. Edwards, *Physical Review*, January 1911.

† Giltay and De Haas, *Proc. Roy. Soc. Amsterdam*, January 1910. See also E. H. Barton and T. F. Ebbelwhite, *Phil. Mag.* September 1910, and C. V. Raman, *Phil. Mag.* May 1911.

Note dated the 8th of August added in proof.

Since the paper was first written, several other interesting effects have been noticed, of which the following is a summary:—

(a) Cyclical forms of vibration of the G-string and belly of a 'cello may also be obtained when the vibrating length is double that required for production of the wolf-note, that is, when the frequency is half that of the wolf-note. In this case, when the pressure of the bow is sufficient to maintain a steady vibration, the second harmonic in the motion of the belly is strongly re-inforced. When the pressure is less than that required for a steady vibration, cyclical changes occur, the principal fluctuations in the motion, both of the string and the belly, being in the amplitude of the second harmonic. In this, as in all other cases, the cyclical changes disappear and give place to a steady vibration, when the bow is applied at a point sufficiently removed from the end of the string. In this particular case, a large, *almost soundless* vibration may be obtained by applying the bow rather lightly and rapidly at a point distant one-fifth or more of the length from the end; the octave is then very weak in the vibration of the string, but may be restored, along with the tone of the instrument, by increasing the pressure of the bow.

(b) The 'cello has another marked point of resonance at 360 vibrations per second. The pitch of this is also lowered by loading the bridge.

(c) When the vibrating length of the G-string or A-string of the 'cello is about a fourth of the maximum or less, cyclical forms of vibration may be obtained at almost any pitch desired, by applying the bow with a moderate pressure rather close to the bridge.

(d) As the frequency of vibration is gradually increased from a value below to one above the wolf-note frequency, the phase of the principal component in the "small" motion at the end of the string, that is also of the transverse horizontal motion of the bridge, undergoes a change of approximately 180°. This is in accordance with theory.

XLIV. *Some Experiments on Residual Ionization.*
By K. H. KINGDON, M.A., *University of Toronto**.

[Plate X.]

I. *Residual Ionization in Acetylene.*

INTRODUCTION.—If dry air be enclosed in a clean zinc vessel and removed from the neighbourhood of all ordinary ionizing agents, it is found, on measuring the conductivity of the gas, that ions are being produced in it at the rate of about 8·7 per c.c. per second. The production of these ions is called natural ionization. If, however, the zinc vessel be surrounded by a water-screen the natural ionization is found to decrease; and if the measurements be made over a considerable body of water, such as the ocean or the great lakes of America, the ionization in air is found to fall to a definite minimum of about 4·4 ions per c.c. per second, which we may call the residual ionization. The difference between these two rates of ionization has been shown to be due to a penetrating radiation from the earth's surface, which can be cut off by a screen of water if sufficient thickness be used. The possible components of the residual ionization appear to be:—(1) a radioactive impurity in the gas, (2) a radioactive impurity in the walls of the ionization-chamber, (3) ionization by the collisions of thermal agitation. In an effort to determine to which of these sources residual ionization should be attributed, McLennan and Treleaven† measured the residual ionization in several gases, the results obtained being as follows:—

Gas.	No. of ions per c.c. per second.
Carbon Dioxide	4·83
Hydrogen	1·11
Ethylene	6·32
Nitrous Oxide.....	5·02
Acetylene	27·00

It will be noticed that in the above list the residual ionization in acetylene is very much higher than in any of the

* Communicated by Professor J. C. McLennan, F.R.S.

† J. C. McLennan and C. L. Treleaven, *Phil. Mag.* xxx. p. 415 (1915).

other gases, a fact which certainly cannot be accounted for on the ground of its density. The present investigation was therefore undertaken to see whether this large ionization could be traced to either of the sources (1) or (3) above. As a result, it has been found that in acetylene made from calcium carbide there is present a slight trace of radium emanation, and this it has been shown accounts for the high residual ionization in the gas.

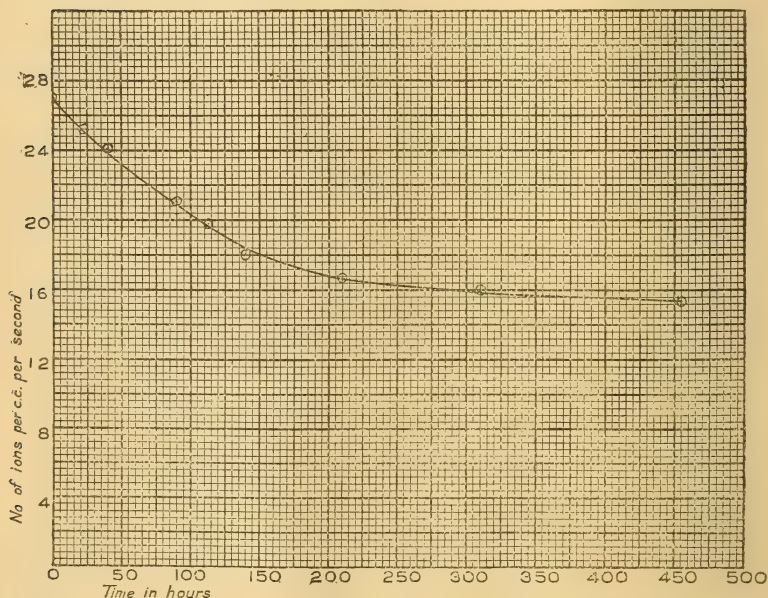
Experiments.—The gas used at first in the experiments with acetylene was taken from a commercial Prest-o-Lite tank. The ionization-chamber was made of zinc because it has been shown that this metal contains smaller traces of radioactive impurities than any other. The external dimensions of the chamber were:—diameter 11.6 cm., length 22.8 cm., and its volume was 2167 c.c. The thickness of the walls was about 3 mm. The chamber was carefully scoured with emery, and washed with dilute hydrochloric acid and water, to remove radioactive deposits. During all the course of the work the chamber was absolutely air-tight. It was provided with a zinc electrode which was connected to a sensitive electrometer in the usual manner. The wall of the chamber was kept at a potential of 240 volts, which ensured that all the currents measured were saturation currents.

The ionization-chamber was filled with acetylene which had been carefully dried and freed from dust, and the number of ions made per c.c. per second was found to be about 20. As this number was considerably less than that previously found by McLennan and Treleaven, the zinc Wolf electrometer (Pl. X. fig. 1) used by them was filled with acetylene from the Prest-o-Lite tank, and the number of ions made per c.c. per second was found to be only 12. The difference between this number and that found with the zinc ionization-chamber was probably due to a radioactive impurity in the walls of the latter. It was thought that the low value of the ionization was due to the commercial acetylene not being pure, and to test this some acetylene was made in the laboratory from calcium carbide, and both the Wolf electrometer and the zinc ionization-chamber were filled with it. The number of ions made per c.c. per second (n) was measured at intervals during a period of about 19 days. The following set of readings was obtained with the Wolf electrometer, and a similar set with the zinc ionization-chamber.

T (hrs.).	<i>n</i> .
0	27.0
22	25.2
40	24.1
88	21.1
113	19.8
139	18.0
209	16.7
310	16.0
454	15.4

A curve plotted from these readings is shown in fig. 2.

Fig. 2.

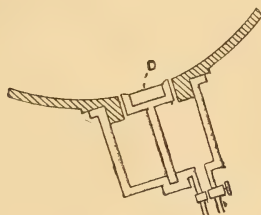


The decrease in the ionization clearly indicates the presence of some radioactive impurity in the gas. From the shape of the curve it may be estimated that the number of ions made per c.c. per second would finally fall to about 15. From this the number of ions per c.c. per second initially due to the radioactive impurity was about 12. As this number of ions had decreased to 6, or to one half, in about 90 hours (3.8 days), it may be concluded that the impurity was radium emanation. In the gas from the Prest-o-Lite tank this

emanation had had ample time to decay, and so the number of ions made per c.c. per second was quite low.

To make certain of the source of the emanation the calcium carbide was tested for radioactivity in the following manner. A metal cap, fig. 3, was made which could be screwed on

Fig. 3.



in place of the drying-tube of the Wolff electrometer. A small brass cup, which was to contain the carbide, was supported on a rod screwed into the end of the cap. The figure shows the cap in place, the top of the cup being flush with the wall of the electrometer. The following measurements were made. The cup was put in place, and the electrometer filled with fresh, dry air. The number of ions made per c.c. per second, n , was then measured. The cup was removed, filled with powdered carbide, and the carbide was covered with aluminium foil $\cdot 0003$ cm. thick. The cup was then replaced in the electrometer, and the number of ions made per c.c. per second again measured. The next day two more measurements of the ionization were made, the first with both brass cup and carbide in place, and the second with them both removed. The air in the electrometer was the same as that used on the first day, and had probably received slight traces of radium emanation from being in contact with the carbide. Hence the values of n obtained on the second day are considerably greater than those obtained on the first day. However, the difference between the two readings taken on either day, which is the important thing here, is almost the same for both days.

First day.	n .	Second day.	n .
Brass cup in place	8.7	Carbide and cup in place ...	12.2
Carbide and cup in place ...	10.6	Both removed	10.4
Difference	1.9	Difference	1.8

These figures clearly indicate that the carbide was slightly radioactive. Any trace of radium emanation present would make itself very apparent in the gas generated from the carbide, and it is to the emanation that the high value of the residual ionization previously obtained was due.

II. *On the possibility of a Portion of the Residual Ionization in Gases being due to the Collisions of Thermal Agitation.*

Introduction.—In the first part of this paper the three possible components of the residual ionization in gases were stated to be:—(1) a radioactive impurity in the gas, (2) a radioactive impurity in the walls of the ionization-chamber, (3) ionization by the collisions of thermal agitation. Now, for the gases air, carbon dioxide, hydrogen, and nitrous oxide, McLennan and Treleaven* have shown that if a clean zinc ionization-chamber is used, and if the gases are properly dried and filtered before admitting them to the chamber, uniform and reproducible values of the residual ionization are obtained for each gas. Hence it does not seem probable that the residual ionization is in general due to the presence of traces of radioactive emanation in the gas. The part played by the second of the above-mentioned components has been quite fully investigated. Recently McLennan and Murray† have shown that by constructing the ionization-chamber of ice, it is possible to obtain a very low value for the residual ionization in air. Their experiments show the great effect of radioactive impurities in the walls of the chamber on the ionization, for with one ice-chamber the number of ions made per c.c. per second was 2·6, while with another the number was 5·5. Although from these experiments it appears possible that the residual ionization, in air at least, may be due entirely to radioactive impurities in the walls of the ionization-chamber, yet it seemed worth while to make some experiments to test for the presence of ionization due to the collisions of thermal agitation. The method was to vary the temperature or the density of the gas in the ionization-chamber, and from the resulting changes in the ionization to see if it was possible to detect the presence of any ionization produced by the collisions of thermal agitation. The results of the experiments in which the density of the gas was varied show that only a part of the residual ionization can be due to the collisions of thermal agitation. The results of the experiments in which the temperature of the gas was varied give

* *Loc. cit.*

† J. C. McLennan and H. G. Murray, *Phil. Mag.* xxx. Sept. 1915.

indications of a small number of ions per c.c. per second produced by the collisions of thermal agitation, but more refined experiments are required to confirm this point. In addition a formula has been derived for the number of thermal collisions in a gas per c.c. per second producing ionization which agrees with the experimental results if the number of such collisions is small.

Theory.—The question of ionization by the collisions of thermal agitation has been investigated theoretically by Langevin and Rey *. In this paper the authors obtained an expression for the number of collisions in a gas per c.c. per second for which the relative velocity of the colliding molecules normal to the sphere of shock was greater than an arbitrary standard. If we denote the number of these "effective" collisions by K , then

$$K = \nu e^{-\frac{1}{2}h\nu v^2},$$

where ν = total number of collisions per c.c. per second,

$$h = \frac{3}{4\epsilon T}, \text{ and } \epsilon = 2.02 \times 10^{-16},$$

v = arbitrary minimum velocity.

According to this formula K would vary very rapidly with the temperature, a prediction which is contradicted by experiment.

Exception was taken to Langevin's work by Wolfke †, who suggested that the potent factor in producing ionization at the collision of two molecules was not their relative velocity normal to the sphere of shock, but rather their relative velocity tangential to it. Indeed he suggested that the normal component would rather prevent ionization by pushing the electron further into the atom, although it is difficult to judge of the value of this suggestion on account of the very conjectural nature of our knowledge of the mechanism of an atom. However, on this ground Wolfke suggested that the number of effective collisions would depend on the relative velocity of the molecules normal to the sphere of collision being less than a certain value, v . The formula obtained for the number of effective shocks is

$$\nu(1 - e^{-\frac{1}{2}h\nu v^2}),$$

where the symbols have the same meaning as before. From this Wolfke calculated that if the collisions in air produce

* Langevin and Rey, *Le Radium*, x. p. 142 (1913).

† Wolfke, *Le Radium*, x. p. 265 (1913).

4 ions per c.c. per second at 17°C. , they will produce 2 at 130°C. , and 6 at -20°C. It should be noted that the above formula only includes the "negative" condition for an ionizing collision, *i. e.* the normal velocity must be below a certain value. A factor representing the "positive" condition should also be introduced, *i. e.* the tangential velocity must be greater than a certain value. To do this we may proceed as follows:—

The expression for the total number of collisions per c.c. per second is obtained by Boltzmann as follows. We assume the presence of two kinds of molecules of masses m and m_1 respectively; n and n_1 are the numbers of each kind per c.c.; $d\omega$ and $d\omega_1$ represent the products of the velocity components for each kind; and f, f_1 , represent for the two kinds of molecules the values of the function

$$n \sqrt{\frac{h^3 m^3}{\pi^3}} e^{-hmc^2}.$$

The conditions of a collision between a molecule m and a molecule m_1 can be characterized by the two parameters b and a defined as follows (fig. 4a):—

Fig. 4a.

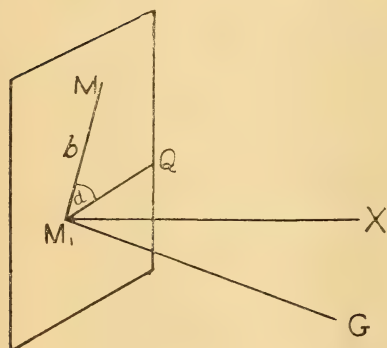
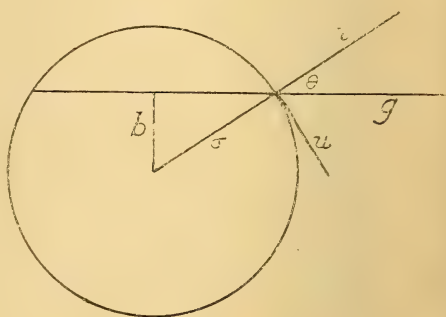


Fig. 4b.



M_1 is the centre of the molecule of mass m_1 . The molecule of mass m moves with a relative velocity g parallel to M_1G , and the projection of the centre of this molecule on the plane P drawn through M_1 perpendicular to M_1G lies at M . The line M_1Q represents the intersection of the planes P and GM_1X . Then $M_1M=b$, and the angle $MM_1Q=a$. The number of collisions per c.c. per second is then

$$\nu = \int f f_1 g b d\omega d\omega_1 db da;$$

or integrating for a from 0 to 2π ,

$$\nu = 2\pi \int b db \int g f f_1 d\omega d\omega_1.$$

Langevin has carried out four of the remaining integrations in such a way as to obtain the result

$$\nu = \frac{2\pi^3 k^2}{hm} \sqrt{\frac{\pi}{2hm}} \int_0^\sigma b db \int_0^\infty g^3 e^{-\frac{1}{2}hmg^2} dg,$$

where σ is the radius of the sphere of action of the molecule and

$$k = n \sqrt{\frac{h^3 m^3}{\pi^3}}.$$

If now we let θ (fig. 4 b) be the angle between g and the normal to the sphere of action, then v the velocity normal to this sphere is equal to $g \cos \theta$ and $b = \sigma \sin \theta$. Making these substitutions in the above integral, we obtain for the total number of collisions per c.c. per second for which the relative velocity normal to the sphere of action is less than a certain v ,

$$\begin{aligned} L &= \frac{2\pi^3 k^2 \sigma^2}{hm} \sqrt{\frac{\pi}{2hm}} \int_0^v v^3 dv \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\cos^3 \theta} e^{-\frac{hmv^2}{2\cos^2 \theta}} d\theta \\ &= \frac{2\pi^3 k^2 \sigma^2}{h^2 m^2} \sqrt{\frac{\pi}{2hm}} \int_0^v v e^{-\frac{1}{2}hmv^2} dv \\ &= \frac{2\pi^3 k^2 \sigma^2}{h^3 m^3} \sqrt{\frac{\pi}{2hm}} (1 - e^{-\frac{1}{2}hmv^2}) \\ &= n^2 \sigma^2 \sqrt{\frac{2\pi}{hm}} (1 - e^{-\frac{1}{2}hmv^2}). \end{aligned}$$

To obtain the number of collisions per c.c. per second for which the relative velocity tangential to the sphere of collision is greater than a certain u , put $u = g \sin \theta$, and $b = \sigma \sin \theta$ in the expression for ν above. Then the number of such collisions is

$$\begin{aligned} M &= \frac{2\pi^3 k^2 \sigma^2}{hm} \sqrt{\frac{\pi}{2hm}} \int_u^\infty u^3 du \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin^3 \theta} e^{-\frac{hmu^2}{2\sin^2 \theta}} d\theta \\ &= \frac{2\pi^3 k^2 \sigma^2}{h^2 m^2} \sqrt{\frac{\pi}{2hm}} \int_u^\infty u e^{-\frac{1}{2}hmu^2} du \\ &= \frac{2\pi^3 k^2 \sigma^2}{h^3 m^3} \sqrt{\frac{\pi}{2hm}} e^{-\frac{1}{2}hmu^2} \\ &= m^2 \sigma^2 \sqrt{\frac{2\pi}{hm}} e^{-\frac{1}{2}hmu^2}. \end{aligned}$$

Hence the probability that for any collision the relative velocity of the molecules normal to the sphere of action shall be less than v is $(1 - e^{-\frac{1}{2}hmv^2})$, since $n^2\sigma^2\sqrt{\frac{2\pi}{hm}}$ is the total number of collisions per c.c. per second. Similarly the probability that the relative velocity tangential to the sphere of action shall be greater than u is $e^{-\frac{1}{2}hmu^2}$. Therefore the probability that both these conditions are fulfilled for a particular molecule is

$$e^{-\frac{1}{2}hmu^2}(1 - e^{-\frac{1}{2}hmv^2}),$$

and the number of such collisions per c.c. per second will be

$$N = n^2\sigma^2\sqrt{\frac{2\pi}{hm}}e^{-\frac{1}{2}hmu^2}(1 - e^{-\frac{1}{2}hmv^2}).$$

Experiments.—The only experimental work which has been done on this subject is by Patterson *, and by Devik †. In Patterson's work the gas was contained in an iron cylinder, and as iron usually contains some radioactive impurity, the number of ions generated per c.c. per second was quite large ($n=61$). He failed to detect any effect of temperature on the ionization up to 400°C ., but it is possible that the effect might have been masked by the largeness of the currents measured. Also, as the air in the receiver was always at atmospheric pressure, its density would decrease as the temperature was raised; this decrease in density would decrease the ionization current due to the earth's penetrating radiation and also that due to a radiation of the β or γ type coming from impurities in the walls of the chamber, both of which form part of the total current measured.

In Devik's experiments the gas was momentarily heated by an adiabatic compression, and the ionization measured at the moment of greatest compression. The only gas which showed any signs of ionization caused by the high temperature (estimated at 900°C .) was antimony hydride.

In view of the methods of the above experiments, it was thought worth while to carry out another investigation in which the following conditions should be satisfied:—(1) the ionization-chamber should be airtight; (2) the residual ionization should be as low as possible so that any change would make itself more apparent; (3) the temperature should be kept constant during the time of each reading. Unfortunately, in order to fulfil these requirements the range of

* Patterson, *Phil. Mag.* vi. p. 231 (1903).

† Devik, *Sitz. d. Heid. Akad. Wiss.* xxiv. (1914).

temperature had to be reduced considerably. The zinc ionization-chamber previously described was used. It was covered with thin asbestos, then wound with nichrome resistance-wire, and packed in magnesia. By passing currents up to 1.5 amperes through this wire, the chamber could be maintained at any temperature between 10° and 100° C. for as long a time as desired. The chamber was absolutely airtight, the wax joints around the electrode and guard-ring being kept cool with a water-jacket. The temperatures were calculated from the changes in pressure.

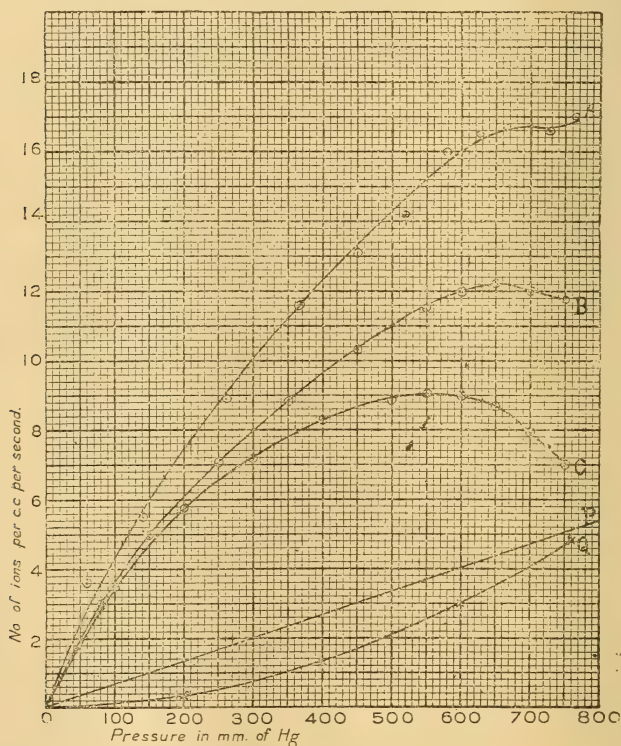
Variation of Ionization with Pressure.—The gases used were carbon dioxide, acetylene, and hydrogen. Several sets of readings were taken with each of these gases at room temperature to show the connexion between ionization and pressure. The readings and curve (fig. 5) shown were obtained with carbon dioxide, and are typical of the others. The ionization shows a slight maximum at 650 mm. pressure, due presumably to a soft radiation from the walls of the chamber.

p (mm.).	n .
764	17.0
726	16.6
667	16.7
626	16.5
579	16.0
519	14.2
450	13.1
366	11.6
262	8.9
141	5.5
60	3.6

We may proceed as follows to see whether this curve gives any indication of the presence of ionization by collision. The possible components of the ionization are—(1) the ionization due to the earth's penetrating radiation, which from the experiments of McLennan and Treleaven will be about 5.1 ions per c.c. per second at 760 mm. pressure; (2) that due to a possible ionization by thermal collisions, which from the same experiments cannot be more than about 4.8 ions per c.c. per second at 760 mm. pressure; (3) that due to any radioactive impurity in the walls of the receiver. Now component (1) will vary directly as the pressure, and may be

represented by the straight line OP in the figure. If we diminish the ordinates of OA by the corresponding ordinates of OP we obtain the curve OB, which is the ionization-pressure curve for the remaining components.

Fig. 5.



Again, on the above theory of ionization by thermal collisions, the number of ions produced per c.c. per second varies as the square of the number of molecules per c.c., and therefore as the square of the pressure. Hence the pressure-ionization curve for this component may be represented by the curve OQ. By diminishing the ordinates of OB by the corresponding ordinates of OQ, the curve OC is obtained, which is the pressure-ionization curve for the radiation from the walls of the chamber. This curve shows a very pronounced maximum at a pressure of 550 mm.; and it is evident that if it is possible for a curve such as OC to represent correctly the pressure-ionization curve for a soft radiation,

then it is possible that about 5 ions per c.c. per second are produced in the gas by the collisions of thermal agitation. Now it is a well-known fact that when a radiation produces ions in a gas, the total number of ions made is the same so long as the radiation is totally absorbed in the gas; also the effect of recombination will be least, and therefore the current will be greatest, when the gas is at such a pressure as just to absorb the rays. Hence it is possible that for a certain range of pressures the current will increase as the pressure decreases, on account of the effects of recombination. The conditions of this experiment are particularly favourable for a large recombination effect, since the direction of the rays is perpendicular to the direction of the electric field, and also since carbon dioxide is a heavy gas. Yet in spite of these arguments it seems improbable that the effects of recombination could cause a rise from 7 to 9 ions per c.c. per second, or an increase of about 33 per cent. in the current, as is the case here. These results would therefore seem to show that the residual ionization cannot be wholly due to the collisions of thermal agitation. They cannot, however, be said to exclude the possibility that a smaller number of ions than 5 per c.c. per second may be due to these collisions.

Variation of Ionization with Temperature.—The gases used were carbon dioxide and acetylene, and the range of temperature was from 18° to 100° C. Great difficulty was experienced in getting reliable sets of readings; for as each set required a period of from six to eight hours, it was quite possible that the leak of the electrometer might vary during this time. As a determination of the leak required that the ionization-chamber be exhausted, and as also it was found that in order to obtain consistent readings the gas had to be allowed to stand in the chamber for some time before commencing readings, it was only possible to obtain one leak reading for each set of temperature readings. All the sets of readings, however, agree in showing that the ionization is practically unchanged from 18° to about 80° C. For temperatures from 80° to 100° some sets of readings show a very marked increase in the current, while in others this increase is very small. The two chief sources of possible error would appear to be thermo-electric currents, and the driving off of minute quantities of radioactive emanation from the walls of the ionization-chamber as its temperature was raised. These errors were guarded against by eliminating possible thermo-junctions, and by exhausting the ionization-chamber while it was heated to 100° . To test for the presence of thermo-electricity the junction of the brass electrode and the

brass connecting wire at the top of the ionization-chamber, which junction was the only one that could possibly serve as a hot junction, was heated with a small flame, the flame was removed, and the earth connexion to the quadrants of the electrometer broken, but no change could be detected in the normal ionization current flowing to the electrometer. It seems likely though that the rise in current sometimes noted at temperatures of about 100° was due to small quantities of emanation being driven off from the walls of the chamber.

It might be interesting to see if the proposed formula for N agrees at all with the results of the temperature experiments. The experiments on pressure show that the number of ions produced per c.c. per second in carbon dioxide at 20° C. and 760 mm. pressure is probably less than four. In the expression for N we have then to assign values to u and v so that the following conditions may be fulfilled:

- (1) N must be about 4 at 20° C. and 760 mm. pressure.
- (2) N must change slowly with the temperature, at least in the region of 20° .

The only physical condition suggesting itself which will fulfil the above requirement is that, for a collision to produce ionization, it must be almost perfectly tangential (this will make the total number of such collisions small), and that the arbitrary minimum tangential velocity of each of the colliding molecules must be about equal to the most probable velocity for a temperature of 20° (this ensures that N shall change slowly with the temperature in this region). Then for carbon dioxide, if we put the minimum relative tangential velocity $u = 2\alpha = 2 \times 3.43 \times 10^4$ cm. per second, and the maximum relative normal velocity $v = 8.97 \times 10^{-10}$ cm. per second, we find that at 20° C., $N = 4$, and at 100° , $N = 5.1$. That is, N changes very slowly as the temperature is raised, which is in qualitative agreement with the experimental results. A more exact application of the formula for N does not seem worth while at the present time, since, for the reasons stated above, the accuracy of the readings does not warrant it. It may, however, be of interest to note that using the above values for u and v , and making changes in h , p , and η to correspond to the rise in temperature, at 302° C. the value for N is 6.4.

Summary.

- (1) It has been shown that the high residual ionization in acetylene prepared from calcium carbide is due to the presence of slight traces of radium emanation.

(2) It has been shown that only a portion of the ultimate residual ionization in gases can be due to the collisions of thermal agitation.

(3) A formula has been devised for the number of collisions per c.c. per second producing ionization which is in qualitative agreement with experimental facts.

In conclusion the author wishes to express his thanks to Professor J. C. McLennan, who suggested the problem, and whose assistance and encouragement have been most valuable.

Physical Laboratory,
University of Toronto.
May 8, 1916.

*XLV. A New Method of Measuring the Refractive Index and Dispersion of Glass in Lenticular or other forms, based upon the "Schlieren-methode" of Töpler. By R. W. CHESHIRE, B.A.**

THE methods most commonly employed for the accurate refractometric examination of optical glass for commercial purposes involve special preparation of the specimen under test. The Pulfrich refractometer, which is probably used in optical workshops more frequently than any other instrument, is designed to give an accuracy in the determination of n_D of one unit in the fourth decimal place, and in the partial dispersions of two or three units in the fifth decimal place. As a preliminary to the examination on this instrument the glass must be cut in the form of a right-angled prism and, of the two faces forming the right angle, at least one must be plane and well polished and the other, in order to secure good definition through the telescope, must meet it in a sharp knife-edge. The commercial accuracy of determination in the optical constants specified above is sufficient for practical designing purposes, and is, in any case, greater than the accuracy with which the refractive index of successive meltings of glass can be reproduced on the large scale.

It may happen, however, that one is confronted with the problem of determining the optical constants of a piece of glass in a form in which it is not possible to examine it on the refractometer. The glass, for instance, may be a small component lens forming part of a microscope objective and

* Communicated by the Director of the National Physical Laboratory.

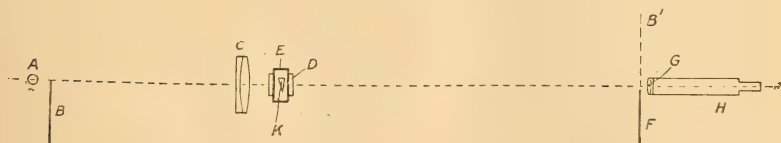
too small therefore to permit of a prism being cut from it with which to conduct the ordinary examination; on the other hand, it may be considered undesirable to deface or injure the lens in any way which would militate against its subsequent use in an optical system. One of the most obvious methods of solving the problem is to measure one or more of the optical lengths associated with the simple lens, the principal focal length or the back focal length for some specified wave-length, and to supplement this information with measurements of the radii of curvature of the lens surfaces and the axial thickness at the vertex. A knowledge of these quantities is theoretically sufficient to enable one to calculate the index of refraction of the glass for the wave-length employed, but a little consideration will suffice to show that the limitations of practice restrict the accuracy possible in the determination of the refractive index to a figure appreciably higher than one in the fourth place of decimals, with a corresponding greater relative inaccuracy as regards the partial dispersions. In the first place, the ordinary type of spherometer in which the three fixed points lie on a circle of diameter $1\frac{1}{2}$ in. to 2 in. is, in many cases, useless for the measurement of the curvatures of the surfaces. The small lenses used in the construction of microscope objectives, eyepieces, and the smaller sized telescope objectives require to be examined on a special optical spherometer designed to deal with these small apertures. Various optical devices involving the use of the lens surfaces as reflectors have been suggested and employed from time to time, but none of these methods aim at or secure an accuracy of the order of 1 in 5000 which is the present desideratum. And it should further be remembered that even this accuracy is not sufficient for the determination of the partial dispersions. Secondly, as regards the determination of focal lengths or of back focal lengths, the presence of spherical aberration in the simple uncorrected lens very seriously limits the reliance to be placed upon the observations. In the most favourable case, that of a convexo-plane lens, the ratio of the longitudinal spherical aberration for the extreme ray to the focal length may be taken to be approximately represented by $(h/f)^2$, where f denotes the focal length and h the semi-linear aperture of the thin lens. But, in order to secure reasonably good definition and an absence of depth of focus, a semi-aperture of at least $f/40$ is advisable, corresponding to axial spherical aberration amounting to about 1 part in 1500. When all these factors are taken into consideration it will be seen that the accuracy to be expected in the determination

of n_D is of the order of one in the third place of decimals rather than one in the fourth.

The method here described was devised with a view to obtaining a direct measurement of the refractive index of the glass, thus obviating the necessity for making intermediate observations which, by their unreliable nature, greatly limit the accuracy finally obtainable. The rationale of the method consists in immersing the glass under examination in a liquid whose refractive index for the specified wave-length can be varied continuously until an equality is attained between the index of the glass and that of the surrounding liquid. The ordinary process can then be applied to measure the refractive index of the liquid on the Pulfrich refractometer. It should be noted that the method may be employed for the refractometric examination of glass in any irregular shape, and not solely of glass in lenticular or prismatic form. It is not claimed that the method is an absolute one for the determination of the refractive index of the glass, as the ultimate accuracy is, of course, that set by the measurements on the refractometer; but it is possible with care to equalize the refractive indices and partial dispersions of the glass and the immersion fluid to at least as high an order of accuracy as that with which they can subsequently be measured on the refractometer.

The "Schlieren-methode" invented by Töpler and described in detail in *Wied. Ann.* vol. cxxxi. p. 33 (see also R. W. Wood, 'Physical Optics,' p. 78) affords an extremely sensitive means of detecting the presence in an otherwise homogeneous transparent medium of striæ or other optical non-homogeneity, and is therefore well suited to the present purpose. The arrangement of the apparatus employed is shown diagrammatically in plan in fig. 1.

Fig. 1.



A vertical straight edge behind which is placed a source of monochromatic light A is represented in plan by B. An image B' of this straight edge is projected by means of a telescope objective C (focal length 125 cm., aperture 8 cm.) into the plane of a second straight edge, F, disposed as shown in front of an observing telescope H, so as to cover about one-half of the full aperture of its objective G. The

various parts of the apparatus are mounted rigidly in any convenient manner, and the two edges and the observing telescope H should be approximately centred with the axis of the principal lens C. The observing telescope, situated at a distance of roughly 5 metres from the lens C, has an aperture of about 3.5 cm. and a magnifying power—in normal adjustment—of $\times 24$. The edge F is secured to a rack-and-pinion motion providing for a slow transverse movement across the front of the observing telescope in a right- or left-hand direction. The desired parallelism of the two edges may be checked either by removing the eyepiece of the observing telescope and looking down the tube, or by examining with a small pocket magnifier the images of the two edges projected into the Ramsden circle of the telescope. A small rectangular glass cell, E (see below), containing the immersion fluid and the glass under test, K, is adjusted in the usual way on the prism, D, of the refractometer (not shown in diagram), which is placed just in front of the lens C and at such a height that the cell is approximately opposite the centre of the lens. Suppose, for the sake of simplicity, that a prism of small angle is being examined and that it is immersed in the fluid contained in the cell with its refracting edge vertical. In general the refractive indices of the glass prism and the surrounding liquid will differ, and the glass will therefore exert a prismatic effect upon light passing through it. The image of the edge B will therefore present a doubled appearance, the image formed by light passing through the glass being displaced laterally by a small amount relatively to the rest of the image depending upon the magnitude of the difference between the two indices. Consequently, on traversing slowly the edge F across the objective of the observing telescope, which will be supposed to be focussed upon the prism, it will be found that the surface of the prism gradually darkens and finally becomes completely black, whilst the side faces and the rest of the cell will show up brightly. On continuing the steady motion of the edge F these faces also will begin to darken, until finally no light passes into the observing telescope and the whole field appears black. The order of these two phenomena is determined by the sign of the difference of the refractive indices of the fluid and prism. If now the refractive index of the immersion fluid be changed somewhat, it will be at once obvious, from the relative interval at which these phenomena occur, whether the difference in the refractive indices has been increased or decreased by the alteration. In the course of a few trials it

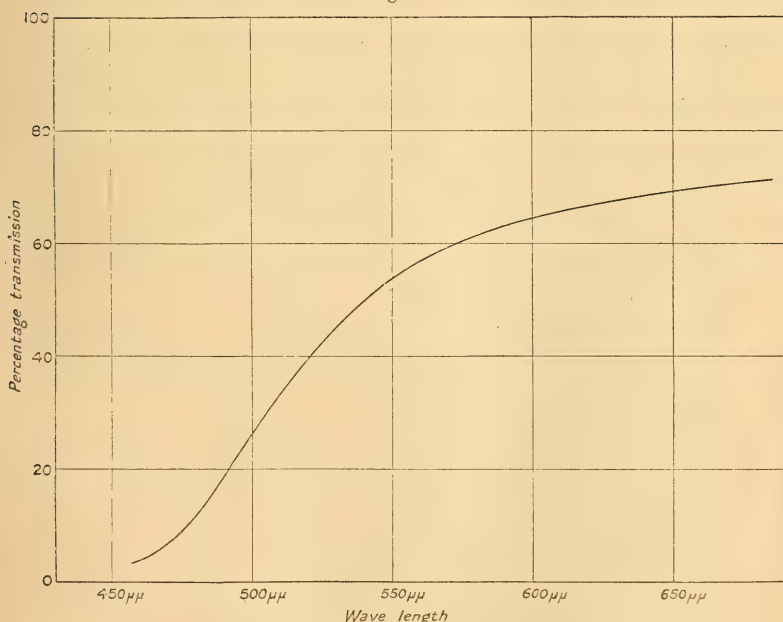
seen that the section represented by BC only is effective. It would of course be possible to utilize the full aperture AB of the refractometer prism by making the width of the cell appreciably greater than the full aperture which it is desired to fill, but, on the other hand, this change would increase the thickness of the layer of immersion fluid through which light has to pass. Unfortunately all the suitable immersion fluids absorb very strongly light at the blue end of the spectrum, and it is therefore of importance to keep the width of the cell as small as possible. The refractive index of the glass forming the base-plate must also be greater than that of any liquid which it is proposed to examine in the cell. Practical constructional difficulties rendered it inadvisable to diminish the thickness of the base-plate of the cell below about 2 mm., and this was therefore the dimension finally adopted. With this thickness and a base-plate of extra dense flint for which $n_D = 1.74$, no trouble was experienced in obtaining sufficient light and aperture to give good settings with the refractometer up to a value $n_D = 1.66$ for the fluid contained within the cell. The deposition of dirt in the right angle formed by the base-plate and the sides, or the presence there of a slight trace of the cement used in building up the cell, would ruin the sharp definition at the edge of a line seen through the telescope of the refractometer and it was therefore considered desirable to chamfer the inner edges of the base-plate over a small fraction of a millimetre, as shown in fig. 2. The upper and lower faces of the base-plate must be as nearly as possible plane parallel, and in the cell built up for the purposes of this investigation by Messrs. Adam Hilger, the plane parallelism of these faces was guaranteed to an accuracy of 5".

The side faces of the cell through which passes the light used to determine the equality or otherwise of the indices of the fluid and the glass under test should be plane polished, but it is not necessary that they should also be parallel: and in order to secure good optical contact between the lower surface of the base-plate and the Pulfrich prism, it is desirable that the former should project very slightly below the level of the side faces of the cell, as shown in the figure. The inside dimensions of the cell as actually constructed were $5 \times 2 \times 1.5$ cm.

The immersion fluid finally selected for use in the cell was an aqueous solution of mercury potassium iodide, more commonly known as Thoulet's solution. An excellent account of the preparation and properties of this solution is given in Johannsen's '*Manual of Petrographic Methods*,' where also will be found a very comprehensive table of

other useful immersion fluids. In its most concentrated form the refractive index of the solution for the D line is 1.72, and the index may be varied continuously down to 1.33 by admixture with increasing proportions of water. The solution possesses the great advantage from the practical point of view in that the excess of water may be driven off by heat and the original solution with a high index of refraction recovered again. Many of the mixtures that have been suggested for the purpose of obtaining a solution of variable refractive index involve the employment of two liquids other than water which cannot be separated and recovered in this way after use. The double iodide possesses the further useful property that it does not attack the Canada balsam used for cementing together the walls of the cell. The main drawback to the use of the solution for the present purpose lies in the fact that it exercises marked absorption at the more refrangible end of the spectrum.

Fig. 3.



The transmission curve for light of various wave-lengths passing through the cell filled with Thoulet solution of refractive index $n_D = 1.51$ was obtained by means of a spectrophotometer, with the result shown in fig. 3.

It will be seen that the percentage transmissions for the four customary lines C, D, F, and G' are approximately 70.

63, 16, and 2 respectively. The transmission curve shown in the figure relates to the Thoulet solution actually employed in the cell for these refractive index determinations. Owing to a tendency of the iodine to separate out from the solution the tint of a solution of given index is apt to vary and to pass from a yellowish-green to a brown colour as this separation increases. The corresponding transmission curves will therefore change slightly, but will retain the same general characteristics as regards the spectrum distribution of intensity in the transmitted light. The original yellow-green solution may always be recovered again by warming the brown solution in the presence of a little clean mercury. In common with all liquids the temperature coefficient of the refractive index of Thoulet's solution is very high, and is of the order of -0.0006 per degree centigrade. It might have been expected that this circumstance would, in itself, preclude the possibility of obtaining any significant result to five figures, but in practice no difficulty is experienced if the apparatus be shielded from direct currents of air. After a balance has been obtained through the observing telescope for the equality of the indices, an interval of not more than a minute need elapse before two or three readings have been taken for the refractive index of the solution, and throughout this interval the solution will not be found to have changed in refractive index to an extent which is appreciable on the Pulfrich refractometer, provided that the solution has previously been allowed to attain the room temperature. In any case two independent observers could, if desired, take simultaneous readings, one on the refractometer, whilst the other views the liquid in the cell through the observing telescope. A much more serious difficulty to be met arises from the unequal evaporation at the surface of the liquid and from the walls of the cell, producing local changes in concentration and giving rise to the presence of striæ, which eventually permeate the whole mass of the fluid and assume most fantastic forms when seen through the observing telescope with the rest of the field dark. Töpler in his original paper describes these striæ and gives an excellent reproduction of their appearance. The growth of these beautiful but very undesirable striæ within the fluid may be prevented by providing the open rectangular cell with a glass cover-plate on the under side of which has been smeared a little vaseline. This has the effect of excluding the air, and after a few moments equilibrium is restored between the vapours in the cell and no further local changes of concentration take place.

The ordinary soda-flame produced by holding a piece of common salt in a Bunsen flame will be found to give sufficient light to enable accurate observations to be taken on the sodium line. If more light should be required for any purpose, an apparatus designed to produce a very intense source of sodium light and described by H. E. Armstrong in the *Proc. Roy. Soc.* vol. lxxxi. (1908) is recommended. It will be found convenient to provide two sources of light, one to be used for obtaining the equality of the indices, and the other to be used in conjunction with the prism-condenser of the refractometer, which is of course swung out of action whilst the first source is being used, to determine the refractive index of the liquid as soon as the equality has been obtained. It is a matter for regret that it has not hitherto been found possible to determine the refractive indices of the glass for the other standard refractometric lines C, F, and G' to the same satisfactory order of accuracy that is possible with the D line. The intensity of the G' line in the hydrogen spectrum given by the usual H-tubes used in refractometry is not great, and the line is by no means an ideal one to use at the best of times. Moreover, as the curve shown in fig. 3 indicates, the immersion fluid itself transmits only a negligible percentage of light of this wave-length. This line therefore must be regarded as quite useless for our present purposes. The position as regards the C and F lines is not quite so unsatisfactory, but even in this case the intensity of the light given by the hydrogen tube is relatively feeble and is not calculated to give results with any certainty. It should be remembered that the adjustment for equality of indices is really a photometric one, and consequently far more light is required than in the case of the usual refractometer readings where the cross-lines in the eyepiece of the refractometer telescope are brought into coincidence with the edge of the line under examination, and the setting is rather of the nature of a geometrical or positional one. The separation of the C and F lines is effected by means of coloured filters, the red filter employed transmitting about 89 per cent. of the C line, and the blue filter about 56 per cent. of the F line. The most satisfactory sources to be used for measurements on the partial dispersions would appear to be the mercury-vapour lamp for a green line $\lambda=546\mu\mu$, and the recently described cadmium arc (H. J. S. Sand, *Phys. Soc. Proc.* xxviii., 1916) for the red cadmium line $\lambda=644\mu\mu$. The

mercury-vapour lamp used in conjunction with the Wratten mercury monochromatic filter provides all the light that is required for the purpose.

Experimental readings have been taken on a small prism with a view to determining the sensitivity of the method for the D line. The glass cell was first cemented on to the Pulfrich prism in the usual fashion with a solution of barium mercuric iodide ($n_D = 1.79$) and adjusted so that the fringes observed in monochromatic light ran parallel to the direction of the incident grazing light. The prism under test was then temporarily cemented to the under surface of the cover-plate with its refracting edge vertical, and the cell was filled with Thoulet solution to such a height that the prism was immersed in the liquid to a depth of about 5 mm. The vaselined cover-plate being appreciably larger than the open top of the cell could then be moved to and fro, and the mixture within the cell well stirred by the prism without admitting air and thereby disturbing the equilibrium and concentration of the immersion solution. A glance through the observing telescope was sufficient to discover whether the stirring had been carried far enough and whether the solution was perfectly homogeneous. The cover-plate was then cautiously withdrawn a little distance to allow a few drops of the concentrated Thoulet solution to be dropped into the mixture contained in the cell, the mixing process was repeated, and observations were then taken through the observing telescope to determine whether the difference between the indices had been increased or decreased. It was found that after four or five trials the indices could be brought so nearly to an equality that one drop of Thoulet solution, or one drop of water, was sufficient to change the sign of the difference. The final adjustment was then made with two solutions, one with an index slightly above, and the other with an index slightly below that of the mixture within the cell. As soon as the equality had been obtained a reading was taken at once on the refractometer for the refractive index of the solution, using the second sodium flame mentioned above in the ordinary manner. A series of determinations of the refractive index of the equalizing Thoulet solution was made in this way, and then the cell was removed from the Pulfrich prism, and the glass prism examined on the refractometer by the usual process and its index determined directly. In this way it was possible to investigate the sensitivity of the apparatus and to check the mean value obtained by a direct reading on the prism.

Eight separate determinations were made by the immersion-fluid method with the following results :—

Number of experiment.	Observed n_D for Thoulet solution.	Difference in fifth place from mean value.
1.....	1.51493	+1
2.....	1.51492	0
3.....	1.51495	+3
4.....	1.51488	-4
5.....	1.51491	-1
6.....	1.51489	-3
7.....	1.51494	+2
8.....	1.51491	-1

Mean value 1.51492.

Average difference from mean value = .00002.

Value obtained for the index of the prism by a direct reading on the refractometer = 1.51490.

An inspection of the results obtained shows that the indices of the glass prism and the surrounding liquid can be equalized to an average accuracy of $\pm .00002$. The accuracy with which the indices may be equalized is clearly proportional to the angle of the prism, or to the angle between the cathetal surfaces if the glass examined is not in the form of a regular prism. In the present instance this angle was about 10° , and an even higher order of repetition accuracy is to be expected in the case of an irregularly shaped piece of glass in which the surfaces of the refracting edge chosen meet at an angle greater than this. It is evident, then, that the "Schlieren-methode" is well adapted for the equalizing of the indices to one or two units in the fifth place, and is therefore all that is required in commercial practice.

It was found that when the straight edge F had been moved into coincidence with the image of the edge B, and even after it had overlapped it slightly, the field could not be obtained absolutely dark, and a small image of the extended light source A persisted in remaining visible in the field of view of the observing telescope. This was eventually traced down to an out-of-focus flare-spot image

of the source of light formed by the objective C about 3 metres away from the observing telescope. Such an image will, of course, always send light into the uncovered portion of the object-glass of the observing telescope in whatever position the edge F may happen to be placed. The intensity of the light that goes to form a flare-spot image, however, is not great and exerts no disturbing effect upon the readings. It is well, however, to shield off the major portion of the extended source of light from the apparatus and, instead of a simple straight edge backed by a large source of light, to use one of the straight edges in a slit about 1 mm. in width. It will be found that this reduces the general illumination to almost negligible proportions. If it be considered worth while the intensity of the flare image can be reduced further by tarnishing the glass-air surfaces of the lens C along the lines pointed out by H. Dennis Taylor in a patent specification No. 29561 (1904), entitled "A method of increasing the brilliancy of the images formed by lenses." A paper on the same subject by F. Kollmorgen in the Transactions of the Illuminating Engineering Society, vol. xi. (1916), might also be consulted. The outer zones of a thin positive lens may be regarded as built up of a number of small prisms oriented with their refracting edges tangentially to the circular boundary of the lens, and no difficulty is therefore to be expected in dealing with such a lens in accordance with the method of this paper. Owing to the pressure of more urgent work, however, it has not been found possible further to develop the method with a view to ascertaining its accuracy when applied to high-power negative lenses with appreciable edge-thickness. It was thought, nevertheless, that an account of what has already been accomplished might prove of some service to others who might be interested in determining all the optical constants of a given system.

The author desires, in conclusion, to place upon record his indebtedness to Dr. Glazebrook, the Director of the National Physical Laboratory, for the facilities placed at his disposal for carrying out this investigation.

XLVI. *The Photoelectric Effect on Thin Films of Platinum.*

By J. ROBINSON, M.Sc., Ph.D., Lecturer in Physics at East London College*.

THE photoelectric current from thin films of platinum does not increase uniformly with the thickness, but varies† as shown in Curve A, fig. 2. This applies whether the incident or emergent effect is considered; *i. e.*, whether the light falls on the platinum film before or after passing through the quartz plate on which the film is deposited. For thicknesses below about 10^{-7} cm., *i. e.* from where the current is a maximum, the emergent effect is larger than the incident effect both as regards current and velocities‡. An explanation of the difference in magnitude of the incident and emergent currents was offered by Partzsch and Hallwachs§, who showed that in the emergent case more light is absorbed than in the incident case. This would offer a full enough explanation if the magnitude of the effects were exactly the same, but it would not explain the difference in velocities. Hence doubt was cast on the result that the velocities do differ. It has since been shown by Stuhlmann||, who used a method similar to the author's, that the emergent velocity is larger than the incident velocity.

Partzsch and Hallwachs' result, that more light is absorbed in the emergent case, shows the necessity for further work to find:—in the first place, what the asymmetry of the currents is when unit quantity of light is absorbed in each case, and in the second place, whether for very thin films the absorption of light varies with the thickness in a way similar to that of the current.

In the experiments to be described, an attempt was made to get some information on these problems. It was considered advisable to make the measurements of the absorption of light on the same films on which the photoelectric measurements were made.

A D E K is a nearly closed metallic cylinder standing on a base D E. There is a small inlet of light at B. The

* Communicated by Prof. C. H. Lees, F.R.S.

† Robinson, *Phil. Mag.* xxv. p. 122 (1913); Werner, *Ark. Mat. Ast. Fys.* viii. no. 27, p. 1 (1913); Stuhlmann & Compton, *Phys. Rev.* ser. 2. ii. pp. 189 & 327 (1913).

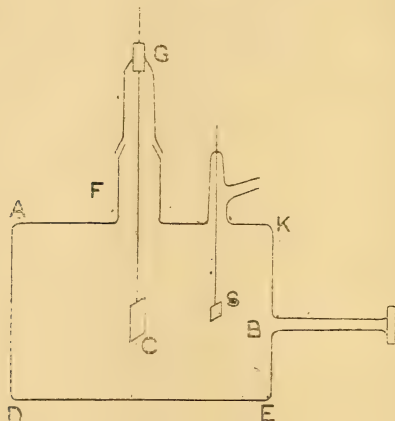
‡ Robinson, *loc. cit.*

§ Partzsch & Hallwachs, *Ann. des Phys.* xli. p. 247 (1913).

|| *Phys. Rev.* iv. p. 195 (1914).

vessel thus acts nearly as a black body, and very little light which enters at B can escape again. Thus it is all absorbed somewhere in the vessel. C is a platinum film on quartz arranged so that a parallel beam of light entering at B falls on it. Let I_0 be the intensity of light entering at B, and

Fig. 1.



hence falling on C. A certain proportion of this, αI_0 , is absorbed by the film, βI_0 is reflected, and γI_0 is transmitted, where $\alpha + \beta + \gamma = 1$. The amount of light $(\beta + \gamma)I_0$ is absorbed by the vessel. The amount of light αI_0 is that which produces the photo current in the film. We need to find i_1 , the photo current from the film, and αI_0 , the light absorbed by the film. The photo current i_1 can be measured by charging the walls of the vessel to a potential to drive the electrons away from the film.

Suppose that I_0 is kept constant, and with a mercury-vapour lamp it can be kept practically constant. Then $\alpha = 1 - (\beta + \gamma)$, where $(\beta + \gamma)I_0$ is the amount of light absorbed by the walls of the vessel. These walls remain constant throughout a series of experiments, and if the photo current from the walls is measured by applying a potential to drive electrons to the film, this current is proportional to the amount of light absorbed by the walls, and hence to $(\beta + \gamma)$. Thus α can be obtained in terms of the photo current of the walls of the vessel. α itself cannot be absolutely determined in each case unless some means is found for measuring the intensity of the light, or unless another set of experiments is made, so that I_0 can be eliminated from the equations.

This was at first attempted by arranging a screen at S of the same material as the walls of the vessel, so that the relative amounts of light falling on the film and the vessel could be altered. This was abandoned, as it was quite sufficient for the present purpose to know how α varies with the thickness.

A quartz tube was cemented to the side tube to let ultra-violet light into the vessel. Another side piece, F, had a ground joint for the purpose of rotating the film C. The rod carrying the quartz plate for the film was insulated at G by an amber plug. The contact between the film and the rod was made as in a former paper, by getting a thick deposit of platinum at an edge of the quartz-plate, laying tinfoil over this so as to allow part of the thick platinum film to be uncovered, and clamping this prepared edge of the quartz to the rod. The films to be tested were deposited over the whole of one side of the quartz plate, and only the centre of the plate was exposed to the light. The plate was of dimensions 10 mm. \times 8 mm. \times 1 mm.

The inside of the vessel was coated with soot from burning camphor in one series of experiments, and in another series consisted of zinc. The platinum film was joined to a pair of quadrants of a Dolezalek electrometer whose sensitiveness was 2000 divisions per volt, when the needle was charged to 100 volts. The current from the film was measured by charging the walls of the vessel to a high positive potential, and that from the walls by charging them to a high negative potential, so that the maximum effects were obtained in each case.

The following results were obtained :—

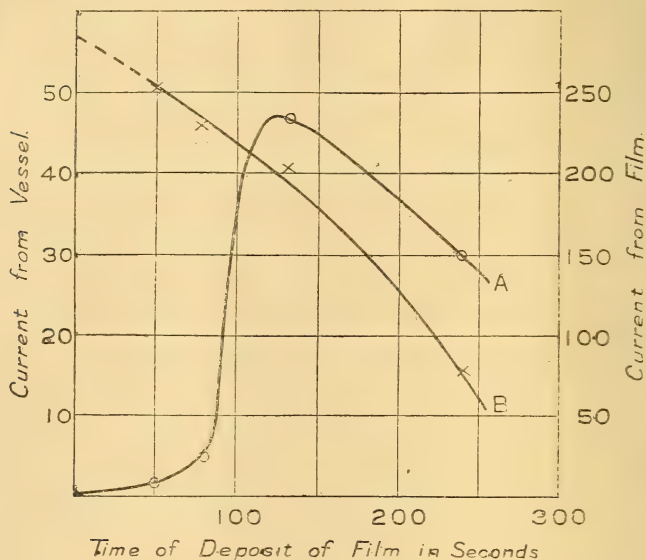
TABLE I.

Thickness of film.	Current from film.	Current from the walls of the vessel.
50 seconds.	8	51
80 "	22	46.5
130 "	235	41
240 "	150	16

The thickness of the film was measured in terms of the length of time to deposit it. The quartz plate was taken out of the vessel in each case and placed in a discharge-tube with a platinum cathode to have the film formed or thickened.

These results are shown graphically in fig. 2.

Fig. 2.



Curve A.—Photo current for various thicknesses of film.

Curve B.—Photo current from the walls of the vessel.

It is seen that the current from the film varies with the thickness in the way already demonstrated*; *i. e.*, it increases gradually with the thickness at first, then it suddenly increases rapidly, after which it falls again. The current from the walls of the vessel diminishes uniformly, and it is proportional to the light absorbed by the vessel. Hence $(\beta + \gamma)I_0$ diminishes uniformly as the thickness of film increases, and hence αI_0 the light absorbed by the film increases uniformly with the thickness. Producing the curve for the current from the vessel backwards, we find the current that would have been obtained had no light been absorbed by the film. This value is 57 in the units employed, and this gives an arbitrary measure of the intensity of light I_0 on entering the vessel. Curve A (fig. 3) shows how the light absorbed by the film varies with the thickness. It was found by subtracting the ordinates of the curve B (fig. 2) from 57.

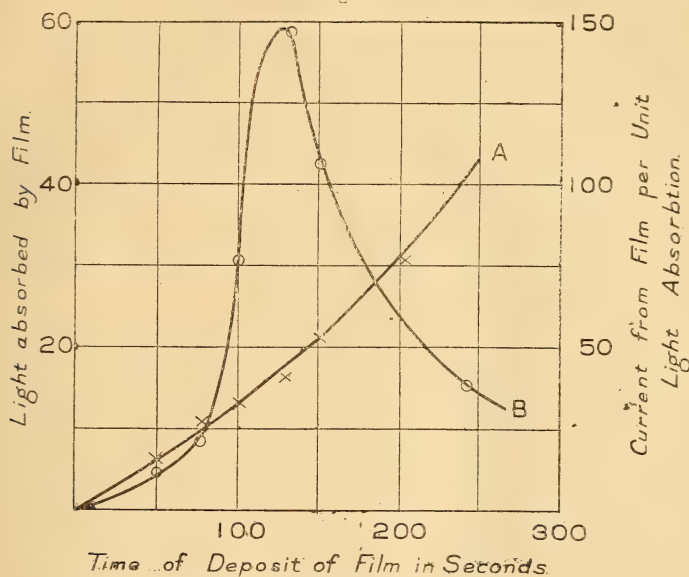
It is at once obvious that the large photo current at the thickness 10^{-7} cm. is not produced by an abnormal absorption of light.

In Table II. values for the current from the film and light

* Robinson, *loc. cit.*

absorbed by the film as obtained from the curve A (fig. 2) and curve A (fig. 3) are given for various thicknesses, as also are the photo currents for unit absorption of light.

Fig. 3.



Curve A.—Intensity of light absorbed by the film.

Curve B.—Photo current for unit intensity of light absorbed.

TABLE II.

Thickness of film in seconds.	Current from film.	Light absorbed.	Current per unit light absorbed.
50	8	6	13.3
80	22	10.5	21
100	100	13	77
130	235	16	147
150	220	21	105
200	178	31	58
240	150	41	37

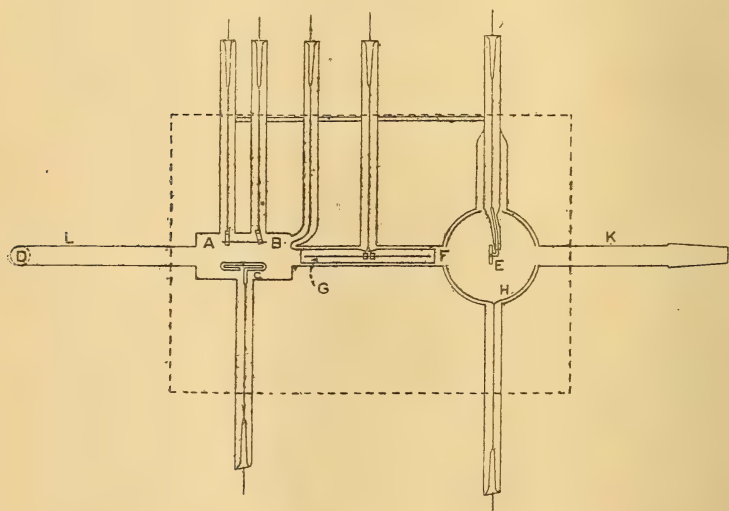
In curve B (fig. 3) are plotted the photo currents per unit absorption of light against the thickness of film, and the maximum value of the current appears still more pronounced. The photoelectric sensitiveness of platinum is thus very large for films of the order of 10^{-7} cm.

June 19, 1916.

XLVII. *Experiments with Electron Currents in Different Gases.* (1) *Mercury Vapour.* By O. W. RICHARDSON, F.R.S., *Wheatstone Professor of Physics, University of London—King's College,* and CHARLES B. BAZZONI, Ph.D., *Instructor in Physics, University of Pennsylvania* *.

IN these experiments electron currents from an incandescent tungsten filament AB (fig. 1) flowed under various potential differences to a cold anode C of nickel wire about 1.5 mm. in diameter. The filament AB was about 1.5 cm. in length and .08 mm. in diameter. The current across the gap AB to C has been measured under various conditions. At the same time the nickel electrode E

Fig. 1.



at the centre of the bulb H, which is covered on the inside with a conducting layer of platinum, enabled the photoelectric effect of the radiation from the discharge to be examined. The connecting tube contained the tubular sheath F and the central wire G. An electric field was applied between F and G to stop the diffusion of ions from ABC to the bulb EH. The apparatus shown in the figure was made of transparent quartz, and could be maintained at various temperatures by means of a specially constructed electric

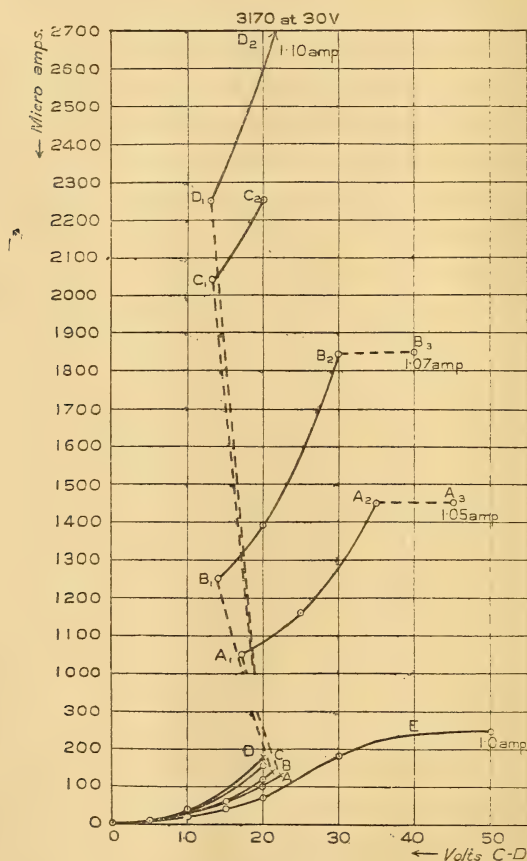
* Communicated by the Authors.

furnace. Fig. 1 exhibits the relative size and position of the metal parts which are of electrical importance. The mercury was supplied from the bottom of the tube LD, which could be heated by a current through a metal spiral outside it. D is a piece of tube 7 cm. long perpendicular to the plane of the rest of the section, and is outside the furnace already referred to whose boundaries are indicated by the dotted lines. In this way a little mercury could be distilled from D into the bulb ABC, and the pressure of the mercury vapour could then be regulated by means of the surrounding electric furnace. The currents across the gap $AB \rightarrow C$ were measured by a microammeter, and the photo-electric currents to E by means of a quadrant electrometer. Beyond K the quartz tube was ground to fit a glass cone, the joint being made tight with a minimum quantity of involatile grease, and precautions were taken to prevent it from becoming warm under any circumstances. Beyond the joint were (1) a side tube leading to a Geissler tube to test the spectrum of the gas present, (2) a U-tube which could be immersed in liquid air, (3) a mercury cut-off, (4) a bulb containing coconut-charcoal in a liquid-air bath, (5) a tube with glass tap leading to Gaede pump, McLeod gauge, &c. So far as we have been able to ascertain, the experimental tube was absolutely air-tight. Before beginning the observations the quartz tube was heated in the furnace, and maintained at a red heat for some time by means of a blowpipe; the filament AB glowed with the charcoal in liquid air, and the pump was running, so as to get all the occluded gases out of the apparatus. This procedure was found satisfactory; but it appeared that continued heating of the filament AB liberated traces of hydrogen for some time after the experiments were begun. After this evolution of gas had ceased it was found that on heating the tube LD lines due to hydrogen and carbon monoxide could be detected. The experiments to be described refer to conditions in which these contaminants were not present, at any rate in sufficient amount for spectroscopic detection, unless the contrary is specifically stated. For the construction of the quartz apparatus we are indebted to the skill of Mr. Reynolds of the Silica Syndicate, Ltd.

The relation between current and potential difference across the gap $AB \rightarrow C$ depends on the pressure and nature of the gas and on the temperature of the filament, which controls the maximum possible electron emission. It may possibly depend on the temperature of the gas as well. The effect of gradually raising the applied potential difference is shown

for a series of different filament temperatures by the curves in fig. 2. These refer to discharges in mercury vapour at pressures comparable with 0.01 mm. At a low filament temperature (1.0 amp. heating current) the current increases

Fig. 2.



more rapidly with the voltage than the first power at low voltages. At higher voltages the rate of increase falls off until finally saturation ensues (curve E, fig. 2). The current only behaves in this way when the temperature of the filament is low. The behaviour at higher temperatures is indicated by curves A, B, C, and D in fig. 2. These curves are all very much alike, and the behaviour below about 20 volts is very similar to that of curve E. With higher voltages it is, however, quite different from that shown by E. Thus with

a heating current of 1.05 amps. ($OAA_1A_2A_3$) when the driving potential reaches 22 volts (A) the current suddenly jumps from 130 microamps. to (A) 1100 microamps. At the same time the resistance of the gap falls; so that a potential difference of only 21 volts is required to maintain the larger current. On increasing the potential after this condition has been established, the current increases uniformly along the line A_1A_2 . Above 35 volts (A_2) another kind of instability appeared, the final reading (A_3) being considerably under that observed when the potential (45 volts) was first applied. This instability is probably due to the discharge wandering inside the tube, but we have not paid much attention to it, as it does not seem to be so interesting as that observed at the lower voltages. On lowering the potential from 45 volts the uniform part of the curve A_2A_1 was found to extend to the left beyond 21 volts; but when the applied potential fell to 17 volts (A_1) the current suddenly dropped on to the lower curve again, and the potential difference rose to 19 volts owing to the increased resistance of the gap $A \rightarrow C$.

The potential differences were tapped off a sliding rheostat in series with a battery of 200 volts with its middle-point earthed. The line ran to the negative end of the filament through a 5000 ohm resistance to prevent the development of an ordinary high-current arc. The potentials were measured by a voltmeter between the negative end of the filament and the earth, to which the anode C was also connected (through the microamperemeter).

The other curves B, C, D hardly call for individual comment. The chief point to note is that the potential difference at which the current jumps from one curve to the other drops uniformly to lower values as the filament temperature is raised and the number of electrons available becomes greater. This is true whether the jump is from the low- to the high-current curve, or *vice versa*.

The magnitude of the "kick" corresponding to the passage from O A to A_1A_2 and *vice versa*, as well as the potential differences under which these changes take place, depends a good deal on the pressure of the vapour as well as on the temperature of the filament. At very low pressures the kick, as measured by the ratio of the currents at A_1 and at A, is small, but sets in at relatively high potential differences; as the pressure increases the ratio of the currents increases to a maximum and subsequently falls off until at pressures comparable with a centimetre of mercury the kick is scarcely noticeable. The potentials at which the kicks take place appear to drop uniformly towards a limit as the vapour

pressure of the mercury rises. These statements are illustrated by the following numbers, in which the values of the vapour pressure of the mercury present are very rough guesses from the temperatures as indicated by a thermometer in contact with the bulb:—

Pressure of Hg Vapour.	Potential at A. volts.	Current at A. microamps.	Potential at A ₁ . volts.	Current at A ₁ . microamps.	Heating Current. amps.
0.0001 cm.	40	406	36	1188	1.05
0.001	26	120	21	792	1.04
0.001	23	198	13.5	3432	1.12
0.05	12.5	36	11.9	180	1.05
1.0	11.7	223	11.7	223	1.07

It will be seen from the numbers in the second and third rows that increasing the filament temperature tends to diminish the minimum potential necessary to maintain the currents on the high-level range. This is shown also by the following numbers:—

Vapour Pressure. cm. of Hg about	Minimum Voltage for A ₁ . volts.	Heating Current. amps.	Current across AB → C. microamps.
(A) < 0.001	18.5	1.05	1584
	14	1.09	2700
	13.5	1.10	2700
(B) > 0.001	15	1.05	1089
	13.2	1.09	2838
	13	1.10	3036
(C) < 0.05	12.1	1.05	192
	12.0	1.10	252
(D) > 0.05	12.0	1.05	210
	11.9	1.10	200
	11.7	1.07	230

The observations under A and C were obtained immediately after taking B and D respectively, and then opening the connexion to the charcoal tube. The pressures in A and C are therefore believed to be respectively somewhat less than those for B and D.

These and other observations indicate that increasing the pressure of the mercury vapour or the temperature of the filament does not lower the potential necessary to maintain

the discharge in the condition corresponding to A_1 indefinitely, but that a limiting minimum value of this potential difference is gradually reached. The lowest potential difference under which we have been able to maintain the discharge in the state A_1A_2 is 11.5 volts. The potential differences are measured from the negative end of the hot filament, so that this is the greatest potential difference between any part of the hot filament and the anode. As about one-tenth of the filament at each end is too cold to emit electrons in appreciable numbers, the greatest potential difference due to the field which is effective in driving the electrons across the gap would be some 0.3 volt less than this. On the other hand, the electrons are emitted with kinetic energy of thermal agitation in accordance with Maxwell's law. The allowance for this is somewhat indefinite, but would probably wipe out the 0.3 volt under consideration. To maintain the discharge in the state A_1A_2 therefore it appears to be necessary that the electrons should be able to acquire an amount of energy equal to that given by falling through a potential difference very close to 11.5 volts. This is nearly equal to the sum of the cathode and anode potential drops in the ordinary luminous high-current mercury arc. According to Aron's observations the former is 5.4 and the latter 7.4 volts.

The emission of light from the discharges under consideration will be dealt with below.

The Low Potential Discharge.

The currents with small potential differences (O A, fig. 2) exhibit characteristics similar to those shown in fig. 2 over a wide range of pressure of mercury vapour and of the temperature of the filament. The behaviour at a pressure of mercury vapour of the order of 1 mm. and with a heating current of 1.06 amp. in the filament is shown in greater detail by the following numbers:—

Volts	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Currents	}	0	1.3	1.8	4.5	6.6	8.8	10.8	13.8	15.3	18.0	21	24	28.3	33.8
(microamps).		0	1.3	1.8	4.5	6.6	8.8	10.8	13.8	15.3	18.0	21	24	28.3	33.8
Currents	}	0	0.8	2.1	3.9	6.0	8.8	10.9	13.8	16.9	20.1	23.6	27.2	31	34.9
calculated ...		0	0.8	2.1	3.9	6.0	8.8	10.9	13.8	16.9	20.1	23.6	27.2	31	34.9

The currents are roughly proportional to the potential difference raised to the power 1.5. This is shown by the numbers in the last row, which are calculated by assuming that the currents vary as $V^{3/2}$, and that the value at 5 volts is correct. Below 7 volts the agreement with this formula

is within the limits set by the errors of measurement, but between 7 volts and 13 volts the observed currents are uniformly below those given by the $V^{3/2}$ law. Above 14 volts the observed currents exceed the values calculated in this way.

It is known that electron currents between a hot and a cold electrode in a very good vacuum vary as $V^{3/2}$ for small values of V , and the phenomenon has been accounted for satisfactorily by Langmuir*. No doubt a similar explanation will apply to those of the present experiments in which the pressures were very low, but it will scarcely cover a case such as that of the numbers in the last table, since an essential feature of Langmuir's explanation lies in the assumption that the electrons move freely between the electrodes under the acceleration due to the electric field without interference or collisions with gas molecules. This condition will certainly be far from being satisfied with gas pressures of the order of a millimetre.

The following considerations show that with monatomic gases a similar law for low potential differences is to be expected even at high pressures. According to the researches of Franck and Hertz, if the potential difference is less than a critical value V_1 characteristic for each monatomic gas, the electrons undergo collisions with the gas molecules without loss of energy; so that if they are assumed to start from the hot electrode ($V=0$) with negligible velocity, their kinetic energy on reaching a point in the field where the potential is V ($V < V_1$) will be given by

$$\frac{1}{2}mv^2 = eV. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Owing to the presence of electrons in the space between the electrodes, V will have to satisfy Poisson's equation

$$\nabla^2 V = 4\pi\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The average velocity of the electrons in the direction of the electric intensity will be governed by the acceleration between successive collisions, so that the current density at any point will be given by the expression †

$$i = \frac{n}{2} \frac{e^2}{m} \frac{\lambda}{v} \text{grad } V = \frac{\rho}{2} \frac{e}{m} \frac{\lambda}{v} \frac{\partial V}{\partial x}, \quad . \quad . \quad . \quad (3)$$

where λ is the mean free path of the electrons in the gas.

* Phys. Review, vol. ii. p. 457 (1913).

† Cf. 'Electron Theory of Matter,' 2nd Ed. p. 410.

If we confine our attention at present to the discharge between two parallel plane electrodes, (2) reduces to

$$\frac{\partial^2 V}{\partial x^2} = 4\pi\rho. \quad (2')$$

Eliminating ρ and v from (1), (3), and (2'),

$$aV^{\frac{1}{2}} = \frac{\partial V}{\partial x} \frac{\partial^2 V}{\partial x^2}, \quad (4)$$

where

$$a = \frac{8\pi}{\lambda} \sqrt{\frac{2m}{e}} i. \quad (5)$$

A first integral of (4) is

$$2aV^{3/2} + C = \left(\frac{dV}{dx}\right)^2. \quad (6)$$

If the currents are a long way from saturation, $\frac{dV}{dx} = 0$, when V and $x = 0$; so that $C = 0$, and

$$(2a)^{\frac{1}{3}}x + C' = 2V^{\frac{1}{2}}. \quad (7)$$

Since $V = 0$ when $x = 0$, $C' = 0$, and if V_l ($V_l < V_1$) is the potential difference between the electrodes, whose distance apart is l ,

$$2a = \left(2 \frac{V_l^{\frac{1}{2}}}{l}\right)^3. \quad (8)$$

Thus

$$i = \sqrt{\frac{e}{2m}} \frac{\lambda}{2\pi l^3} V_l^{3/2}, \quad (9)$$

and the current varies as the potential difference raised to the power 1.5.

So far we have only considered parallel plane electrodes. Other cases lead to differential equations which, up to the present, have proved intractable. The relation $i \propto V^{3/2}$ can, however, be shown to be independent of the shape, size, and relative position of the electrodes by a general argument. In general, the relations between v , V , ρ , and the coordinates are governed by equations (1), (2), and (3). Consider any, the same, geometrical system under two different potential differences. Let v , V , and ρ be the variables at any given point in the one case, and v' , V' , and ρ' in the other. Then the dashed and undashed variables respectively will separately satisfy equations (1), (2), and (3). If possible let

$V' = nV$ at any point and $i' = n^s i$ at the same point, where n is constant and s is to be determined. From (1)

$$v = \left(\frac{2Ve}{m}\right)^{\frac{1}{2}} = \left(\frac{V'}{n}\right)^{\frac{1}{2}} \left(\frac{2e}{m}\right)^{\frac{1}{2}} \dots \dots \dots (10)$$

From (3)

$$n^{-s} i' = i = \frac{\rho}{2} \frac{e}{m} \lambda \left(\frac{n}{V'}\right)^{\frac{1}{2}} \left(\frac{m}{2e}\right)^{\frac{1}{2}} \frac{1}{n} \text{grad } V'$$

and

$$i' = \frac{\rho'}{2} \frac{e}{m} \lambda \left(\frac{1}{V'}\right)^{\frac{1}{2}} \left(\frac{m}{2e}\right)^{\frac{1}{2}} \text{grad } V'.$$

Thus

$$\rho' = n^{s-\frac{1}{2}} \rho. \dots \dots \dots (11)$$

But from (2)

$$4\pi\rho' = \nabla^2 V' = n \nabla^2 V = 4\pi n\rho,$$

and

$$\rho' = n\rho. \dots \dots \dots (12)$$

For (11) and (12) to be compatible

$$s = 3/2. \dots \dots \dots (13)$$

Corresponding to any solution $V = f(x, y, z)$ of the equations there will be a solution for which $V' = nV$, $\rho' = n\rho$, and $i' = n^{3/2}i$. If the total potential difference is changed then the current will vary as the potential difference raised to the power 1.5.

When the maximum potential difference V_i exceeds the critical value V_1 , the problem becomes more complicated. Since the electrons lose all their kinetic energy by collisions at the critical value eV_1 the equation (1), for values of V lying between V_1 and $2V_1$, will have to be replaced by

$$\frac{1}{2}mv^2 = e(V - V_1) = eU \dots \dots \dots (14)$$

In equations (2) and (3) the variable V can be replaced by the new variable U . Thus the differential equation for the space between the critical equipotential surface and the anode is the same as before, the only change being that the potential differences are now measured from the critical value. From (2), (3), and (14),

$$2aU^{3/2} + C = \left(\frac{\partial U}{\partial x}\right)^2 \dots \dots \dots (15)$$

The further treatment of the problem depends very much on what one supposes to take place at the critical impacts. The work of Franck and Hertz has shown that at this stage the electrons lose all their energy, and that some positive

ions are apparently liberated. According to Bohr's theory the last named effect is either illusory or is a secondary phenomenon. For the present we shall assume for simplicity that the only effect influencing the motion of the electrons at the critical equipotential surface is the loss of kinetic energy. We propose to defer the consideration of the more complicated case, in which positive ions are liberated, until later, when we hope to be able to submit further experimental results bearing on the question at issue.

Turning to equation (3), and keeping to the case of parallel planes, since i is the same everywhere, we see that $\frac{1}{v} \frac{dV}{dx} \frac{d^2V}{dx^2}$ must have the same value on each side of the critical equipotential surface. Since v vanishes at the critical surface, either $\frac{dV}{dx}$ or $\frac{d^2V}{dx^2}$ must also vanish. But $\frac{dV}{dx}$ is not zero for $V < V_1$, and if $\frac{dV}{dx}$ were not continuous, there would be a finite charge on a surface in the gas, and this would not be in equilibrium under the forces to which it is subjected. Thus $\frac{dV}{dx}$ must be continuous at the critical surface and $\frac{d^2V}{dx^2}$ must vanish for $V = V_1$. Considering the solution for $V < V_1$, we see from (6) that the value of $\frac{dV}{dx}$ at the critical surface (since $C=0$ in this region) is given by

$$\left(\frac{dV}{dx}\right)_{V=V_1} = (2a)^{\frac{1}{3}} V_1^{\frac{1}{3}} \dots \dots \dots (16)$$

Since U vanishes at this surface C in (15) is equal to $2aV_1^{3/2}$, and

$$(2a)^{\frac{1}{3}} dx = \frac{dU}{(U^{3/2} + V_1^{3/2})^{\frac{1}{3}}} \dots \dots \dots (17)$$

Thus

$$(2a)^{\frac{1}{3}} x + C' = V_1^{\frac{1}{3}} F\left(\frac{U}{V_1}\right) = V_1^{\frac{1}{3}} F(z), \dots \dots (18)$$

where

$$F(z) = z - \frac{2}{15} z^{5/2} + \frac{1}{18} z^4 - \frac{28}{891} z^{11/2} + \dots \dots \dots + (-1)^n \frac{1.4.7 \dots (3n-2)}{3^n \lfloor n} \frac{2}{3n+2} z^{\frac{3n+2}{2}} + \dots \dots (19)$$

Since U and z vanish when $V = V_1$ and $x = x_1$, we have

$$C' = -(2a)^{\frac{1}{2}} x_1 = -2V_1^{\frac{1}{2}}, \quad . \quad . \quad . \quad (20)$$

from the solution for the space between the cathode and the critical surface. Thus if the distance between the cathode and the anode is l and the potential difference between them V_l ,

$$(2a)^{\frac{1}{2}} = \frac{V_1^{\frac{1}{2}}}{l} \left\{ 2 + F\left(\frac{V_l - V_1}{V_1}\right) \right\}$$

and

$$i = \frac{\lambda}{16\pi} \left(\frac{e}{2m}\right)^{\frac{1}{2}} \frac{V_1^{3/2}}{l^3} \left\{ 1 + \frac{V_l}{V_1} - \frac{2}{15} \left(\frac{V_l}{V_1} - 1\right)^{5/2} + \frac{1}{18} \left(\frac{V_l}{V_1} - 1\right)^4 - \frac{28}{891} \left(\frac{V_l}{V_1} - 1\right)^{11/2} + \dots \right. \\ \left. + \dots + (-1)^n \frac{1.4.7 \dots (3n-2)}{3^n n} \frac{2}{3n+2} \left(\frac{V_l}{V_1} - 1\right)^{\frac{3n+2}{2}} + \dots \right\}^3 \quad (21)$$

This formula makes the currents for values of V between V_1 and $2V_1$ increase with V less rapidly than the requirements of the $V^{3/2}$ formula which holds for $V < V_1$. The calculated falling off is indicated roughly by the following numbers:—

$V_l/V_1 \rightarrow$	1.25	1.5	1.8	2.0
$i/i_1' \rightarrow$.73	.57	.4	.4

The values of i/i_1' are the currents calculated by (21) divided by the corresponding currents calculated on the assumption that the $V^{3/2}$ law which holds for $V < V_1$ is valid also for $V > V_1$. The calculated falling off is much greater than that shown by the results on p. 431. This might be expected, as the theoretical conditions are very imperfectly realized in the experiments. Apart from the quite different system of electrodes used, there was a drop of potential down the cathode due to the heating current of about 2.5 volts. This makes the effective critical potential about 1.5 volts too high. The sharpness of the effects arising from the loss of energy at the critical value also tends to be obliterated, owing to the occurrence of the initial distribution of velocity which will prevent the average kinetic energy from actually vanishing at any point. The larger observed currents might also be attributed to the liberation of positive ions at the critical collisions; since we should expect any considerable development of positive ions to permit a much closer approach to saturation, owing to the consequent reduction of ρ in equation (2).

Emission of Radiation.

At low pressures of the order of 0.001 mm. when the kick from O A to A_1A_2 (fig. 2) does not set in until potential differences exceeding 30 volts have been applied, we have not been able to detect any evidence of the presence of a visible discharge either on the low range O A or on the high range A_1A_2 . The tests were carried out by visual observations, both directly and through blue glass, with appropriate diaphragms to eliminate the glare from the filament, and by examination of the tube with a large direct-vision spectroscope by Hofmann. It may be that there is a faint luminous discharge which is swamped by the diffused light coming from the hot filament; but, if so, its intensity must be very low. Under these conditions there is, however, an emission of ultra-violet radiation, whose presence can be detected by its effect on the electrode E (fig. 2). This is shown by the following examples:—

(a) At p = about 0.001 mm. and a heating current of 1.05 amps. through the filament the thermionic current at 40 volts was 429 microamps. At this potential (corresponding to A, fig. 2) the current suddenly jumped to 1056 microamps. at 39.5 volts on the higher range. The point corresponding to A_1 was located at 36 volts. No effect on the electrometer connected to E could be detected on the low-current range corresponding to O A below 30 volts, but at 36 volts 5 divisions in 30 seconds were obtained. On the range corresponding to A_1A_2 a deflexion of 50 divs. in 30 secs. at 40 volts was observed. Evidently a large increase in the ultra-violet emission sets in with the kick on to the higher range.

(b) At a pressure similar to that in the experiment just described, and with a heating current of 1.07 amps., the current was found to jump up at 39 volts and down at 31 volts. At 39 volts the current was about 280 microamps., and it jumped to 1848 microamps. at 36 volts. The following observations up to 30 volts were taken on the low-current range, and that at 40 volts on the high-current range:—

Volts across gap A B \rightarrow C (fig. 1)...	15	20	25	30	40
Electrometer deflexions in 30 secs....	0	0	5	13	200

In this case also there is a big increase in the ultra-violet emission when the high-current discharge sets in, but there was a distinct effect which set in between 20 and 25 volts on the low-current range. This effect may have been present in experiment (a): but under the conditions of that experiment it would have been too small to detect with any certainty. It is difficult to be sure that these small effects

are not due to the presence of traces of contaminants such as hydrogen, to which the spectroscopic test is not very sensitive at these low pressures. No hydrogen lines could be detected in the mercury spectrum under the conditions of experiment (b).

At pressures greater than about 0.01 mm. the jump to the higher range was always accompanied by the establishment of a visible glow discharge. The spectrum of the glow is that of the mercury arc. The behaviour of the luminosity when the potential across the gap is changed presents very interesting features. We have not been able to observe any luminosity except that due to the light coming from the hot filament when the discharge is on the low-current range. On the high-current range at the lowest potential which would maintain the discharge a faint glow appeared close to the surface of the anode. On raising the potential slightly the glow grew out from the anode into the surrounding space. A further slight increase of the potential difference, in some of the experiments at least, caused the glow to spring away from the anode. In this condition there was a bright patch in the space between the electrodes with a dark space round both the anode and the cathode. A further small increase of potential difference made the glow settle round the cathode. After this stage was reached the luminosity was greatest near the cathode. Further increase of potential difference caused the luminosity to spread out further from the cathode and to increase in intensity. In this condition there was a gradual falling off in intensity towards the anode, and the luminosity did not appear to have a sharp boundary.

At high pressures the changes in the potential difference which are necessary to cause the glow to move over from the anode to the cathode are quite small, being comparable with 1 volt. Observations similar to the following were frequently made with pressures of mercury vapour of the order of 1 cm. With 0.98 amp. heating current and the quartz tube at 225° C. the following observations of the luminosity and discharge current at different potentials were taken:—

Volts.....	11	12	14	25	50
Luminosity	None.	On anode.	Crossed to cathode.	Filled whole space.	Filled whole space.
Current (microamps.)	16	84	184	315

With a larger heating current (1.05 amps.), but under conditions otherwise similar to the foregoing, the following observations were taken:—

Volts.....	11	12	13.2	25	50
Luminosity	None.	On anode.	Crossed to cathode.	Filled whole space.	Filled whole space.
Current (microamps.)	20	124	544	884

The smallest potential difference which we have recorded between the stage when the glow starts on the anode and that at which it crosses to the cathode is 0.5 volt, the largest about 6 volts. These observations were made with a high pressure of mercury vapour comparable with 1 cm. The smallest potential differences were required with the smallest filament temperatures and correspondingly small discharge currents, and *vice versa*. In this respect the behaviour of the glow is similar to that of the positive column in the ordinary low-pressure discharge-tube, which is gradually forced back on to the anode as the strength of the discharge current is increased.

We have examined the glow carefully with the spectro-scope in the initial stages to see if there is any difference in the potentials necessary to excite the different lines of the mercury spectrum. We have not been able to detect any such effect, either when the glow just starts or when it is made to die out. The strong lines of wave-lengths 4358 and 5460 have been examined with especial care in this respect. So far as we are able to observe, the potentials required to excite these two lines are identical, certainly to within one-tenth of a volt. The equivalent voltage difference of the two lines by the quantum relation is about eight-tenths of a volt. This indicates that the lines of the mercury arc spectrum are not excited separately by single electron impacts, one for each line, but suggests that they are the result of a complex change set up in the mercury atom, and affecting a large number of electrons simultaneously.

The value 11.5 volts of the smallest potential difference which will start the glow is less than the value 12.5 volts given by McLennan and Henderson* for this quantity. The precise interpretation of this voltage is a difficult matter without a knowledge of the distribution of potential along the discharge. If it is interpreted as the potential difference equivalent to the energy necessary for an electron to disrupt the mercury atom or the mercury ion in a definite way, we have to face the difficulty that, according to the experiments of Franck and Hertz, it is impossible for an electron travelling through mercury vapour to acquire kinetic energy in excess of that equivalent to 4.9 volts. This might be overcome if there is a sharp drop of potential sufficiently close to the cathode to carry a considerable number of electrons through the critical region without atomic impact, on the further assumption that the collisions become elastic again after passing the critical velocity.

* Roy. Soc. Proc. A. vol. xci. p. 485 (1915).

On testing with the electrode E (fig. 1) a vigorous emission of ultra-violet light was found to set in simultaneously with the appearance of the glow. With pressures sufficiently high to give a visible discharge under the comparatively low potential differences, we have not been able to detect any ultra-violet emission at potentials under that at which the glow sets in. In fact, we have never been able to discover any effect on the nickel electrode E with potentials across $AB \rightarrow C$ of less than 11.5 volts. Presumably this is due to the absorption of the single line 2536 of the low-voltage spectrum by the mercury vapour in the connecting tube; but it may be that the frequency of this line is not high enough to excite the photoelectric emission from the nickel electrode used. We have not tested this point.

With high pressures there is no discontinuity in the current when the luminosity sets in, such as is observed at lower pressures. This is indicated by the numbers in the last row of the table on p. 430.

A number of observations giving results of interest were taken in the early stages of the experiment before the tungsten filament had been thoroughly glowed out. Under these conditions the discharge in the Geissler tube showed the spectrum of hydrogen quite strong, but no other lines except those due to mercury. It was then found that with a low pressure of mercury vapour a considerable emission of ultra-violet light set in at a potential difference considerably under that necessary later, when the hydrogen had been got rid of. This effect set in quite consistently between 16 and 17 volts, and at first increased rapidly with rising potential difference. The rate of increase then fell off until, finally, the effect varied little, if at all, with the potential difference. This is illustrated by the following numbers taken with a mercury pressure of about 0.001 mm. and an uncertain amount of hydrogen:—

Volts across $AB \rightarrow C$ 0 5 10 15 16 17 20 25 30 35 40 55 65

Microamps across $AB \rightarrow C$. 0 13 31 45 52 60 68 70 70 71 ...

Photoelectric current from E (scale divs. in 30 sec.).	}	0	0	0	0	0	5	11	25	33	50	58	75	76

It will be observed that as the potential rises the discharge current becomes constant sooner than the photoelectric emission. The thermionic currents were too small for the kick to develop under the conditions of these observations.

Wheatstone Laboratory,
King's College.



Fig. 1.



Fig. 2.

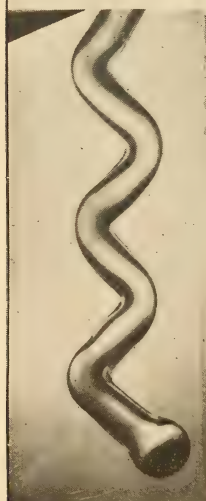


Fig. 3

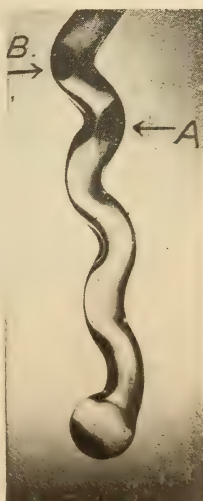
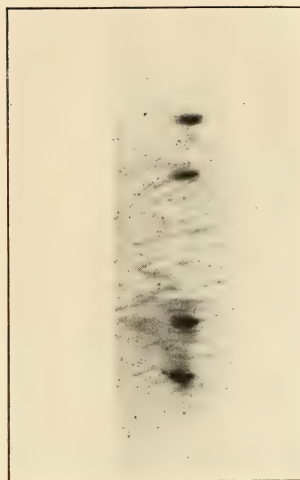


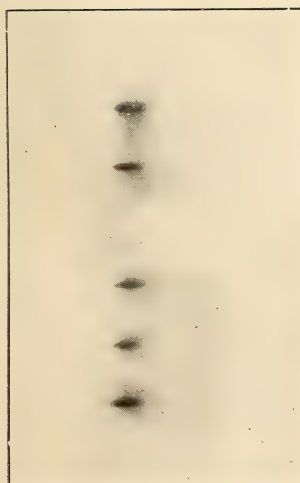
Fig. 4.



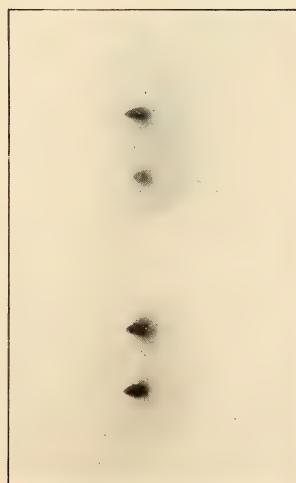
Photographic Plate 30.



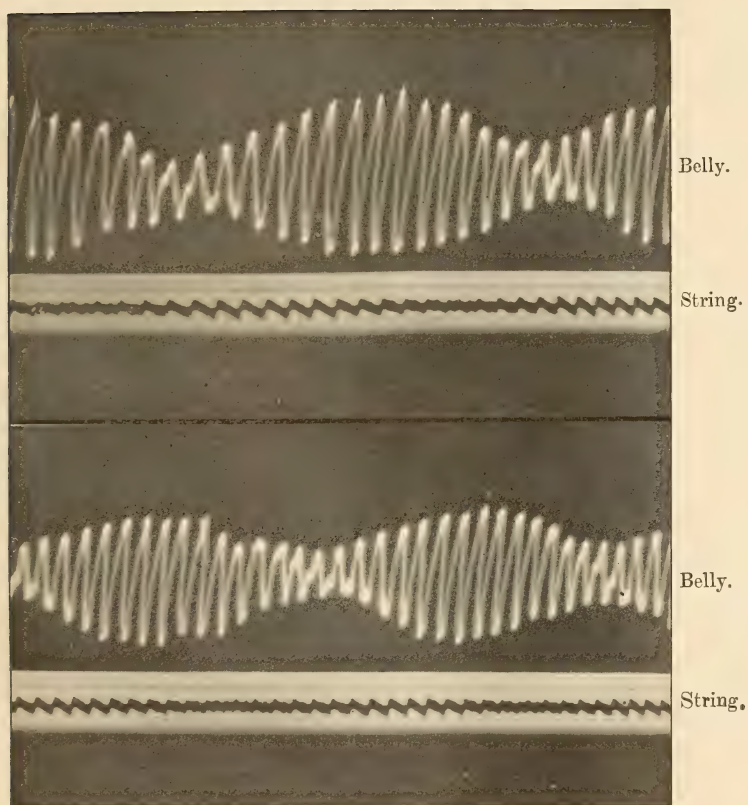
Photographic Plate 33.



Photographic Plate 26.



Photographic Plate 28.



TIME AXIS —————→

Simultaneous Vibration-Curves of Belly and String of Violoncello
at the "Wolf-note" pitch.



FIG. 1.

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AND
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[SIXTH SERIES.]

NOVEMBER 1916.

XLVIII. *The Mobility of the Negative Ion.*
By S. RATNER, of the University of Petrograd*.

IT is well known that in general the mobility (k) of a gaseous ion varies inversely as the pressure (p), so that the product kp is constant for a given gas, but at low pressures the mobility of the negative ion becomes abnormally great, and kp increases rapidly with diminution of pressure. The phenomenon of the abnormal mobility of the negative ion has been studied by a large number of experimenters. Langevin †, Kovarik ‡, Lattey §, and others could observe the abnormal increase of the mobility only in the case when the pressure was reduced below a certain value (75–200 mm.) and therefore assumed the existence of a “critical” pressure at which the abnormality sets in. Frank ||, and later Haines ¶, have shown that in carefully purified nitrogen, argon, helium, and hydrogen, the mobility of the negative ion is abnormally great, even at atmospheric pressure. Kovarik and Lattey (*loc. cit.*) drew attention to the fact that at low pressure the mobility of the negative ion varies with the electric force. J. S. Townsend** has shown that the mobility of the negative ion may in general be expressed as a function

* Communicated by Sir J. J. Thomson, O.M., F.R.S.

† P. Langevin, *Annales de Ch. et de Physique*, xxviii. (1903).

‡ Kovarik, ‘The Physical Review,’ xxx. p. 415 (1910).

§ R. T. Lattey, *Proc. Roy. Soc. A*, lxxxiv. p. 173 (1910).

|| Frank, *Verh. d. D. Phys. Ges.* xii. pp. 221 & 613 (1910).

¶ Phil. Mag. xxx. p. 503 (1915).

** J. S. Townsend, ‘Electricity in Gases.’ Oxford, 1915.

of $\frac{X}{p}$, where X is the electric force and p is the pressure, and that for large values of $\frac{X}{p}$ the mobility begins to diminish with increase of the force. Experiments were carried out in order to determine whether the mobility of the positive ion also departs from the inverse pressure law. Todd * has found that at pressures below 1 mm. the mobility of the positive ion also becomes abnormally great; a comparatively small increase of the mobility of the positive ion was observed also by J. S. Townsend †, at higher pressures, but these results are contradicted by other experimenters.

The study of the mobility of ions leads us to results which may throw considerable light upon the nature of an ion and should be thoroughly carried out in all possible directions. Unfortunately the methods usually employed are but little adapted for this purpose when the mobility becomes abnormally great. Rutherford's, Langevin's, and Zeleny's methods and their modifications serve only in the case when the velocity of the ion does not exceed the order of $10^3 \frac{\text{cm.}}{\text{sec.}}$

and therefore can hardly be used for a complete study of the mobility of the negative ion. Chattock's method involves other difficulties, as in this case the mobility has to be investigated under very unfavourable conditions, *i. e.* when the electric force acting on the ion is very strong and not uniform. The only suitable method for measuring large ionic velocities is that given by J. S. Townsend, depending on the action of a magnetic force on the motion of a stream of ions.

In the present paper a new method of measuring ionic mobilities is described and some results are given.

Method and Apparatus.

The method may be considered as a modification of that given by Chattock ‡, and is based on the production of a wind in an ionized gas when acted on by an electric field. In the case of the discharge from a point to a plate, Chattock deduced the following expression for the mobility (k) of an ion in terms of the discharge current (c), the corresponding wind-pressure (P), and the distance between the plates (d):

$$k = d \frac{c}{P}.$$

* G. Todd, *Phil. Mag.* xxii. p. 791 (1911).

† J. S. Townsend, *Phil. Mag.* July, 1914.

‡ Chattock, *Phil. Mag.* (5) xlviii. p. 401 (1899).

In a previous note the author * has shown that if two parallel plates are immersed in an ionized gas, a small potential difference established between the plates produces a wind-pressure large enough to be measured by a special gauge †, which will be described later on. When a surface ionization of one sign close to one of the plates is produced, the density of electrification, as well as the strength of the field, is uniform throughout the volume between the plates (if the disturbing effect of the ions is neglected), and in this case Chattock's formula may be deduced in a very elementary way. Let P be the total surface electrification on the plate, X the strength of the field, w the velocity of the ions, c the total ionization current, P the corresponding wind-pressure, and d the distance between the plates. Then Pw is the quantity of electricity streaming through a plane parallel to the plates in unit time and $c = Pw$. When the ions move with a uniform velocity their drag on the gas is equal to the product of the total electrification between the plates and the strength of the field ‡. Hence $P = d\rho X$.

These two equations give

$$\frac{w}{X} = d \frac{c}{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\frac{w}{X}$ is the mobility of an ion. In the case when the velocity w of an ion is not proportional to the electric force X , $\frac{w}{X}$ is what is called the abnormal mobility of an ion.

The experiments may be carried out at any pressure, the potential difference between the plates varying over a very wide range, viz. from zero up to that required for a spark-discharge between parallel-plate electrodes, and, as will be shown later, by this method ionic velocities may be measured of the order of from 10 to $10^7 \frac{\text{cm.}}{\text{sec.}}$.

The apparatus used is shown in figs. 1, 2, and 3. Two brass plates A and B (fig. 1) are mounted on a wooden base ST. The circular plate A, which consists of two connected metal sheets 8 cm. in diam., is supported by a rod f sliding in a brass collar e and is insulated by an

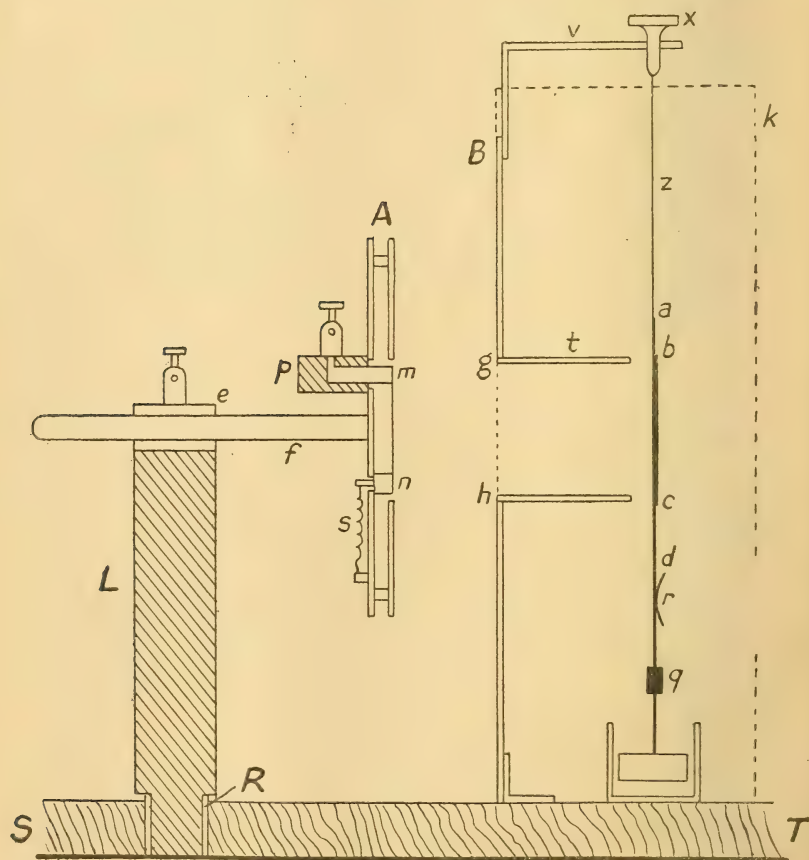
* *Comptes Rendus*, clviii. p. 565 (1914).

† I wish to thank Prof. Debierne for suggesting to me the idea of this gauge.

‡ This is experimentally proved by the fact, that using the above formula Chattock deduced for the mobility of an ion the same value as given by other methods.

ebonite column *L* fixed in an earthed guard-ring *R*. The plate is cut out in the middle in the way shown in figs. 1 and 3, and in this gap a strip of thin platinum foil *mn* is stretched in the plane of the plate. One of the extremities of the strip is insulated from the plate by the ebonite plug *p*,

Fig. 1.



while the other extremity is brought into contact with the plate and is fixed on a sliding metal support *n* pulled by a weak spring *s*. The strip can be heated to any desired temperature by passing a current through it from an insulated battery of accumulators kept at the same potential as the plate *A*. The plate *B*, 8 cm. wide and 14 cm. high, has

cut out of its centre a circular hole 3 cm. in diam. opposite to the strip of platinum and covered with metal gauze *gh*. The gauge employed is shown in figs. 1 and 2. A vane *abcd* is cut out in the form shown in fig. 2 from a sheet of aluminium 0.1 mm. thick, and is suspended by a fine brass ribbon *z* attached to a torsion head *X* which may slide on a rod *v* projecting from the plate *B*. On its lower part the

Fig. 2.

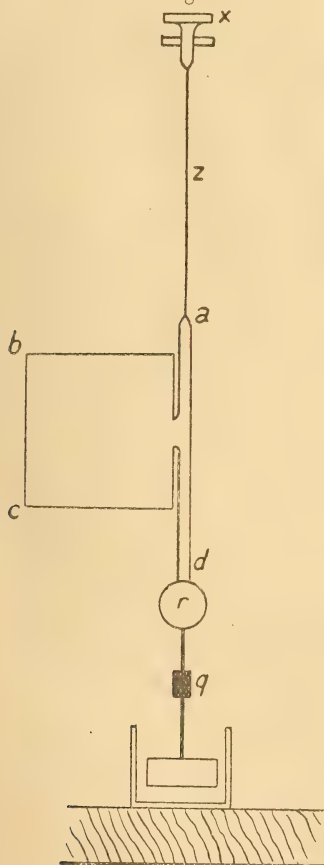
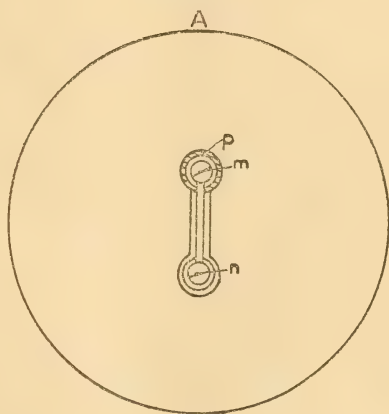


Fig. 3.



vane is provided with a mirror *r*, a small weight *q*, and an arrangement for damping. The earthed cage *k* forms an electrostatic protection for the vane. The apparatus is placed on a brass plate pierced by insulated electrodes and is covered with a bell-jar provided with a window of plane glass, so that the image formed by the mirror may be undistorted. Before being admitted into the bell-jar the

air passes through calcium chloride, concentrated sulphuric acid, and glass-wool. The plate A may be earthed or charged to a desired potential by means of a battery of small cells, one terminal of which is earthed, while the plate B is connected to a galvanometer provided with a series of shunts, by means of which any current between 10^{-9} and 10^{-5} ampere could be measured. The strip of platinum, coated with a mixture of barium oxide and aluminium phosphate, emits when heated a copious supply of negative or positive ions, according to the direction of the electric field between the plates. The motion of these ions, as shown above, produces between the plates a stream of gas which passes by inertia through the grating gh and a small tube t , and imparts to the vane a deviation measured in the ordinary way by means of the mirror r .

When the platinum strip is heated, no electric force being applied between the plates, a convection current due to the heating of the strip is produced in the gas, and the image from the mirror changes its zero position on the scale. For each temperature of the strip a correction of the zero-point of the apparatus is therefore necessary, and precautions must be taken in order to diminish, as far as possible, these displacements of the zero. The suspension of the vane should not be too sensitive, the platinum strip has to be very narrow and placed in a vertical line, as shown in the figure, and the vane should not be placed close to the edge of the tube t , but about 5 millimetres from it. The ionization current, as well as the corresponding wind-pressure, may be varied over a wide range by changing the temperature of the strip, or the electric force between the plates.

It is seen from equation (1) that for a given distance d between the plates the mobility of an ion $\frac{w}{X}$ is proportional to $\frac{c}{P}$, the ratio between the ionization current and the corresponding wind-pressure. A great number of preliminary experiments were carried out in order to verify the above equation. As at high pressures the mobility of an ion is known to be constant over a wide range of the strength of the field, the ratio $\frac{c}{P}$, according to equation (1), must also be constant under the same conditions. Further, supposing $\frac{c}{P} = k_{(-)}$ for negative ions and $\frac{c}{P} = k_{(+)}$ for positive, $\frac{k_{(-)}}{k_{(+)}}$ must be the ratio between the mobilities of the negative and the positive ion.

A sample of the results of these experiments is shown in Table I., where p is the pressure of air under the bell-jar, v the potential in volts applied to the plate A, and c and P are given in arbitrary units.

TABLE I.

$p=600 \text{ mm.}, d=20 \text{ mm.}$							
v negative.	c .	P .	$\frac{c}{P}=k_{(-)}$.	v positive.	c .	P .	$\frac{c}{P}=k_{(+)}$.
280	18	11	1.64	320	26	22	1.18
400	45	27	1.66	400	34	28	1.22
„	20	12	1.66	600	75	65	1.16
600	66	41	1.61	800	71	60	1.18
1000	102	63	1.62	1000	66	56	1.18
„	28	17	1.65	„	31	26	1.19
Mean value of $k_{(-)}$ 1.64				Mean value of $k_{(+)}$ 1.19			

$$\frac{k_{(-)}}{k_{(+)}} = \frac{1.64}{1.19} = 1.38.$$

It is seen from the Table that $\frac{c}{P}$ is constant within the limits of experimental error and that the value of $\frac{k_{(-)}}{k_{(+)}} = 1.38$ is in good agreement with the well-known ratio between the mobilities of the negative and the positive ion in air. Experiments of this kind, together with some other control experiments, which will be described later, prove the efficiency of the method and show that the results are reliable.

It is easy to see that the wind-pressure produced by the stream of gas can be measured by the gauge only in arbitrary units and, consequently, the absolute value of the mobility cannot be deduced from equation (1). This, however, is not necessary for the purpose of this work, as in the results given below the abnormal mobility of the negative ion is expressed in terms of its normal mobility—a constant determined to a high degree of accuracy by other methods.

Results.

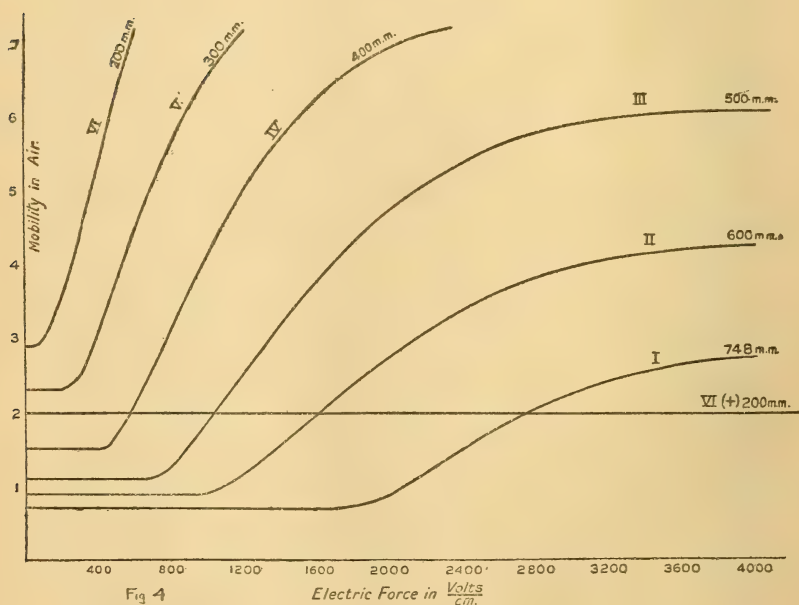
1. The mobility of an ion $\frac{w}{X}$, as measured in arbitrary units by the ratio $\frac{c}{P}$, is constant over a certain range of the

applied electric force X , which is in agreement with the well-known experimental facts. But in the case of negative ionization, when X is gradually increased the experiments show that at a certain value of $X = X_1$ depending upon the pressure of the gas a lack of proportionality between c and P suddenly comes in in such a way that $\frac{c}{P}$ begins to increase.

Equation (1) shows that in this case the velocity of an ion increases more rapidly than the electric force, *i. e.*, that its mobility becomes abnormally great. The value of this critical electric force, X_1 , decreases rapidly with diminution of the pressure, so that at pressures below 200 mm. the same effect is reached with comparatively small forces when the ionic velocities are small enough to be measured by Langevin's or Rutherford's method.

In the curves I., II., III., IV., V., and VI. (fig. 4), the

Fig. 4.



mobilities $\frac{w}{X}$ of the negative ion at pressures of 748, 600, 500, 400, 300, and 200 mm. respectively, are plotted against the electric force X , and the curve VI₍₊₎, which is a straight line parallel to the axis of X , shows the mobility of the

positive ion at 200 mm. pressure. The mobilities are given in arbitrary units, the mobility of the negative ion at atmospheric pressure being equal to 0.72 in these units. The Table II., corresponding to curve I., is given to show the procedure of the experiments.

TABLE II.

$p=748 \text{ mm.}, d=10 \text{ mm.}$							
$v.$	$c.$	P.	$\frac{c}{P}$	$v.$	$c.$	P.	$\frac{c}{P}$
80	10	14	.71	2000	52	60	.88
200	32	44	.73	2120	24	21	1.14
600	46	63	.73	2200	48	39	1.23
1000	38	53	.72	2400	71	47	1.51
1400	51	69	.74	2600	65	34	1.91
1600	26	36	.72	3000	89	40	2.23
1720	31	43	.73	3400	76	30	2.53
1800	35	46	.76	4000	92	34	2.77
1880	43	54	.80				

The curves show that when the critical value of the electric force is reached the mobility of the negative ion begins to increase, at first rapidly. At atmospheric pressure the mobility is doubled when the electric force is increased from

1800 to 2400 $\frac{\text{volt}}{\text{cm.}}$; at 400 mm. pressure the mobilities at 450 and 2000 $\frac{\text{volt}}{\text{cm.}}$ are as 1 to 5. With larger forces

the mobility increases less rapidly, and it may be noticed from the curves that it apparently tends to reach a constant value. The curves show also that at high pressures the velocity of

the negative ion cannot be expressed as a function of $\frac{X}{p}$.

The determination of the critical electric force X_1 at a given pressure involves great difficulties. At high pressures the result of experiments carried out under the same conditions varies from day to day to the extent of 20 per cent. The amount of moisture present in air does not apparently affect the results, although in these experiments the air was never dried with special care. The effect may be due to

some other impurities in the air, and this supposition is supported by the fact that, as shown below, even imperceptible traces of chloroform vapour change completely the aspect of the mobility curves. At lower pressures, when the critical force \bar{X}_1 is small, another difficulty arises, as the ionization-current and the corresponding wind-pressure decrease with the potential difference between the plates and become, in this case, too small to be measured with accuracy.

In the Table III. the values of \bar{X}_1 and $\frac{\bar{X}_1}{p}$ at different pressures are given, \bar{X}_1 being the mean approximate value from a great number of experiments.

TABLE III.

p .	\bar{X}_1 .	$\frac{\bar{X}_1}{p}$.
750	1800	2.4
600	1000	1.7
500	700	1.4
400	450	1.1
300	250	.8
200	100	.5
150	60	.4
100	30	.3
75	20	.26

It is seen from the Table that $\frac{\bar{X}_1}{p}$ decreases rapidly with diminution of pressure, so that at high pressures the negative ion may attain comparatively large velocities before it begins to assume an electronic state, and that is the reason why the abnormal mobility could not be observed at high pressures by other methods. It is, however, worth noticing that even at atmospheric pressure the largest velocity attained by the normal negative ion, viz. $3200 \frac{\text{cm.}}{\text{sec.}}$, is small compared with its velocity of thermal agitation.

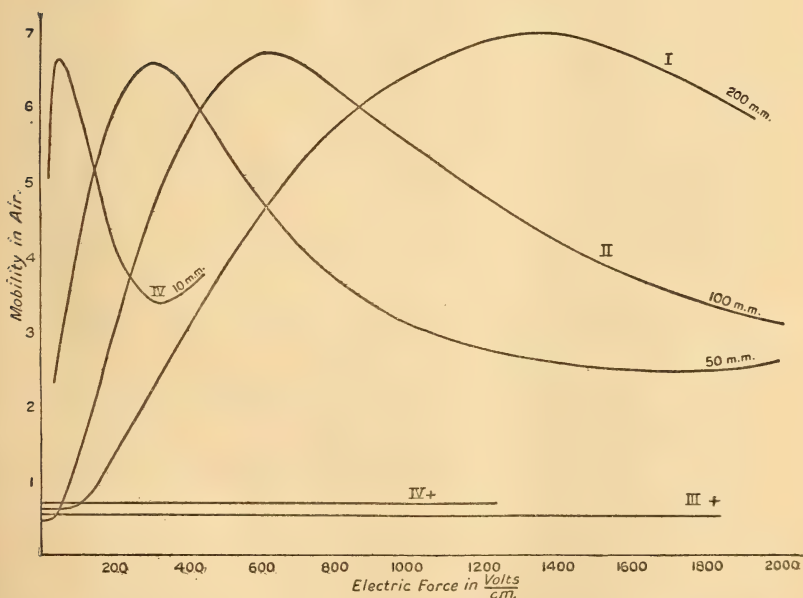
2. Interesting mobility curves are obtained with larger values of $\frac{\bar{X}}{p}$. It has been already noticed from the curves (fig. 4) that at large electric forces the mobility tends to remain constant. When the force is still further increased

the mobility of the negative ion attains a maximum value and begins to diminish. This phenomenon could be predicted from Langevin's formula for the mobility of a charged particle whose mass is small compared with that of a mole-

cule: $\frac{w}{X} = \frac{e}{m} \cdot \frac{l}{u}$, where e and m are the charge and the

mass of the particle, l its free path, and u its velocity of agitation. The mass associated with the negative ion gradually diminishes with increase of electric force, and at this stage of evolution of an ion its mobility increases with the force. When, finally, the electronic state is reached by the ion, its mobility follows the above equation, and since the velocity of agitation of an electron increases with the electric force*, its mobility begins to decrease†.

Fig. 5.



This decrease in the mobility of the negative ion is shown in curves I., II., III., and IV. (fig. 5), for pressures of 200, 100, 50, and 10 mm. respectively. The curves III.₍₊₎ and IV.₍₊₎ show the mobility of the positive ions at 50 and 10 mm.

* J. S. Townsend and H. T. Tizard, Proc. Roy. Soc. A, lxxxviii. p 336 (1913).

† A full discussion of this question will be found in J. S. Townsend's book, 'Electricity in Gases.' Oxford, 1915.

pressure. In this figure the mobilities at different pressures are given in different units in order to avoid using too small a scale. In the curves I. and II. the mobility at any point may be given in terms of the normal mobility represented as straight lines in the beginning of the curves, while in the curves III. and IV. it may be calculated from the mobility of the positive ion at the same pressures. The curves show that the velocity of the ions always increases with increasing forces, even in the case when their mobility diminishes.

By their maximum points the curves are divided into two parts, the first part corresponding to the gradual dissociation of the ion and the second to the pure electronic state of an ion. The value of $\frac{X}{p}$ corresponding to the maximum

mobility is constant at pressures up to 200 mm. and is equal to 6.2, which is very large compared with 0.2—the value given by J. S. Townsend in case of carefully dried air.

The rate of decrease of the mobility, as seen from the curves, diminishes with the increase of the force, and at sufficiently large values of $\frac{X}{p}$ the mobility tends to reach a

minimum value. At low pressures this bend in the curve is well marked, as shown by curve IV. Experiments with still larger values of $\frac{X}{p}$ are made impossible by the luminous

discharge which takes place under these conditions between the heated strip and the grating. It is possible that the luminous discharge is preceded by a feeble ionization by collision, which might be responsible for this bend of the curves. Further experiments in this direction are now in progress.

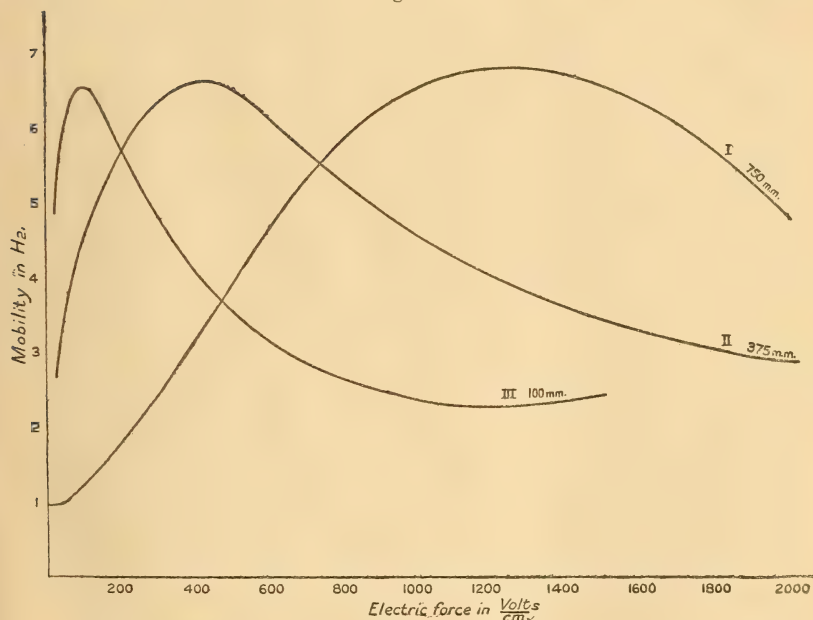
With regard to positive ions, the experiments show that at pressures down to 5 mm. and with very large forces the mobility remains constant.

3. *Experiments in hydrogen.*—The same experiments were carried out in hydrogen—a gas in which the negative ion is known to exist in the electronic state under ordinary conditions. The hydrogen was generated in a Kipp apparatus from hydrochloric acid and zinc and purified by passing slowly through NaOH, KMnO_4 , and H_2SO_4 . It was always contaminated with small quantities of impurities, since the brass-plate and the bell-jar enclosing the apparatus contained a great number of different joints which allowed a constant small leakage of air. The mobility curves in hydrogen show the same characteristic features as in the case of air. The

critical value X_1 of the electric force, at which the mobility of the negative ion becomes abnormal, is very small in hydrogen and increases with the amount of impurities to such an extent that it may serve as a good criterion for testing the purity of hydrogen.

The curves I., II., and III. (fig. 6) represent the mobility

Fig. 6.



of the negative ion at pressures 750, 375, and 100 mm. respectively, in the purest hydrogen obtained for these experiments. The critical value X_1 of the electric force is

about $40 \frac{\text{volt}}{\text{cm.}}$ at atmospheric pressure, and at lower pressures

it becomes too small to be measured by this method. The

value of $\frac{X}{p}$ corresponding to the maximum point of the

mobility curves is very large, varying from 1.8 at atmospheric pressure to 1.1 at 100 mm., and with further increase of the electric force the mobility tends, as in air, to attain a

minimum value. These high values of $\frac{X}{p}$ are probably due

to the impurities contained in the hydrogen. Experiments

in purer hydrogen and nitrogen, as well as in carefully dried air, are now in progress.

4. *Experiments in heavy gases.*—Experiments were also made in order to test the mobility of the negative ion in heavy gases, such as the vapours of chloroform, carbon tetrachloride, and methyl iodide*. The mobility in these gases was found to be constant for the negative as well as for the positive ion, the mobility curves being straight lines parallel to the X axis even at low pressures (down to 5 mm.) and with large electric forces.

A striking effect illustrating the influence of impurities on the mobility of the negative ion was observed during these experiments. Even after the chloroform was removed and the apparatus, as well as the bell-jar and the pump, was thoroughly cleaned and freed from traces of this gas, no further experiments in other gases were possible for many days. It was found that chloroform vapour present in air to an amount as small as 1 part in 1,000,000 changes completely the aspect of the mobility curves.

Control Experiments and Sources of Error.

1. It is important for the accuracy of the experiments, that the total wind-pressure produced by the ions should be given by the gauge, and this is secured if all the ions reach the plate B within the area of the grating. In order to ascertain this the grating was insulated from the plate by a narrow air-gap, and the galvanometer connected by means of a suitable key, either with the plate or with the grating. It was found that even at the largest distance between the plates, and under conditions in which the lateral diffusion of ions is abnormally great, the total current is received by the grating, the charge picked up by the plate being imperceptible.

2. The temperature of the gas close to the heated strip must be very high, and therefore the mobility of the ion at the moment when it leaves the strip must be large. In order to ascertain the magnitude of this source of error, curves representing the mobility of an ion as a function of the electric force were drawn for distances between the plates varying from 5 to 45 mm. No difference between them could be observed, which shows that the path traversed by

* In these experiments the silvered surface of the mirror was attacked by the halogens. I wish to thank Mr. E. Everett for suggesting to me the idea of platinizing the mirror and for setting up the cathode-ray apparatus necessary for this operation.

the ion with a larger velocity is small compared with the distance between the plates and may be neglected.

3. It was supposed at first that the electric field between the plates was uniform. This, however, is not the case, the electrification between the plates being large enough in these experiments to disturb considerably the uniformity of the field. Unfortunately, this source of error cannot be eliminated or even diminished, since the ionization-current must be considerable in order to produce a perceptible wind-effect. It is useless to increase the sensitiveness of the gauge beyond certain limits, as the electric wind produced must be large compared with the convection current due to the heated strip. There is, however, sufficient evidence to show that the results are but little affected by this source of error. In almost all the experiments described the ionic velocities were very large, so that the density of electrification between the

plates was considerably reduced. The equation $\frac{w}{X} = d \frac{c}{P}$,

as shown above, was found to be true over a wide range of X , c , and P . The mobility of the negative ion was shown to be a function of the electric force only, and not to change with the distance between the plates, which would be the case if this source of error was great.

4. The platinum strip freshly coated with salts gives off, when heated for the first time, a considerable amount of smoke (consisting probably of charged particles), which make the results inconsistent. In this case the strip has to be strongly heated at reduced pressure and in a strong electric field before measurements are taken.

Electric Wind in case of Ionization of both signs.

It seemed to be of interest to study the pressure of the electric wind in the case of ionization of both signs, when the wind is produced by positive and negative ions moving in opposite directions. For this purpose a small cell containing 25 mgr. of radium bromide and provided with a thin mica window was placed in the centre of the plate A in place of the platinum strip, and the gas between the plates strongly ionized by the α rays. The plate A could be positively or negatively charged, the plate B being earthed, and the wind-pressure measured in the usual way by the gauge. In this case the pressure of the wind at any point between the plates is a resultant of two opposite forces produced by the motion of positive and negative ions.

In the curves (+) and (—) (fig. 7) the wind-pressure is plotted against the electric force at 450 mm. pressure, in the case when the ions moving towards the grating are positive and negative respectively. These curves are in full

Fig. 7.

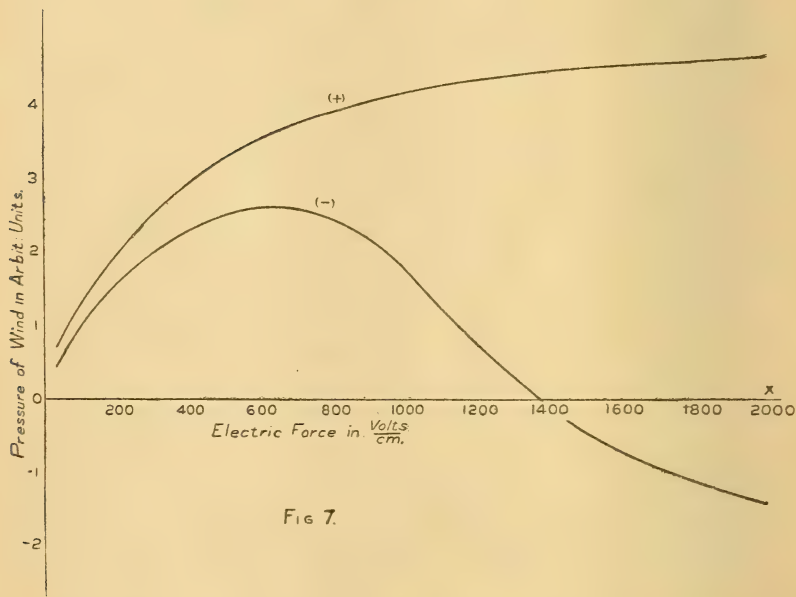


FIG 7.

agreement with the results relating to the mobility of the negative ion given above. For small forces the wind-pressure is small, as the ions in this case recombine on their way and traverse only a small part of the distance between the plates. The wind-pressure increases with the force until the saturation current is reached, and with further increase of the force a striking difference between the two curves is observed. Under these conditions the mobility of

the negative ions, as measured by the ratio $\frac{c}{P}$, begins rapidly

to increase, and since c is constant in these experiments, the wind-pressure P produced by them must decrease at the same rate; while the mobility of the positive ions, and consequently the wind-pressure due to them, is not affected by the increase of the force. When the plate A is negatively charged and the negative ions are moving towards the grating, the wind-pressure produced by them diminishes

with the strength of the field, and when the opposite force due to the positive ions begins to prevail, the pressure is reversed and assumes a negative value. In the case when the positive ions are moving toward the grating the wind-pressure produced by them remains constant, but since the opposite force due to the negative ions gradually diminishes, the resultant pressure measured by the gauge increases with the electric force. At atmospheric pressure the negative direction of the electric wind can be produced only with large electric forces exceeding $3000 \frac{\text{volt}}{\text{cm.}}$, at 50 mm. pressure a force of $80 \frac{\text{volt}}{\text{cm.}}$ is required for the same effect.

It is of interest to note that with small electric forces applied within the plates the wind-pressure increases with diminution of pressure of the gas down to a certain limit. This is due to the fact that the ionization-current, which is in this case far from saturation at atmospheric pressure, gradually approaches it when the pressure is reduced.

It would not be out of place to consider these results in connexion with the observations made by Joly* on the motion of radium in an electric field. Almost all the peculiar properties of radium described in that paper are easily explained, in the light of these results, by the electric wind produced by radium.

On the Nature of the Negative Carrier.

In the discussion of the results given above the existence was assumed of a transition stage between a negative ion and an electron, at which the average mass associated with the ion gradually diminishes with increasing electric forces. Whilst these experiments were in progress a new paper on the mobility of the negative ion appeared †, in which a radically different point of view as to the nature of the ion is put forward. According to E. M. Wellisch there are two distinctly different kinds of negative carriers in a gas: normal ions and free electrons, the proportion of the latter increasing with diminution of pressure; the conception of an intermediate stage between an ion and an electron is erroneous and is due merely to the attempt at averaging the different properties of these two kinds of ions.

* Joly, *Phil. Mag.* vii. p. 303 (1904).

† E. M. Wellisch, *Amer. Jour. of Science*, xxxix. p. 583 (1915).

Equation (1) may be written in the form $\frac{P}{c} = d \frac{X}{w}$, which shows that the wind-effect produced by the ions is inversely proportional to their mobility. According to Wellisch's theory the wind-pressure measured by the gauge in these experiments is mostly due to the slow normal ions, the effect produced by the free electrons being very small. The ratio $\frac{P}{c}$, as shown above, diminishes with increasing electric forces, and reaches in some experiments 5 per cent. of its initial value. It is necessary therefore to suppose that the proportion of free electrons increases with the electric force, so that, finally, the amount of normal ions becomes vanishingly small. Wellisch, however, states that the proportion of free electrons is independent of the strength of the field, and is always less than 50 per cent. of the total number of the negative carriers.

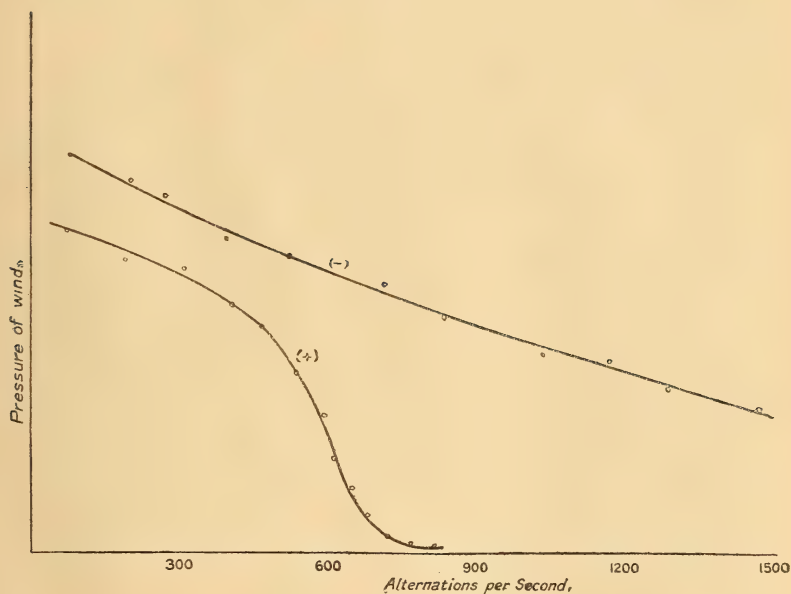
In order to verify the results of Wellisch, experiments were carried out which were based on the principle of separating, by means of an alternating electric field, the normal ions from the free electrons, and measuring the wind-effect due to one of these two kinds of ions separately. It appeared necessary for this purpose to study at first the wind-effect produced by the negative ions in an alternating field under conditions in which the mobility is normal, *i. e.* when, according to Wellisch, all the ions are normal ions. The alternating field between the plates A and B was established by means of a high-speed commutator which could produce up to 2000 alternations per second*, the potential difference between the plates being kept constant. The experiments show that when the number of alternations is gradually increased, the wind-pressure at first slowly diminishes and then rapidly falls down to a very small value. If the wind-pressure is plotted against the number of alternations per unit time, the curves obtained are all of the type of curve (+) (fig. 8), and simple calculations show that the sudden drop in the wind-pressure corresponds to the number of alternations at which the ions emitted by the heated strip begin not to reach the grating.

Now, if we repeat the same experiments under the conditions, in which the mobility of the negative ion is abnormal, *i. e.* when, according to Wellisch, the wind-pressure is mostly produced by the normal ions and only to a small extent by the free electrons, the following observations are to be

* The commutator was that used by G. Todd, Phil. Mag. xxii. p. 791 (1911).

expected: The wind-pressure should at first slowly diminish with increasing rate of alternations, and when the number of alternations per second is large enough to prevent the normal ions from reaching the grating, a sudden drop in the wind-pressure should be observed.

Fig. 8.



The experimental results are, however, altogether different from what might be expected if Wellisch's theory were true. No drop can be observed in the wind-pressure when the frequency of alternations reaches the critical value, which decisively proves that the wind is produced only by one kind of ions whose mobility (as measured by the ratio $\frac{c}{P}$) is intermediate between that of a normal ion and of a free electron. The curve (-) (fig. 8) was drawn at 100 mm. pressure, the distance between the plates being 3 cm. and the p.d. between them 200 v.; 900 alternations per second is enough under these conditions to separate the normal ions from the electrons; the curve, however, shows that the wind-pressure continues to diminish slowly with further increase of the rate of alternation. The curve (+) shows the wind-pressure under the same conditions, but in the case when it is produced by positive ions.

The experiments of Wellisch were carried out on the supposition that the mobility of the negative ion is independent of the electric force applied; this, however, is shown to be true only for comparatively small electric forces. The study of Wellisch's paper leaves but little doubt that the sudden increase in the mobility of the negative ion with increasing forces is responsible for the characteristic bend in the curves, which have led him to the erroneous conclusion of the existence of two different kinds of negative ions.

Thus, for a satisfactory explanation of the results given above, we must assume the gradual diminution of the average mass of the negative ion with increasing electric forces. The conception of "the average mass associated with an electron" is, however, complicated, owing to the uncertainty as to the nature of an ion, and may be interpreted in a way different from that adopted in this paper. According to the interpretation recently given by Sir J. J. Thomson * the electron may easily escape from the system of molecules constituting the ion and travel a certain distance in a free state, until it is once more attached to a molecule, and so on, so that the negative carrier makes its way through the gas, partly as a free electron and partly as a normal ion. From this point of view there is no "transition stage" between an ion and an electron, and the increase in the mobility of the negative ion, as well as the apparent diminution of the mass associated with it, show only that under certain conditions the proportion of time during which the ion is moving as a free electron increases with the electric force.

Summary.

1. A new method of measuring ionic mobilities in gases is described.

2. At a given pressure the mobility of the negative ion is shown to be constant only over a certain range of electric forces applied. With increasing electric forces X a certain value of $X = X_1$ is reached when the mobility of the ion begins to increase rapidly. The value of the critical force X_1 increases rapidly with the pressure, being equal to about $1800 \frac{\text{volt}}{\text{cm.}}$ at atmospheric pressure.

3. The mobility of free electrons was measured and shown to diminish with increase of the electric force.

4. The mobility of the negative ion in hydrogen and in some heavy gases was measured.

* Phil. Mag. xxx. p. 321 (1915).

5. No abnormality in the mobility of the positive ion could be observed at pressures down to 5 mm. and with very large electric forces.

6. The pressure of the electric wind in case of ionization of both signs was studied.

7. The recent theory of E. M. Wellisch with regard to the nature of the negative ion was investigated and shown to be erroneous.

These experiments were to a great extent carried out in the Physical Laboratory of the Polytechnic Institute in Petrograd. I am indebted to Prof. A. F. Joffe for allowing me to work in this laboratory and to Mr. J. S. Shcheglaëff for putting the necessary apparatus at my disposal.

In its final stages the work was completed at the Cavendish Laboratory, Cambridge, and I wish to take this opportunity of thanking Prof. Sir J. J. Thomson for permission to use the laboratory and for his interest of the progress of this work.

Cavendish Laboratory,
August 1916.

XLIX. *The Equilibrium of the Magnetic Compass in Aeroplanes.* By S. G. STARLING, B.Sc., West Ham Municipal Technical Institute*.

THE following investigation of the question of the behaviour of the aeroplane compass was undertaken at the suggestion of Mr. A. J. Hughes. It appears that the deviation of the compass when under acceleration causes grave errors in estimating the course, and that these differ considerably from one course to another. The greatest errors occur when the turn is made, as the course is changed, and as these errors may arise from a variety of causes, it becomes necessary to find the true position of equilibrium of the compass-card, due to the magnetic effect of the earth's field upon the needle.

The calculation of the couples acting on the needle presents great difficulties, as these depend upon the position of the needle at every instant, but if the equilibrium position at each instant is found, it can then be seen to what extent it would be disturbed, and this has led to suggestions which would cause the equilibrium position to be maintained true magnetic N. and S.

There is, of course, a tilting of the compass-card due to

* Communicated by Prof. A. W. Porter, F.R.S.

acceleration when the machine is increasing or decreasing in speed. This causes the card to dip towards the fore part during acceleration and to the aft during retardation. The resulting disturbance of the card will be a maximum when flight is magnetic East or West, and this would only occur during straightforward flight and chiefly near the ground, when the compass readings are not particularly important. By far the most serious errors of the compass are due to the acceleration towards the centre of the circle when a turn is made, and it is with these errors that the present paper is concerned.

As the aeroplane turns it is tilted towards the centre of the turn exactly as any other vehicle, and the resultant of gravitational and centrifugal forces may still be in the median plane of the machine, so that a plumb-line on it would still be perpendicular to the base-board, or, on the other hand, this resultant may be inclined to it. In the latter case, it would be possible to devise a compensator in the nature of a magnet or electric coil under gravitational control which would produce a magnetic field opposite to that causing the disturbance of the needle. This was tried and found to produce no effect, either of correction or disturbance, and it is therefore concluded that, as the aeroplane turns, the banking is such that the resultant weight of every part of it is still perpendicular to the base-board. Thus the card of the compass, which is free to take its new position under gravitation and centrifugal force, will still remain parallel to the normally horizontal glass of the compass-bowl. The machine then goes round a bend just as a large conical pendulum would do.

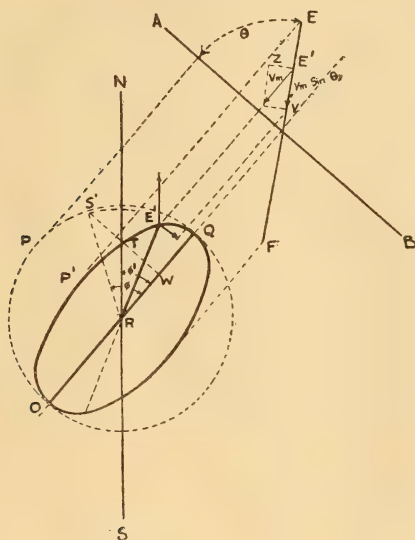
Any disturbances due to motion of the liquid in the bowl and friction at the pivot are now, for the moment, put on one side, and the equilibrium of the needle under magnetic forces alone considered.

Let H be the horizontal component of the earth's magnetic field, and V its vertical component. Then, $\tan d = \frac{V}{H}$, where d is the magnetic dip.

In fig. 1, AB is a truly horizontal plane shown in elevation and EF the compass-card inclined at angle θ to the horizontal. $OP'Q$ is the plan of the compass-card, and OPQ the card if imagined to be rotated through θ° into the horizontal plane. NS is the magnetic meridian, so that NRQ is the true compass-course (ϕ'), and $S'RQ$ is the angle between the magnetic meridian and the course RQ , as measured on the card, and is called ϕ .

For simplicity the pole of the needle is supposed to be at the edge of the card and the strength m , and the radius of the card l , but this assumption does not affect the result.

Fig. 1.



Then $TW = RW \cdot \tan \phi'$,
and $SW = RW \cdot \tan \phi$.

$$\therefore \frac{TW}{SW} = \frac{\tan \phi'}{\tan \phi}; \text{ but } \cos \theta = \frac{TW}{SW}.$$

$$\therefore \cos \theta = \frac{\tan \phi'}{\tan \phi},$$

a result which will be required shortly.

Vertical component of earth's field.—There will be a vertical force V_m acting downwards upon the N. pole of the needle and, of course, an equal and opposite upward force upon the S. pole.

This has a component $E'Z$ perpendicular to the plane of the card, which has no turning effect upon the needle, and a component $E'Y = V_m \sin \theta$ in the plane of the card, and perpendicular to OQ .

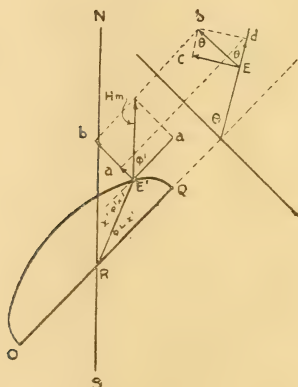
Horizontal component of earth's field.—The horizontal force H_m on the pole E' may be resolved into two horizontal

components (fig. 2), $E'a = Hm \cos \phi'$ in the plane of the compass-card, and parallel to OQ and $E'b = Hm \sin \phi'$, not in the plane of the card. The latter is resolved into $E'c$ normal to the card, and $E'd = Hm \sin \phi' \cos \theta$ in the plane of the card and perpendicular to OQ .

Equilibrium of the needle.—There are now three forces acting on each pole of the needle, namely, $Vm \sin \theta$, $Hm \cos \phi'$, and $Hm \sin \phi' \cos \theta$.

In fig. 2, x' is the angle between the magnetic meridian

Fig. 2.



and the vertical plane through the axis of the magnet, while x is the corresponding angle in the plane of the card, that is, $S'RE'$ (fig. 1). x is therefore the deviation of the compass from magnetic North, as seen by an observer in the aeroplane.

If $2l$ be the length of the magnetic needle, the magnetic moment is $2l \cdot m$, and the couples due to the three components of field acting on the needle are

$$2l \cdot m \cdot V \sin \theta \cos (\phi - x), \quad 2l \cdot m \cdot H \cos \phi' \sin (\phi - x),$$

and $2l \cdot m \cdot H \sin \phi' \cos \theta \cos (\phi - x),$

and the needle will be in equilibrium if

$$H \sin \phi' \cos \theta \cos (\phi - x) = V \sin \theta \cos (\phi - x) + H \cos \phi' \sin (\phi - x).$$

But $\frac{V}{H} = \tan d.$

$$\therefore \cos \phi' \tan (\phi - x) = \sin \phi' \cos \theta - \tan d \cdot \sin \theta,$$

or $\tan (\phi - x) = \cos \theta \tan \phi' - \frac{\tan d \sin \theta}{\cos \phi'}. \quad (1)$

For any angle of dip (d), compass course (ϕ'), and tilt θ , the value of $(\phi - x)$ can then be found, and since ϕ is calculated from $\cos \theta = \frac{\tan \phi}{\tan \phi'}$, x then becomes known.

In the northern hemisphere, when the aeroplane makes a turn from N. to E. and continues from E. to S., the tilt of the machine is downwards towards the south, and the plane of the card approaches the position of being perpendicular to the earth's resultant magnetic field, but the two turns N. to E. and E. to S. are symmetrical, the deviations being in opposite directions in these two quarter turns.

In going from S. to W. and W. to N. to complete the circle, the card is tilted towards the north, that is, its plane comes more nearly into the direction of the earth's resultant magnetic field, and the control of the needle is always increased by the tilt, and the deviations due to tilt are again symmetrical in the turns S. to W. and W. to N., and are in opposite directions in the turn.

The equation of equilibrium for this southerly half of the complete circle is slightly different from the former, as the N.

Fig. 3.

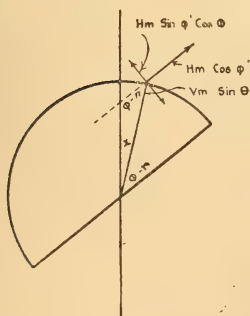
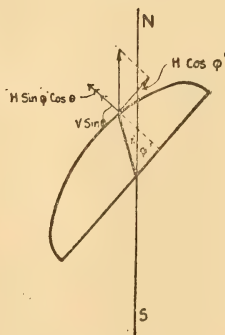


Fig. 4.



polar half of the card is dipping. In fig. 4 the turn S. and W. has just begun and the equilibrium equation is :

$$H \cos \phi' \sin (\phi + x) = V \sin \theta \cos (\phi + x) \\ + H \sin \phi' \cos \theta \cos (\phi + x),$$

or

$$\tan (\phi + x) = \frac{\tan d \cdot \sin \theta}{\cos \phi'} + \cos \theta \tan \phi'. \quad (2)$$

It is easier for these courses to reckon ϕ and ϕ' from the south, although in plotting the results they are considered to be 180° up to 360° E. of N.

Calculation of results.—The following Tables are obtained for a position where the dip is 67° , so that

$$\tan d = \tan 67^\circ = 2.36.$$

For values of θ less than 15° it is taken that ϕ and ϕ' are equal, but for values of θ above 15° , the value of ϕ is calculated from

$$\cos \theta = \frac{\tan \phi'}{\tan \phi}, \text{ or } \tan \phi = \frac{\tan \phi'}{\cos \theta}.$$

It must also be noticed that on the turns N. to E. or E. to S., that is, when the north side of the card is upwards, there may be a position for which $x > \phi$, and the N. pole of the magnet is below the horizontal line OQ (fig. 1). In this case $(\phi - x)$ becomes $(x - \phi)$. This is, in fact, generally the case; for even when $\theta = 20^\circ$, we obtain by putting $\phi = x$ in (1), that

$$\frac{\tan d \sin \theta}{\cos \phi'} = \cos \theta \tan \phi'.$$

$$\sin \phi' = \tan d \cdot \tan \theta,$$

$$= 2.36 \tan 20^\circ,$$

$$= 0.860,$$

$$\text{or } \phi' = 60^\circ \text{ approx.}$$

For values of θ greater than 23° , x is always greater than ϕ' for

$$\sin \phi' = 2.36 \tan 23^\circ,$$

$$= 2.36 \times 0.4245 = 1.0.$$

$$\therefore \phi' = 90^\circ.$$

Thus if θ is more than 23° , $\tan d \tan \theta$ must be greater than unity, and there is no possible value for ϕ' , such that $\phi' = x'$, which means that x' must always be greater than ϕ' , and the North polar end of the needle is below the horizontal OQ from the start of the turn. This appears also from the curves. As an example, the calculation for a tilt of $\theta = 70^\circ$ is given, and only a summary of the results for the other angles of tilt from 10° to 85° .

N. to E., $\theta = 70^\circ$.

ϕ' .	$\frac{2.36 \sin 70^\circ}{\cos \phi'}$.	$\cos 70^\circ \tan \phi'$.	$\tan (x - \phi)$.	$(x - \phi)$.	$\tan \phi = \frac{\tan \phi'}{\cos 70^\circ}$.	ϕ .	x .
0°	2.22	0	2.22	66°	0	0°	66°
10	2.25	0.060	2.19	65	0.514	27	92
20	2.36	0.129	2.23	66	1.06	47	113
30	2.56	0.197	2.36	67	1.69	59	126
40	2.90	0.287	2.61	69	2.45	68	137
50	3.46	0.407	3.05	72	3.49	74	146
60	4.43	0.592	3.84	75	5.06	79	154
70	6.48	0.94	5.54	80	8.08	83	163
80	12.8	1.93	10.9	85	16.6	86	171
85	25.5	3.92	21.6	87	33.4	88	175

S. to W., $\theta = 70^\circ$.

ϕ' .	$\frac{2.36 \sin 70^\circ}{\cos \phi'}$.	$\cos 70^\circ \tan \phi'$.	$\tan (\phi + x)$.	$(\phi + x)$.	ϕ .	x .
0°	2.22	0	2.22	66°	0°	66°
10	2.25	0.060	2.31	67	27	40
20	2.36	0.129	2.49	68	47	21
30	2.56	0.197	2.76	70	59	11
40	2.90	0.287	3.19	73	68	5
50	3.46	0.407	3.87	76	74	2
60	4.43	0.592	5.02	79	79	0
70	6.48	0.94	7.42	82	83	-1
80	12.8	1.93	14.7	86	86	0
85	25.5	3.92	29.4	88	88	0

θ .	10°.		15°.		20°.			23°.			30°.			40°.		
ϕ' .	x .		x .		ϕ .	x .		ϕ .	x .		ϕ .	x .		ϕ .	x .	
	N.	S.	N.	S.		N.	S.		N.	S.		N.	S.		N.	S.
0	22	22	31	31	0	39	39	0	43	43	0	50	50	0	58	58
10	24	21	34	28	11	44	34	11	49	37	11	57	43	13	69	47
20	24	19	37	25	21	49	29	21	54	32	23	66	35	25	79 $\frac{1}{2}$	38
30	26	17	38	22	31	53	25	31	59	27	34	75	28	37	91 $\frac{1}{2}$	29
40	23	14	39	18	42	58	19 $\frac{1}{2}$	42	65	21	44	81	22	47	102	23
50	18	11	38	14	52	60	15	52	71	16	54	93	17	58	115 $\frac{1}{2}$	15
60	17	8	36	11	61	62	12	61	75	13	64	104	11	66	128	11
70	13	6	29	7	71	59	7 $\frac{1}{2}$	71	81	8	72	119	8	74	152	7
80	7	3	17	3 $\frac{1}{2}$	81	47	3	81	86	3 $\frac{1}{2}$	81	143	4	82	161	4
85	4	1	9	2	85	30	2	85	86	2	86	161	1	86	170	2

θ .	60°.			70°.			80°.			85°.		
ϕ' .	ϕ .	x .		ϕ .	x .		ϕ .	x .		ϕ .	x .	
		N.	S.		N.	S.		N.	S.		N.	S.
0	0	64	64	0	66	66	0	67	67	0	67	67
10	19½	82½	45½	27	92	40	45	112	23	64	131	3
20	36	100	31	47	113	21	64	132	5	77	145	-9
30	49	113	20	59	126	11	73	142	-3	81	150	-11
40	59	125	13	68	137	5	78	150	-6	84	155½	-12
50	67	136	8	74	146	2	82	156	-7	86	160½	-11
60	74	147	4½	79	154	0	84	161	-5	87	164½	-9
70	80	158	2	83	163	-1	86	167	-4	88	169	-6
80	85	169½	1	86	171	0	88	173	-2	89	175	-3
85	87	174	1	88	175	0	89	177	-1	89½	177½	-1

The values of x , the deviation of the compass from true magnetic north as seen upon the card, are now plotted against ϕ' the true compass-course of the aeroplane, the values of x when ϕ' is greater than 90° being derived from the Tables, as before described.

Discussion of Curves.—It will be seen (fig. 5) that, as the machine turns from N. to E., the equilibrium position of the compass may be anything from true magnetic N. to true magnetic S., according to the angle of tilt at which the machine takes the curve. For tilt below 23° (that is, the complement of the angle of dip) the error never reaches 90° , but at a tilt of 23° the compass-card is perpendicular to the total earth's magnetic field when the machine is flying E. Consequently, there is no controlling field whatever, and the card will then set anywhere. This vanishing of control is shown by the instability of the compass, as indicated by the fact that the 23° curve bends sharply up (or down) to reach the 180° or (0°) point, when the magnetic course reaches 90° . It is a general rule that the steeper the curve at any point, the less will be the stability of the compass-card at that point.

For tilts above 23° , it is obvious that the compass becomes exactly reversed when flying E., for the direction of the resultant magnetic field of the earth is now pulling the needle into the reverse of its usual direction with respect to the bowl (fig. 6). This is indicated by the fact that the curves for tilts above 23° run to the 180° point on the course 90° . It may also be noticed that as the tilt approaches 90° the error on the 0° course approaches 67° , in fact, at this point the card approaches in action to a dip circle.

On the courses 180° to 360° the deviation due to tilt is very much less than for the 0° to 180° courses, the reason

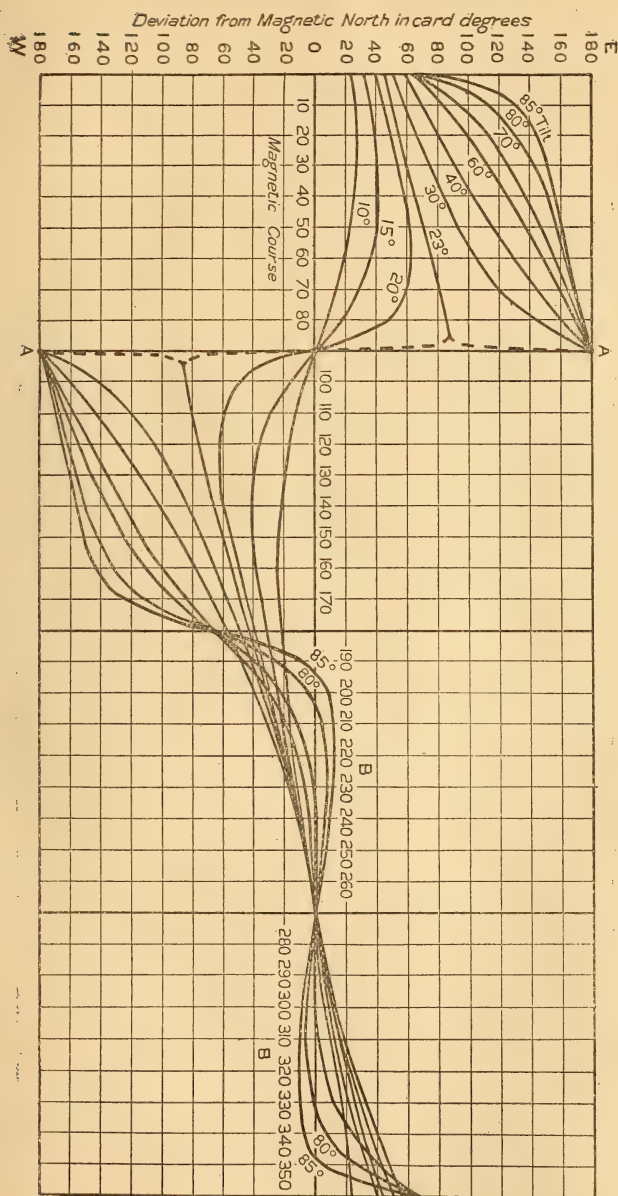
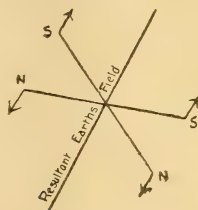


Fig. 5.

being that the card comes more nearly into parallelism with the earth's resultant magnetic field. The card is more stable and the period of oscillation should be diminished. The

deviation becomes reversed at B and B (fig. 5), for a reason to be explained later.

Fig. 6.



In actual flight the positions indicated by the curve will never be actually reached because of the changing course of the machine, but they give an indication of the movement, since they show the point towards which the card tends to move at each course.

For example, with a tilt of 80° or 85° , there is a powerful couple acting on the needle, tending to reverse its direction, and this acts for a large part of the whole turn.

Effect of liquid in Bowl.—In the case chosen for a complete circular turn in the direction N.—E.—S.—W.—N., the deviations alternate in direction, being first E., then W., again W., and finally E. If the rotation of the machine tends to drag the card after it by reason of the viscosity of the liquid in the bowl, this will sometimes increase the error and sometimes diminish it. Thus on the N. to E. (0° to 90°) turn, the effect of the liquid will be to increase the error, and it might be expected that this is the worst of all the turns—in fact, the card would probably go round faster than the machine if the tilt is great.

On the E. to S. (90° to 180°) course, the action of viscosity would be to check the motion of the card due to magnetic effects, and this course should not be so bad as the first. Of the last two, the liquid tends to correct the card from 180° to 270° and to assist in disturbing it from 270° to 360° .

Method of correcting error.—It is clear from the preceding, that the vertical component of the earth's magnetic field (V) is responsible for a large part of the disturbance. If, therefore, this could be balanced, an improvement would be effected. Now, the component of V in the plane of the compass-card is $V \sin \theta$, and therefore an arrangement which will produce a field at the card equal and opposite to this is required. A pair of bar magnets NS (fig. 7) attached to a frame, which can rotate about an axis passing through the compass and in the median line of the aeroplane, produce a vertical field, say F , at the compass. On rotating these about the axis through an angle equal to the angles of tilt, a field $F \sin \theta$ will be produced in the plane of the card. If

this is adjusted to be equal and opposite to $V \sin \theta$, it will be so for all angles of tilt. It is suggested that this rotation of the pair of magnets might be produced by attachment with the steering-gear or with a stabilizer.

Fig. 7.

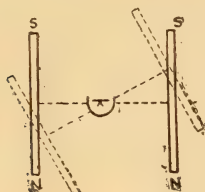
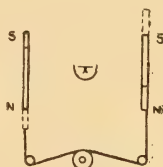


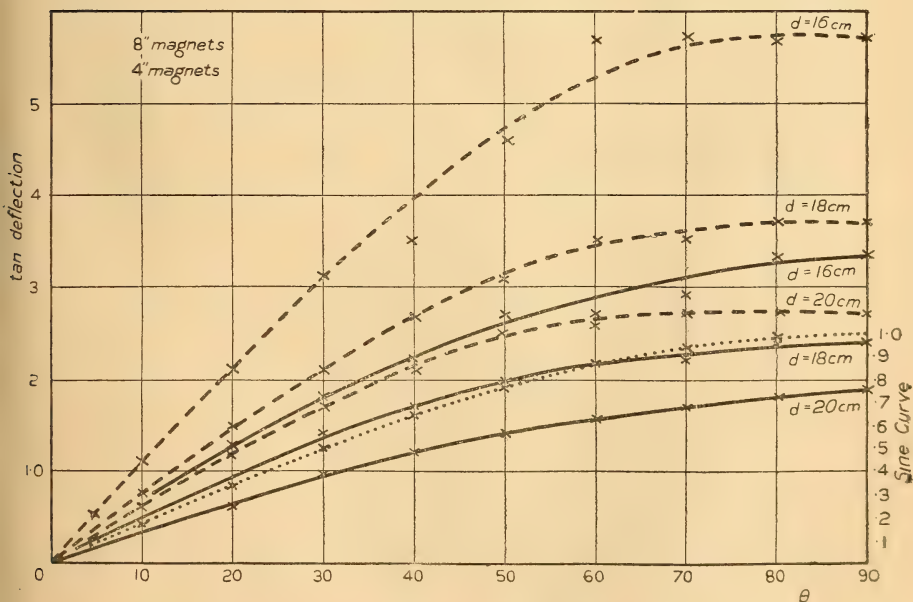
Fig. 8.



As there are certain mechanical difficulties in the way of this, it may be preferable to produce the correcting field by raising one magnet and lowering the other, the magnets being connected together by a cord passing over pulleys (fig. 8).

In order to ascertain whether this correction (fig. 7) is suitable for all angles of tilt, the field in the plane of the card

Fig. 9.



for a travel of 12 cm. on either side of the zero position for a pair of magnets 20 cm. long at a distance of 23 cm., one on either side of the compass, is drawn (fig. 9). A sine curve is

shown in dotted line alongside of it. From the parallelism of the two, it is apparent that a fair approximation to a proper correction might be obtained, if the movement of the magnets could be arranged to be proportional to the tilt of the machine.

Effect of earth's horizontal component alone.—If the earth's vertical component of magnetic field could be entirely eliminated, the errors of the compass due to tilt would be reduced, but not entirely eliminated. On putting $V=0$ in the equation from which equations (1) and (2) are derived, we have

$$\cos \phi' \sin (\phi - x) = \sin \phi' \cos \theta \cos (\phi - x),$$

$$\text{or} \quad \tan (\phi - x) = \cos \theta \tan \phi'. \quad . \quad . \quad . \quad . \quad (3)$$

It is now possible to find x in a manner somewhat similar to the former, for the values of ϕ and of $\cos \theta \tan \phi'$ are already known, so that $(\phi - x)$ and x can be tabulated. These are now done for the values of tilt 20° , 40° , 60° , 80° , and 85° .

θ .	20° .				40° .				60° .				80° .				85° .			
ϕ' .	$(\phi - x)$.	ϕ .	x .		$(\phi - x)$.	ϕ .	x .		$(\phi - x)$.	ϕ .	x .		$(\phi - x)$.	ϕ .	x .		$(\phi - x)$.	ϕ .	x .	
0	0	0	0		0	0	0		0	0	0		0	0	0		0	0	0	
10	9	11	2		8	13	5		5	$19\frac{1}{2}$	$14\frac{1}{2}$		2	45	43		1	64	63	
20	19	21	2		16	25	9		10	36	26		4	64	60		2	77	75	
30	28	31	3		24	37	13		16	49	33		6	73	67		3	81	78	
40	38	42	4		33	47	14		23	59	36		8	78	70		4	84	80	
50	48	52	4		42	58	16		31	67	36		11	82	71		6	86	80	
60	58	61	3		53	66	13		41	74	33		17	84	67		9	87	79	
70	69	71	2		65	74	9		54	80	26		25	86	61		13	88	75	
80	79	81	2		77	82	5		71	85	14		45	88	43		27	89	62	

These curves (fig. 10) are now plotted to the same scale as the previous set. They then make it clear that whereas the total disturbance is semicircular, being of the same sign (positive) from 270° to 90° and negative from 90° to 270° , the disturbance due to the tilt for horizontal component of the earth's field acting alone is quadrantal. It is the combination of the semicircular disturbance due to the earth's vertical component and the quadrantal disturbance due to the horizontal component that gives the want of symmetry about the 0° and 180° ordinates in the first set of curves. It is also seen why the curves at B become negative, for here the quadrantal effect predominates, at the large tilts the disturbing effects of the earth's vertical field being small, since the card approaches to parallelism with the resultant field.

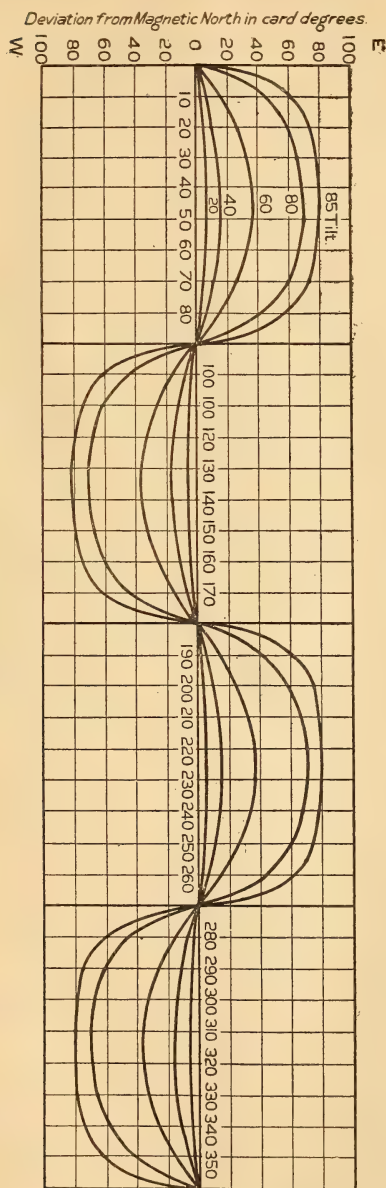


Fig. 10.

Suggested quadrantal compensation.—To correct perfectly for the quadrantal error, soft iron is necessary. It must also be possible to vary the intensity of its effect as the tilt changes. These conditions would be fulfilled by a pair of

soft iron bars, placed under or over the compass-bowl and hinged together (fig. 11), and so arranged that a cord would

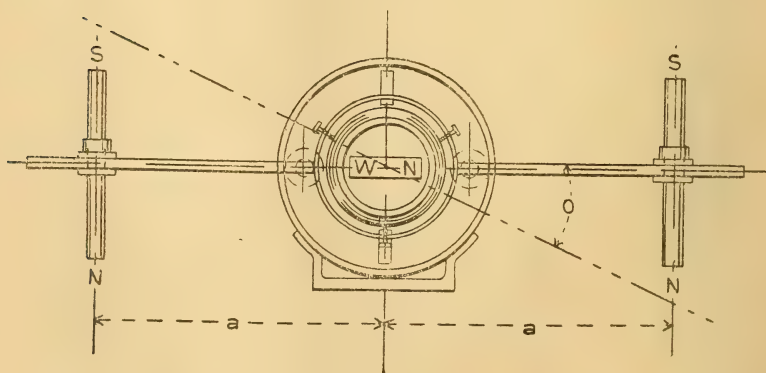
Fig. 11.



raise the ends of the bars N S as the tilt increases. As no measurements are available, it is not possible to say that the raising and lowering could be made to give a correcting field which would vary suitably with the tilt; but the correction would be of the right sign and would approximate to a sine curve as the course changes. As the curves of quadrantal deviation are somewhat similar to sine curves, it might be possible to obtain an approximate correction in this way.

In the event of a quadrantal correction of this kind being too small to be effective, it is also suggested that a pair of soft iron blocks placed one on either side of the compass-bowl would, by the attraction between the needle and them, produce a deviation of the right character. It would thus be necessary to move them nearer to the bowl as the machine tilts, the distance away when the machine is level being so great that they do not appreciably affect the position of the compass-card.

Fig. 12.



Measurements on corrector of type fig. 7.—Two vertical magnets, one on either side of the compass-bowl, are carried by an aluminium bar (fig. 12). The whole can rotate about

the compass-bowl of an aero-compass. Two sets of magnets are used, one, a pair of 8-inch magnets, and the other, 4-inch magnets. The lubber line being set N. and S., the magnets are placed in position one at a time and their level adjusted with the arm horizontal, until the card is in equilibrium in the magnetic meridian.

For various tilts of the arm θ , the deflexions are observed. The tangents of the deflexions are plotted against θ , thus giving a number proportional to the deflecting couple. The values of θ and of $-\theta$ are observed, but the deflexions for the latter are so near the former that, after the first observations, only those for one direction of rotation of the arm ($+\theta$) were observed.

For values of θ near 90° the deflexion approaches 90° , and the readings of the deflexion become difficult to make with sufficient precision to give the tangent of the deflexion accurately.

8-inch magnets.											
$d=16$ cm.				$d=18$ cm.				$d=20$ cm.			
θ .	Compass card.	Defl.	tan defl.	θ .	Card.	Defl.	tan defl.	θ .	Card.	Defl.	tan defl.
0	180°	0	0	0	180	0	0	0	180	0	...
5	206	26	0.48	5	199	19	0.34	5
10	228	48	1.1	10	215	35	0.70	10	210	30	0.58
20	244	64	2.1	20	236	56	1.5	20	230	50	1.2
30	252	72	3.1	30	245	65	2.1	30	240	60	1.7
40	254	74	3.5	40	250	70	2.7	40	244	64	2.1
50	258	78	4.7	50	252	72	3.1	50	248	68	2.5
60	260	80	5.7	60	254	74	3.5	60	249	69	2.6
70	260	80	5.7	70	254	74	3.5	70	250	70	2.7
80	260	80	5.7	80	255	75	3.7	80	250	70	2.7
90	260	80	5.7	90	255	75	3.7	90	250	70	2.7

4-inch magnets.											
θ .	Card.	Defl.	tan defl.	θ .	Card.	Defl.	tan defl.	θ .	Card.	Defl.	tan defl.
10	211	31	0.6	10	205	25	0.41	10	200	20	0.36
20	233	53	1.3	20	225	45	1.0	20	213	33	0.65
30	241	61	1.8	30	234	54	1.4	30	224	44	0.97
40	246	66	2.2	40	240	60	1.7	40	230	50	1.2
50	250	70	2.7	50	243	63	2.0	50	235	55	1.4
60	250	70	2.7	60	246	66	2.2	60	238	58	1.6
70	251	71	2.9	70	246	66	2.2	70	240	60	1.7
80	253	73	3.3	80	247	67	2.4	80	241	61	1.8
90	253	73	3.3	90	247	67	2.4	90	242	62	1.9

With a limiting tilt of the aeroplane of 90° , the vertical component of the earth's magnetic field produces its maximum effect, the magnetic intensity being $0.18 \tan 67^\circ = 0.18 \times 2.3$. For $\theta = 90^\circ$, the deflecting couple should equal this (0.18×2.3); but the field due to the compensating magnets is then $0.18 \times \tan(\text{deflexion})$, therefore for proper compensation $\tan(\text{deflexion}) = 2.3$.

This condition is seen from the curves (fig. 9, p. 471) to be very nearly fulfilled with the 4-inch magnets at a distance $d = 18$ cm. The sine curve having maximum value 2.5 is plotted as a line of dots, and is shown to be very nearly coincident with the corrector curve; showing that the compensation for vertical component would be very nearly perfect with 4-inch magnets at 18 cm. Any discrepancies are due to the fact that the field produced by the compensating card is not uniform, and the card magnets are of considerable size.

L. *Napier's Logarithms: the Development of his Theory.*
By Professor H. S. CARSLAW, Sydney, N.S.W.*

§ 1. INTRODUCTORY.

THIS paper deals with Napier's idea of a logarithm. In my view there are three distinct stages in the development of this idea in his work. In the first he is concerned with a one-one correspondence between the terms of a Geometrical Progression and the terms of an Arithmetical Progression. There are traces of this in the *Constructio* † in his use of the series

$$10^7, 10^7 \left(1 - \frac{1}{10^7}\right), 10^7 \left(1 - \frac{1}{10^7}\right)^2, \&c.,$$

and in the word logarithm itself, derived from $\lambda\acute{o}\gamma\omicron\varsigma$ $\acute{\alpha}\rho\iota\theta\mu\acute{o}\varsigma$, and generally taken to mean "the number of the ratios." In the second he has passed from this correspondence, and his logarithms are given by the well-known kinematical definition, which forms the foundation of the theory of the

* Communicated by the Author. Read to the Roy. Soc. New South Wales, Aug. 2, 1916.

† The *Mirifici Logarithmorum Canonis Constructio* was published in 1619, two years after Napier's death, but had been written several years before his *Mirifici Logarithmorum Canonis Descriptio*, published in 1614. I shall refer to these works as the *Constructio* and the *Descriptio*. The *Descriptio* was translated into English by Wright (1616), and Filipowski (1857), the *Constructio* by Macdonald (1889). The former is a rare book, both in the original and in translation. Several of the more important pages of the latter are reproduced in the 'Napier Tercentenary Memorial Volume,' Plates I.-VI. (London, 1915).

Constructio. In the third, referred to in the Appendix to the *Constructio*, he has reached the idea of a logarithm as defined by the property :—

The logarithms of proportional numbers have equal differences, with the additional condition that the values of the logarithms of two numbers are given.

In the second and third stages he has obtained, what we would now call, a function of the independent variable—the number—, but the function of the third stage is more general than that of the second, which it includes as a special case.

If this view is correct, the statement that “Napier’s theory rests on the establishment of a one-one correspondence between the terms of a geometric series and the terms of an arithmetic series” * should not be taken too literally. Further, the custom of employing the term “Napier’s logarithms” to describe only the logarithms of his *Canon* is unfortunate. It will be seen in the course of this paper that logarithms to the base 10—as we now know them—are Napier’s logarithms just as much as the logarithms of his *Canon*.

THE FIRST STAGE.

§ 2. The idea that multiplication and division could be reduced to addition and subtraction by the correlation of a geometrical series and an arithmetical series was not a new one. Aristotle was familiar with it, and since his time many mathematicians had returned to it. If we take the series

$$\begin{array}{ccccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \dots & 15. & \dots \\ 2, & 4, & 8, & 16, & 32, & 64, & 128, & 256, & \dots & 32768, & \dots \end{array}$$

the product of 128 and 256 in the geometrical series can be read off as 32768, which corresponds to 15, the sum of 7 and 8, in the arithmetical series.

The Swiss Bürgi in his *Arithmetische und Geometrische Progress Tabulen* †, constructed some time between 1603 and 1611, but first published in 1620, used the series

$$\begin{array}{cccc} 10 \times 0, & 10 \times 1, & 10 \times 2, & 10 \times n, \dots \\ 10^8, & 10^8 \left(1 + \frac{1}{10^4}\right), & 10^8 \left(1 + \frac{1}{10^4}\right)^2, & 10^8 \left(1 + \frac{1}{10^4}\right)^n \dots \end{array}$$

* Cajori, ‘The American Mathematical Monthly,’ vol. xxiii. p. 71 (1916).

† A facsimile of the title-page of Bürgi’s work, and of one of the pages of his Tables, will be found in the ‘Napier Tercentenary Memorial Volume,’ Plates XII. and XIII. Comparison with the references in Cantor’s *Geschichte der Mathematik*, Tropicke’s *Geschichte der Elementar-Mathematik*, and Braunmühl’s *Geschichte der Trigonometrie* will show that in none of these is the title quoted correctly.

His tables cover the range 10^8 to 10^9 , and for all practical purposes are as satisfactory as Napier's Table of Logarithms of 1614. If Napier had simply used the idea of the correspondence between the terms of a geometrical series and the terms of an arithmetical series, his work could not be regarded as so great an advance upon Bürgi's as it really is.

But it is clear that at the beginning of his labours, which extended over a period of 20 years, Napier's mind was working on the same lines as Bürgi's, and that he used the series

$$\begin{array}{ccc} 0, & 1, & 2, \dots \\ 10^7, & 10^7\left(1 - \frac{1}{10^7}\right), & 10^7\left(1 - \frac{1}{10^7}\right)^2, \dots \end{array}$$

The geometrical series occurs in the *Constructio*. He employed it in the calculation of his logarithms, but neither then, nor later, are his logarithms the terms of the corresponding arithmetical series. His word logarithm (see above, § 1) is evidently a survival of this stage of his work.

Napier meant his Tables to be used in calculations involving the trigonometrical ratios. In his time, the sine, cosine, &c., were lines—or, more exactly, the measures of lines—in a circle of given radius. Napier took the radius as 10^7 . It may be that Bürgi chose 10^8 in his Tables for a similar reason. With our notation, Napier's numbers would correspond to 7-figure Tables of Natural Sines, &c. If greater accuracy were required, the radius was taken as 10^{10} , and sometimes even a higher power of 10 was used. These sines, &c., following Glaisher*, we shall refer to as line-sines, &c.

THE SECOND STAGE.

§ 3. Napier opened out entirely fresh ground when he passed to his kinematical definition of the logarithm of a sine or number. By this definition he associated with the sine, as it continually diminished from 10^7 for 90° to zero for 0° , a number which he called its logarithm; and the logarithm continually increased from 0 for the sine of 90 to infinity for the sine of 0° .

The fundamental proposition in Napier's theory in the

* The Quarterly Journal of Pure and Applied Mathematics, vol. xlv. p. 125 (1915). To this paper I am indebted, not only for a most convenient notation for the different systems of logarithms, but also for an account of Speidell's work, hitherto inaccessible to me.

Descriptio (1614) and the *Constructio* (1619) is to be found in Prop. 1 of the *Descriptio* :—

“The logarithmes of proportionall numbers and quantities are equally differing”*.

And in Section 36 of the *Constructio* it appears as :—

“The logarithms of similarly proportioned sines are equi-different.”

Glaisher has introduced a convenient notation $\text{nl}_r x$ for Napier’s logarithm, in this system, when the radius is 10^r . He also uses $\text{Sin}_r x$ for the line-sine of the angle x , when the radius is 10^r , and he keeps the symbol $\sin x$ for the sine in the modern sense of the term. With this notation we have

$$\sin x = \frac{\text{Sin}_r x}{10^r}.$$

In this paper I follow his notation, and $\log_e x$ is used in its modern sense for the logarithm of x to the base e , the system commonly called hyperbolic logarithms.

The fundamental theorem, referred to above, can now be stated as follows :—

If $a : b = c : d$, then

$$\text{nl}_r a - \text{nl}_r b = \text{nl}_r c - \text{nl}_r d. \quad (1)$$

Also we are given that

$$\text{nl}_r 10^r = 0. \quad (2)$$

Napier’s *Canon* consists of a Table of Logarithms in which (1) and (2) are satisfied. His definition of the logarithm by means of the velocities of two points moving in two different lines leads us to the formula

$$\text{nl}_r x = 10^r \log_e \left(\frac{10^r}{x} \right).$$

But, of course, neither this, nor the fact that his function $\text{nl}_r x$ has -1 for its differential coefficient, when $x = 10^r$, could be known in his time.

THE THIRD STAGE.

§ 4. Since $uv : u = v : 1$, we have

$$\text{nl}_r (uv) - \text{nl}_r u = \text{nl}_r v - \text{nl}_r 1.$$

Thus

$$\text{nl}_r (uv) = \text{nl}_r u + \text{nl}_r v - \text{nl}_r 1,$$

and it must be remembered that $\text{nl}_r 1$ is not zero.

When $r = 7$, $\text{nl}_r 1 = 161180896 \cdot 38$ (cf. *Constructio*, Section 53).

Similarly, $\text{nl}_r (u/v) = \text{nl}_r u - \text{nl}_r v + \text{nl}_r 1.$

* In quoting the *Descriptio* I follow Wright’s version, and for the *Constructio* I adopt Macdonald’s.

Thus multiplication and division are changed into addition and subtraction. But the logarithms of numbers with the same figures in the same order cannot be read off from one another, since in this system,

$$\text{nl}_r(10^m a) = \text{nl}_r a - m(\text{nl}_r 1 - \text{nl}_r 10),$$

and $\text{nl}_r 1 - \text{nl}_r 10 = 23025842 \cdot 34$ (cf. *Constructio*, Section 53).

It is obvious that if a system of logarithms could be devised in which the logarithm of unity is zero and the logarithm of 10 is unity, the calculations would be immensely simplified, and the table curtailed; because one of the chief defects of Napier's *Canon*, as well as of Bürgi's Tables, was that, if the numbers did not come within the range covered by it, more or less awkward calculations were needed to overcome this difficulty.

Napier's *Canon* was first printed in the *Descriptio* (1614). After his death in 1617 the *Constructio* was published by the care of his son. It had been written several years before the *Descriptio*. To this work was added an Appendix, by the hand of Napier himself, "On the Construction of another and better kind of Logarithms, namely one in which the Logarithm of unity is 0." This Appendix begins with the words:—

"Among the various improvements of Logarithms, the more important is that which adopts a cypher as the Logarithm of unity, and 10,000,000,000 as the Logarithm of either one tenth of unity or ten times unity. Then, these being once fixed, the Logarithms of all other numbers necessarily follow."

It is clear from Napier's words that, when he wrote the Appendix, not only did he see the advantage of such a system, but he was in a position to draw up a Table of Logarithms in which these conditions would be satisfied. Indeed, he gives three distinct methods of finding these logarithms. The kinematical definition of the logarithm was superseded, and the correspondence between the terms of a geometrical series and the terms of an arithmetical series was left far behind. This is the third and final stage of his work.

BRIGGS AND NAPIER.

§ 5. In the change from the logarithms of the *Canon* to this "better kind of logarithms" Briggs was associated with Napier; but, chiefly because of the unsatisfactory account of the matter given by Hutton in his 'History of Logarithms'*,

* Hutton's 'Tracts on Mathematical and Philosophical Subjects,' vol. i. Tract 20.

the share of the former in the discovery has been exaggerated. The fault is not due to Briggs; and though his reference to the question in the preface to the *Arithmetica Logarithmica* (1624) is familiar, I reproduce it again here:—

“I myself, when expounding publicly in London their doctrine to my auditors in Gresham College, remarked that it would be much more convenient that 0 should stand for the logarithm of the whole sine, as in the Canon Mirificus, but that the logarithm of the tenth part of the whole sine, that is to say, 5 degrees 44 minutes 21 seconds, should be 10,000,000,000. Concerning that matter I wrote immediately to the author himself; and as soon as the season of the year and the vacation time of my public duties of instruction permitted, I took journey to Edinburgh, where, being most hospitably received by him, I lingered for a whole month. But as we held discourse concerning this change in the system of logarithms, he said that for a long time he had been sensible of the same thing, and had been anxious to accomplish it, but that he had published those he had already prepared, until he could construct tables more convenient, if other weighty matters and his frail health would permit him so to do. But he conceived that the change ought to be effected in this manner, that 0 should become the logarithm of unity, and 10,000,000,000 that of the whole sine; which I could not but admit was by far the most convenient of all. So, rejecting those which I had already prepared, I commenced, under his encouraging counsel, to ponder seriously about the calculation of these tables.”

Napier also mentions his discovery of the new system in the dedication of his *Rabdologia* (1617) in a passage quoted in my previous paper*.

It will be seen from Briggs's own words that the modification which he suggested to Napier was to keep the logarithm of the radius as zero, but to take the logarithm of one-tenth of the radius as 10,000,000,000. His reference to the *Canon* is sufficient to show that he does not look upon the radius as unity. In the construction of the Tables of Logarithms, after Napier's death, he takes it as 10^{10} , and it

* See also Macdonald's English translation of the *Constructio*, p. 88. This paper may be regarded as a supplement to a paper entitled “The Discovery of Logarithms by Napier of Merchiston,” *Journ. of Proc. Roy. Soc. N.S.W.* vol. xlviii. p. 43 (1914), which deals chiefly with the construction of Napier's *Canon*. I take this opportunity of amplifying, and to some extent correcting, the references in that paper to Briggs's share in the discovery of the “better kind of logarithms.” A paper covering much the same ground as the above will be found in ‘The Mathematical Gazette,’ vol. viii. (1915).

is for this reason that the characteristics 9, 8, &c., are to be found in the logarithms of the sines, &c.

Using the notation $\text{bl}_r x$ for the logarithm of x in the system suggested by Briggs when the radius is 10^r , we have

$$\text{bl}_r a - \text{bl}_r b = \text{bl}_r c - \text{bl}_r d,$$

when $a : b = c : d$.

Also $\text{bl}_r 10^r = 0$, and $\text{bl}_r 10^{r-1} = 10^{10}$.

In this system we have

$$\text{bl}_r (uv) = \text{bl}_r u + \text{bl}_r v - \text{bl}_r 1,$$

$$\text{bl}_r (u/v) = \text{bl}_r u - \text{bl}_r v + \text{bl}_r 1.$$

Also $\text{bl}_{10} 10^{10} = 10 \text{bl}_{10} 10 - 9 \text{bl}_{10} 1 = 0$.

$$\text{bl}_{10} 10^9 = 9 \text{bl}_{10} 10 - 8 \text{bl}_{10} 1 = 10^{10}.$$

Thus $\text{bl}_{10} 10 = 9 \times 10^{10}$ and $\text{bl}_{10} 1 = 10 \times 10^{10}$.

The advantage of the new system consists in the fact that the logarithms of numbers with the same figures in the same order could be read off from each other, since we have

$$\text{bl}_r (10^m a) = \text{bl}_r a - m \times 10^{10}.$$

§ 6. The change upon which Napier had resolved, previous to Briggs's visit, was a much more important one. He "conceived that the change ought to be effected in this manner, that 0 should be the logarithm of unity, and 10,000,000,000 the logarithm of the whole sine." And in the Appendix we see that he often passes from logarithms of sines, and drops all reference to the radius. In the new system, logarithms were to be *defined* by the relations :—

If $a : b = c : d$, then

$$\text{nl} a - \text{nl} b = \text{nl} c - \text{nl} d$$

with $\text{nl} 1 = 0$ and $\text{nl} 10 = 10^{10}$.

It need hardly be added that 10^{10} was taken for the logarithm of 10 instead of unity, for the same reason that 10^7 (or 10^{10}) was taken for the radius in dealing with the trigonometrical ratios.

Later, Briggs takes the logarithm of 10 as unity, and introduces the notation of decimal fractions in his Tables, a notation employed, probably for the first time, by Napier himself.

If this account of the growth of the idea of a logarithm in Napier's work is correct*, it seems unfortunate that the

* See also Gibson's paper in the 'Napier Tercentenary Memorial Volume,' pp. 111–137.

term Napier's logarithms is usually confined to the logarithms of his *Canon*. His "better kind of logarithms" actually consists of the logarithms now in daily use—the logarithms which we call logarithms to the base 10. In some text-books they receive the awkward name of Briggsian logarithms. Certainly Briggs calculated them, and the rapidity and industry with which he performed this immense work in computation will always be the admiration of mathematicians. But the discovery of the system was Napier's, and the logarithms are as much Napier's logarithms as those of his *Canon*.

SPEIDELL'S NEW LOGARITHMES (1619).

§ 7. In most accounts of the discovery of logarithms reference is made to Speidell's 'New Logarithmes' (London, 1619), and it is stated that they contain the first table of logarithms to the base e *. Attention is also usually called to the fact that, while logarithms to the base e are frequently spoken of as Napierean logarithms, they are quite different from the logarithms of Napier's *Canon*; and it is pointed out that the place of the number e in the theory of logarithms and the possibility of defining logarithms as exponents were discoveries of a much later day. These two statements, at first sight, seem inconsistent. A word or two regarding Speidell's system will make the matter clearer, and will also confirm the view I have taken above as to Napier's final conception of the logarithm.

Speidell's 'New Logarithmes,' like Napier's *Canon*, refer to the trigonometrical ratios. Using Glaisher's notation $\text{sl}_r x$ for Speidell's logarithm of x when the radius is 10^r , we have

$$\text{sl}_r x = 10^{r+1} - \text{nl}_r x.$$

It follows that

$$\text{sl}_r (uv) = \text{sl}_r u + \text{sl}_r v - \text{sl}_r 1,$$

$$\text{sl}_r (u/v) = \text{sl}_r u - \text{sl}_r v + \text{sl}_r 1,$$

and $\text{sl}_r 1$ is not zero.

The sole advantage of this system was that it avoided the use of negative quantities in calculation with logarithms. Such quantities were then outside the range of the "vulgar and common arithmetic."

* In Glaisher's paper already referred to, he published the interesting discovery that an Appendix (1618) to Wright's English translation of the *Descriptio* contains a table of hyperbolic logarithms by an anonymous author, whom Glaisher identifies with Oughtred.

Since
$$\text{nl}_r x = 10^r \log_e \left(\frac{10^r}{x} \right),$$

we have
$$\text{sl}_r x = 10^{r+1} + 10^r \log_e \left(\frac{x}{10^r} \right).$$

Thus
$$\begin{aligned} \text{sl}_r \text{Sin}_r x &= 10^{r+1} + 10^r \log_e \left(\frac{\text{Sin}_r x}{10^r} \right), \\ &= 10^r (10 + \log_e \sin x). \end{aligned}$$

In a sense Speidell's 'New Logarithmes' may be said to be hyperbolic logarithms, but the sense is the same as that in which the logarithms of Napier's *Canon* are sometimes said to be logarithms to the base e^{-1} . But this is a misuse of the term *. Still Speidell's logarithms of sines, from the accident that the sine is now used in a different sense, have actually the same figures as our hyperbolic logarithms of sines.

In the 'New Logarithmes' (1619) he takes the radius as 10^5 , so that these tables give

$$\text{sl}_5 \text{Sin}_5 x = 10^5 (10 + \log_e \sin x).$$

§ 8. But subsequently Speidell *did* publish a table of hyperbolic logarithms of numbers, which gives the values of $10^6 \log_e x$ for the numbers 1 to 1000. This table probably appeared either separately, or attached to an impression of the 'New Logarithmes,' in 1622 or 1623. In this system he takes

$$\text{sl}_r x = \text{nl}_r 1 - \text{nl}_r x.$$

It follows that

$$\text{sl}_r (uv) = \text{sl}_r u + \text{sl}_r v,$$

$$\text{sl}_r (u/v) = \text{sl}_r u - \text{sl}_r v;$$

and since

$$\text{nl}_r x = 10^r \log_e \left(\frac{10^r}{x} \right),$$

we have

$$\text{sl}_r x = 10^r \log_e x.$$

But it is clear that in both Speidell's systems of logarithms the connexion with hyperbolic logarithms is accidental, and the same is true of the logarithms discovered by Glaisher, to which reference is made at the beginning of this section.

Like Napier and Briggs, Speidell sees that the fundamental property, that the logarithms of proportional numbers have

* Cf. Glaisher, *loc. cit.* p. 146, footnote.

equal differences, can be taken as the starting-point of the theory; and that, if the logarithm of unity is zero, the logarithms of the product and the quotient of two numbers are, respectively, the sum and difference of their separate logarithms.

THE DIFFERENTIAL EQUATION SATISFIED BY THE LOGARITHM OF x .

§ 9. We have seen that the theory of the different systems of logarithms described in the previous pages rests upon the fundamental property:—

If $a : b = c : d$, then

$$\lambda(a) - \lambda(b) = \lambda(c) - \lambda(d),$$

where $\lambda(x)$ stands for the logarithm of x .

The function $\lambda(x)$, therefore, satisfies the equation

$$\lambda(x+h) - \lambda(x) = \lambda\left(1 + \frac{h}{x}\right) - \lambda(1).$$

∴

$$\frac{\lambda(x+h) - \lambda(x)}{h} = \frac{1}{x} \left\{ \frac{\lambda\left(1 + \frac{h}{x}\right) - \lambda(1)}{\frac{h}{x}} \right\}.$$

Proceeding to the limit $h \rightarrow 0$, of course keeping x fixed we have

$$\lambda'(x) = \frac{A}{x}, \text{ where } A = \lambda'(1).$$

Therefore $\lambda(x) = A \log x + B$,

and the system is made definite by adding two other conditions.

In Napier's *Canon*, writing ρ for the radius, we have

$$\text{nl } x = A \log x + B,$$

with $\text{nl } \rho = 0$, and $\text{nl}' \rho = -1$.

Therefore $\text{nl } x = \rho \log_e \left(\frac{\rho}{x}\right)$.

In Briggs's modification of the system, we have

$$\text{bl } x = A \log x + B,$$

with $\text{bl } \rho = 0$ and $\text{bl } \frac{\rho}{10} = 10^{10}$.

Thus

$$\text{bl } x = 10^{10} \frac{\log_e \left(\frac{\rho}{x} \right)}{\log_e 10} = 10^{10} \log_{10} \left(\frac{\rho}{x} \right).$$

And Napier's final form is, of course,

$$\text{nl } x = 10^{10} \log_{10} x.$$

Bürgi's *Arithmetische und Geometrische Progress Tabulen* also come under the same law. If the terms in the Arithmetical Progression are taken as the logarithms of the terms in the Geometrical Progression, and $\text{Bl } x$ stands for what I may call Bürgi's logarithm of x , we have

$$\text{Bl } x = 10 \frac{\log_e \left(\frac{x}{10^8} \right)}{\log_e \left(1 + \frac{1}{10^4} \right)} = 10 \log_{1 + \frac{1}{10^4}} \left(\frac{x}{10^8} \right),$$

for

$$x = 10^8 \left(1 + \frac{1}{10^4} \right)^s,$$

s being any positive integer.

Finally, treating Napier's series

$$\begin{array}{ccccccc} 0, & & 1, & & 2, & \dots\dots & \\ 10^7, & 10^7 \left(1 - \frac{1}{10^7} \right), & 10^7 \left(1 - \frac{1}{10^7} \right)^2, & \dots\dots & & & \end{array}$$

in the same way, and denoting this logarithm by $\text{Nl } x$, we have

$$\text{Nl } x = \frac{\log_e \left(\frac{x}{10^7} \right)}{\log_e \left(1 - \frac{1}{10^7} \right)} = \log_{1 - \frac{1}{10^7}} \left(\frac{x}{10^7} \right),$$

for

$$x = 10^7 \left(1 - \frac{1}{10^7} \right)^s,$$

s being any positive integer.

Sydney, August 1916.

In fact, squaring (A) and remembering that $n_1^2=1$, we have

$$r_1^2 = r^2 + 4(n_1 r)^2 - 4(n_1 r)^2 = r^2,$$

identically. Thus, r being a unit vector, so is also r_1 . Notice in passing that

$$\Omega_1 \Omega_1 = \Omega_1^2 = 1 - 4n_1 \cdot n_1 + 4(n_1 \cdot n_1)^2 = 1 - 4n_1 \cdot n_1 + 4n_1 \cdot n_1,$$

i. e. $\Omega_1^2=1$, which is an obvious property.

Let, now, the ray r_1 , reflected from the first mirror, impinge upon a second mirror whose normal is n_2 , again a unit vector. Then the ray r_2 , reflected from the second mirror, will be

$$r_2 = \Omega_2 r_1 = \Omega_2 \Omega_1 r,$$

where $\Omega_2 = 1 - 2n_2 \cdot n_2$; and since Ω_2 applied to the vector $\Omega_1 r$ gives the same result as $\Omega_2 \Omega_1$ applied to r (associative property), no separating signs are needed. Thus the reflector of the double mirror 1, 2 is simply $\Omega = \Omega_2 \Omega_1$. Similarly, if r_2 impinges upon a third mirror, the reflected ray is $r_3 = \Omega r$, where $\Omega = \Omega_3 \Omega_2 \Omega_1$, and so on.

Thus the reflector of a multiple mirror consisting of any number κ of plane mirrors, taken in the prescribed succession 1, 2, 3, ... κ , is

$$\Omega = \Omega_{\kappa} \dots \Omega_3 \Omega_2 \Omega_1 = \prod_{i=\kappa}^{i=1} \Omega_i, \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\Omega_i = 1 - 2n_i \cdot n_i, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

n_i being the unit normal of the i th mirror*. If r is the incident and r' the finally reflected ray, we have

$$r' = \Omega r, \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

and, by what has been said before,

$$r'^2 = r^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If, therefore, r is a unit vector, so is r' , and if θ be the angle between the incident and the finally reflected ray, we have simply

$$\cos \theta = r \Omega r. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

* Notice that, for a given multiple mirror, Ω will, in general, be different according to the order of succession of the component reflexions, that is, of the suffixes i . In short, the "products" $\Omega_3 \Omega_2 \Omega_1$, etc., are associative, but, in general, *not commutative*. They become so only for certain multiple mirrors, *i. e.* for some particular arrangements of the component mirrors. If a finite beam of parallel rays r impinges upon, say, a triple mirror, some pencils may be reflected as $\Omega_3 \Omega_2 \Omega_1 r$, others as $\Omega_3 \Omega_1 \Omega_2$, and so on. Thus the several reflected pencils will, in general, diverge from one another.

In particular, when the arrangement of the component mirrors is such that

$$\Omega = -1,$$

then every incident ray will be sent back parallel to its own path. Such multiple mirrors are called *central* mirrors.

By a well-known theorem of vector algebra, the general reflector (1), being a pure versor, could always be expressed by $\Omega = \mathbf{a} \cdot \mathbf{i} + \mathbf{b} \cdot \mathbf{j} + \mathbf{c} \cdot \mathbf{k}$, where both \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{i} , \mathbf{j} , \mathbf{k} are some normal systems of unit vectors, both right-handed, or both left-handed. No use, however, will be made here of this fundamental property, since obviously the most natural entities to represent the properties of any multiple mirror are the normals $\mathbf{n}_1, \mathbf{n}_2$, etc. themselves. These appear in Ω as dyads, such as $\mathbf{n}_1 \cdot \mathbf{n}_1$ or $\mathbf{n}_2 \cdot \mathbf{n}_1$, etc., or as scalar products $n_1^2 = 1$, $\mathbf{n}_1 \mathbf{n}_2 = \cos(\mathbf{n}_1, \mathbf{n}_2)$, and so on. In certain cases it may be advantageous to employ the unit edges of consecutive mirrors, *i. e.* apart from the scalar factors $\sin(\mathbf{n}_1, \mathbf{n}_2)$, etc., the vector products $\mathbf{V}\mathbf{n}_1 \mathbf{n}_2$, and so on.

The utility of the above method of treatment, in which the clumsy and often unmanageable formulæ of spherical trigonometry are replaced by the simple operator (1), *needing no drawings whatever*, will best be exhibited on a number of examples. We shall begin with the simplest case of a double mirror and then proceed to more complicated ones. In each case the procedure will consist in simply "multiplying" out the dyads $\mathbf{n} \cdot \mathbf{n}$ contained in the component reflectors. And in doing so we have only to remember that juxtaposed vectors, not separated by dots, are fused into ordinary scalar products. Thus, $\mathbf{n}_1 \cdot \mathbf{n}_1 \mathbf{n}_2 \cdot \mathbf{n}_2 = \mathbf{n}_1(\mathbf{n}_1 \mathbf{n}_2) \cdot \mathbf{n}_2 = a_{12} \mathbf{n}_1 \cdot \mathbf{n}_2$, where $a_{12} = \mathbf{n}_1 \mathbf{n}_2 = \cos(\mathbf{n}_1, \mathbf{n}_2)$. In short, the "product" of any number of dyads is always a dyad, including an ordinary scalar factor. In what follows we shall employ the general notation

$$\mathbf{n}_i \mathbf{n}_j = \cos(\mathbf{n}_i, \mathbf{n}_j) = a_{ij} = a_{ji} \quad . \quad . \quad . \quad (5)$$

The incident ray or operand \mathbf{r} need not be written out in each case; it is enough to deal with the operators Ω_i and with their resultants, *i. e.* with the reflectors themselves. All of the operations involved being associative as well as distributive, the multiplication will be done as in ordinary algebra, the only precaution (owing to non-commutativity) being the preservation of order.

Double mirror.—The unit normals of the component mirrors being $\mathbf{n}_1, \mathbf{n}_2$, and $\mathbf{n}_1 \mathbf{n}_2 = a_{12} = a$ the cosine of their

included angle, we have, by (1), (2),

$$\Omega = \Omega_2 \Omega_1 = 1 - 2[\mathbf{n}_1 \cdot \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_2] + 4\mathbf{a} \cdot \mathbf{n}_2 \cdot \mathbf{n}_1, \quad (6)$$

or, introducing the vector $\mathbf{p} = \mathbf{n}_1 - 2\mathbf{a}\mathbf{n}_2$,

$$\Omega = 1 - 2\mathbf{p} \cdot \mathbf{n}_1 - 2\mathbf{n}_2 \cdot \mathbf{n}_2. \quad (7)$$

The meaning of this operational equation is seen at once by remembering that $\mathbf{r}' = \Omega \mathbf{r}$; thus

$$\mathbf{r} - \mathbf{r}' = 2(\mathbf{r}\mathbf{n}_1)\mathbf{p} + 2(\mathbf{r}\mathbf{n}_2)\mathbf{n}_2, \quad (7a)$$

i. e. whatever the incident ray \mathbf{r} , the vector $\mathbf{r}' - \mathbf{r}$ is normal to the common edge of the two mirrors. In other words, the projections of \mathbf{r} and \mathbf{r}' upon the edge are equal to one another.

From (6) we see that, in general, $\Omega_2 \Omega_1$ differs from $\Omega_1 \Omega_2$, since the last term $4\mathbf{a}\mathbf{n}_2 \cdot \mathbf{n}_1$ is not symmetrical. Thus, a beam of parallel rays \mathbf{r} (broad enough to impinge upon both mirrors) is split by the double mirror into two beams \mathbf{r}' , \mathbf{r}'' oblique to one another, e. g. such that

$$\mathbf{r}' - \mathbf{r}'' = 4\mathbf{a}[\mathbf{n}_2 \cdot \mathbf{n}_1 - \mathbf{n}_1 \cdot \mathbf{n}_2]\mathbf{r}.$$

In particular, if the double mirror is *orthogonal*, we have $\mathbf{a} \cdot \mathbf{n}_1 \mathbf{n}_2 = 0$, and, independently of the order of reflexions,

$$\Omega = \Omega_2 \Omega_1 = \Omega_1 \Omega_2 = 1 - 2\mathbf{n}_1 \cdot \mathbf{n}_1 - 2\mathbf{n}_2 \cdot \mathbf{n}_2.$$

But since $\mathbf{n}_1 \perp \mathbf{n}_2$, we have

$$1 = \mathbf{n}_1 \cdot \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_2 + \mathbf{e} \cdot \mathbf{e},$$

where \mathbf{e} is a unit vector along the common edge of the two mirrors*. Therefore, for an *orthogonal* double mirror,

$$\Omega = -[1 - 2\mathbf{e} \cdot \mathbf{e}], \quad (8)$$

that is to say, the reflexion from such a double mirror is equivalent to the reflexion from a simple mirror whose normal is \mathbf{e} , followed by a simple reversal (-1). This is valid for *any* incident ray \mathbf{r} . More especially, if the incident ray is normal to the edge, or $\mathbf{r} \cdot \mathbf{e} = 0$, we have $\mathbf{r}' = \Omega \mathbf{r} = -\mathbf{r}$, that is to say, the ray is sent back parallel to itself. The latter property is familiar from ordinary geometrical constructions.

* If \mathbf{i} , \mathbf{j} , \mathbf{k} be any normal system of unit vectors, the dyadic $\mathbf{i} \cdot \mathbf{i} + \mathbf{j} \cdot \mathbf{j} + \mathbf{k} \cdot \mathbf{k}$ is equivalent to 1 , or is, in Gibbs's nomenclature, an *idemfactor*: $[\mathbf{i} \cdot \mathbf{i} + \mathbf{j} \cdot \mathbf{j} + \mathbf{k} \cdot \mathbf{k}]\mathbf{r} = \mathbf{r}$, for any \mathbf{r} . Notice that, since \mathbf{e} appears only through the dyad $\mathbf{e} \cdot \mathbf{e}$, the sense of \mathbf{e} is, obviously, a matter of indifference.

Returning to the general formula (6) we have for the angle θ between the rays \mathbf{r} and \mathbf{r}' , $\cos \theta = \mathbf{r} \Omega \mathbf{r}$, i. e.

$$\frac{1}{2}(1 - \cos \theta) = r_1^2 + r_2^2 - 2ar_1r_2,$$

where $r_1 = \mathbf{r} \mathbf{n}_1$, $r_2 = \mathbf{r} \mathbf{n}_2$ are the projections of the incident ray upon the mirror normals. On the other hand, since $\mathbf{r} = r_1 \mathbf{n}_1 + r_2 \mathbf{n}_2 + (\mathbf{r} \mathbf{e}) \mathbf{e}$ and \mathbf{r} is a unit vector, we have

$$r_1^2 + r_2^2 + 2ar_1r_2 = 1 - (\mathbf{r} \mathbf{e})^2,$$

so that the last equation can be written

$$\frac{1}{2}(1 - \cos \theta) = 1 - 4ar_1r_2 - (\mathbf{r} \mathbf{e})^2. \quad . \quad . \quad (9)$$

This gives the angle θ for any double mirror and for any \mathbf{r} .

From this general formula we see at once that there is no such double mirror which would send back (parallel to its own path) *every* incident ray, in short, that there are no central double mirrors. In fact, $\cos \theta = -1$ would mean $4ar_1r_2 + (\mathbf{r} \mathbf{e})^2 = 0$, and this cannot be fulfilled for all directions of \mathbf{r} .

Triple mirror.—The reflector in this case is $\Omega = \Omega_3 \Omega_2 \Omega_1$, that is, the product of $\Omega_3 = 1 - 2\mathbf{n}_3 \cdot \mathbf{n}_3$ into the operator (6) of the preceding section. Writing, therefore, $\mathbf{n}_1 \mathbf{n}_2 = a_{12}$, etc., we have, for any triple mirror (the order of reflexions being 1, 2, 3),

$$\begin{aligned} \Omega = \Omega_{123} = 1 - 2[\mathbf{n}_1 \cdot \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_3] \\ + 4[a_{12}\mathbf{n}_2 \cdot \mathbf{n}_1 + a_{23}\mathbf{n}_3 \cdot \mathbf{n}_2 + a_{31}\mathbf{n}_3 \cdot \mathbf{n}_1] - 8a_{12}a_{23}\mathbf{n}_3 \cdot \mathbf{n}_1. \end{aligned} \quad (10)$$

Here again, the third and fourth terms being non-symmetrical, an incident beam of parallel rays will give rise to six reflected beams $\Omega_{123}\mathbf{r}$, $\Omega_{132}\mathbf{r}$, etc., which will, in general, not be parallel to one another. These reflected beams become parallel to one another, i. e. Ω becomes independent of the order of reflexions 1, 2, 3, when, and only when,

$$a_{12} = a_{23} = a_{31} = 0,$$

i. e. when the three component mirrors are *perpendicular* to one another. In that case, \mathbf{n}_1 , \mathbf{n}_2 , \mathbf{n}_3 being a triad of normal unit vectors, the dyadic $\mathbf{n}_1 \cdot \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_3$ becomes an idemfactor or 1, and therefore,

$$\Omega = -1; \mathbf{r}' = -\mathbf{r}. \quad . \quad . \quad . \quad (11)$$

That is to say, every incident ray is sent back parallel to itself. The orthogonal triple mirror is a *central* mirror.

Returning to the general triple mirror, let r_1 , r_2 , r_3 be the

direction cosines of the incident ray, *i. e.* the projections of \mathbf{r} upon $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$. Then the direction cosines r_1', r_2', r_3' of the reflected ray \mathbf{r}' will be, by (10),

$$\left. \begin{aligned} r_1' &= r_1(1-2r_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ r_2' &= r_2(1-2r_2+4a_{12}r_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ r_3' &= r_3(1-2r_3+4a_{23}r_2+4a_{31}r_1-8a_{12}a_{23}r_1). \end{aligned} \right\} \dots \quad (12)$$

These scalar formulæ can at once be used for numerical calculation.

The angle $\theta = (\mathbf{r}, \mathbf{r}')$ is given by $\cos \theta = \mathbf{r} \Omega \mathbf{r}$, *i. e.* by (10),

$$\frac{1 - \cos \theta}{2} = r_1^2 + r_2^2 + r_3^2 - 2[a_{12}r_1r_2 + a_{23}r_2r_3 + a_{31}r_3r_1] + 4a_{12}a_{23}r_3r_1.$$

On the other hand, we have, by squaring

$$\mathbf{r} = r_1\mathbf{n}_1 + r_2\mathbf{n}_2 + r_3\mathbf{n}_3^*,$$

$$r^2 = 1 = r_1^2 + r_2^2 + r_3^2 + 2[a_{12}r_1r_2 + a_{23}r_2r_3 + a_{31}r_3r_1],$$

and therefore, for any incident ray, whose order of reflexions is 123,

$$\frac{3 - \cos \theta}{4} = r_1^2 + r_2^2 + r_3^2 + 2a_{12}a_{23}r_3r_1. \quad . \quad . \quad (13)$$

From this general formula, which enables us to calculate at once the angle θ for any incident ray, we can see also that the orthogonal mirror is the only possible central mirror. In fact, the right-hand member of (13) becomes equal to 1 (*i. e.* $\cos \theta = -1$) for *any* direction of \mathbf{r} , when, and only when, $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are normal to one another.

If the three mirrors constitute a *regular pyramid*, *i. e.* if

$$a_{12} = a_{23} = a_{31} = \cos \omega, \text{ say, } . \quad . \quad . \quad . \quad (14)$$

then the reflector (10) becomes

$$\begin{aligned} \Omega &= 1 - 2[\mathbf{n}_1 \cdot \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_3] \\ &\quad + 4a[\mathbf{n}_2 \cdot \mathbf{n}_1 + \mathbf{n}_3 \cdot \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_1] - 8a^2\mathbf{n}_3 \cdot \mathbf{n}_1, \end{aligned} \quad (10a)$$

and the formula (13) for the angle $\theta = (\mathbf{r}, \mathbf{r}')$,

$$\frac{3 - \cos \theta}{4} = r_1^2 + r_2^2 + r_3^2 + 2 \cos^2 \omega r_1r_3, \quad . \quad (13a)$$

for any incident ray, the order of succession of the reflexions being 123, for (10a), and either 123 or 321 for (13a). If the order is 231 (or 132) or 312 (or 213), we have only to write, in the last term of (13a), r_2r_1 and r_3r_2 respectively.

* We assume, of course, that $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are not coplanar, *i. e.* that the three reflecting planes constitute a pyramid, not a prism.

More especially, if the incident ray is *equally inclined* to the three reflecting planes of the regular pyramid, we have

$$r_1 = r_2 = r_3 = \frac{1}{\sqrt{3}} \cot \frac{\omega}{2},$$

and (13 a) becomes

$$\frac{3 - \cos \theta}{4} = \left(1 + \frac{2}{3} \cos^2 \omega\right) \cot^2 \frac{\omega}{2}. \quad \dots (13 b)$$

In this case the angle θ is independent of the order of reflexions, as was to be expected. The reflected beams, although not parallel to one another, are equally inclined to, and symmetrically disposed around, the direction of the incident beam \mathbf{r} . These reflected beams coincide in direction when, and only when, $\omega = 90^\circ$, *i. e.* when the mirror becomes an orthogonal and, therefore, a central mirror.

Further discussion of the above formulæ and the construction of similar ones for quadruple and more complicated mirrors are left to the reader. Here but two further remarks on the general reflector Ω :—

Reversal of the order of reflexions.—Let \mathbf{r} be the incident ray, \mathbf{r}' the finally reflected ray when the order of reflexions is 123... κ , and \mathbf{s}' the finally reflected ray when the order of reflexions is κ ...321. Then, if $\Omega = \Omega_\kappa \dots \Omega_3 \Omega_2 \Omega_1$, as in (1),

$$\mathbf{r}' = \Omega \mathbf{r}, \quad \mathbf{s}' = \mathbf{r} \Omega,$$

whence

$$\mathbf{r} \mathbf{r}' = \mathbf{r} \Omega \mathbf{r}, \quad \mathbf{s}' \mathbf{r} = \mathbf{r} \Omega \mathbf{r},$$

and therefore, for any multiple mirror,

$$\mathbf{r}' \mathbf{r} = \mathbf{s}' \mathbf{r}, \quad \dots \dots \dots (14)$$

while $\mathbf{r}' \mathbf{s}' = \mathbf{r} \Omega^2 \mathbf{r}$. That is, the reflected rays \mathbf{r}' , \mathbf{s}' , although not parallel to one another, are always *equally inclined to the incident ray* \mathbf{r} . The equality $\theta_{123} = \theta_{321}$, exhibited by (13 a) is but a special instance of this general property.

Images of given objects.—Hitherto we have considered \mathbf{r} and \mathbf{r}' as determining the directions of the incident and the reflected rays. In order to obtain the image of a given point-object, let the end-point of the vector \mathbf{r} , drawn from a fixed origin O, determine the position of the object, and the end-point of \mathbf{r}' , drawn from the same origin, the position of the image. Then, in the case of a simple mirror, we have again

$$\mathbf{r}' = \Omega_1 \mathbf{r} = [1 - 2\mathbf{n}_1 \cdot \mathbf{n}_1] \mathbf{r},$$

provided that O is a point of the reflecting plane itself.

Similarly, in the case of a double mirror,

$$\mathbf{r}' = \Omega_2 \Omega_1 \mathbf{r},$$

if O is taken on the common edge of the component mirrors. Thus, in the case of any multiple mirror, we have, as before,

$$\mathbf{r}' = \Omega \mathbf{r}, \quad \Omega = \prod_{i=\kappa}^{i=1} \Omega_i,$$

where $\Omega_i = 1 - 2\mathbf{n}_i \cdot \mathbf{n}_i$, provided that all the component mirrors have a *common* point of intersection, and that this point is taken as the *origin* of the vectors \mathbf{r} , \mathbf{r}' .

Under these circumstances, therefore, the treatment of point-objects and their images is formally the same as that of incident and reflected rays.

September 5, 1916.

LII. *On an X-Ray Vacuum Spectrograph.* By MANNE SIEGBAHN, *Dr. phil.*, and EINAR FRIMAN, *Dr. phil.**

IN order to examine the high frequency spectra of the elements by long wave-lengths the authors have had a vacuum spectrograph built. Hereby, as our former measurements † have shown, the following conditions must be satisfied. First, the crystal must be movable, as otherwise irregularities in the structure of the crystal may be of great influence (comp. Rutherford and Andrade and E. Wagner). Secondly, in order to get a good resolving power, besides using a fine slit it is of great importance to focus the rays.

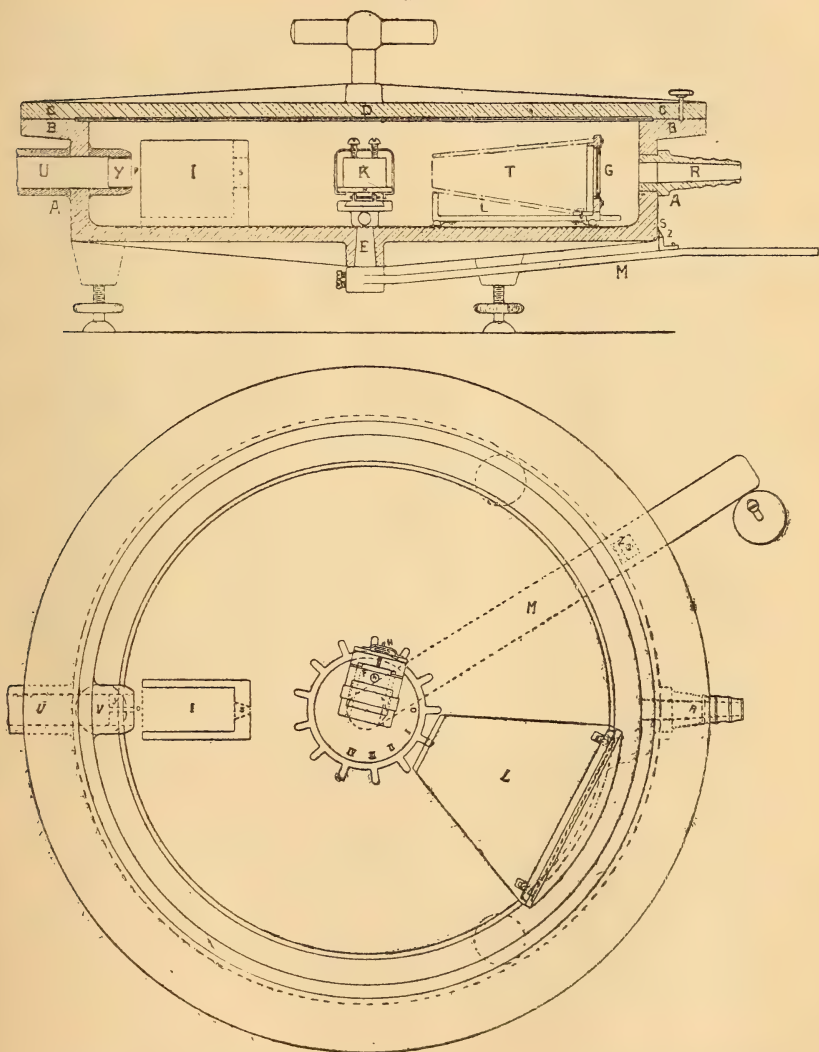
The apparatus built on this principle is shown in fig. 1, in both horizontal and vertical section. The spectrograph consists of a round metallic box of 6 mm. thickness, a height of 8 cm. and an inner diameter of 30 cm. The upper part BB, 3.5 cm. broad, is carefully plane-ground, as well as the corresponding part CC of the cover D. This as well as the bottom is furnished with radial reinforcements in order to resist the pressure better. The cover has a handle and a screw, the latter being used to lift it after the air has been admitted. The box is supported by three set-screws. In the middle of the bottom there is a conical hole with a metallic cone E, fitting well in it. This cone, being kept in the hole by a ring screwed into the box, after lubricating can be turned without the air passing through. The crystal table, placed

* Communicated by the Authors.

† Phil. Mag. vol. xxxi. p. 403 (1916); vol. xxxii. p. 39 (1916).

on the cone, consists of a slide F, that can be moved by means of the screw H. On the slide the table N is placed, that can be turned round the axis *a* with the aid of the

Fig. 1.



screw *h*. This arrangement allows of a careful adjustment of the crystal *K*. This is attached to the table by means of a bow and two screws. The lower part of the cone is joined to a metallic arm *M*, about 20 cm. long, with an index *Z*.

This moves along a scale *S*, that indicates the turning angle of the crystal directly in degrees. The arm *M* can be moved to and fro by a slowly rotating eccentric-pulley. Through the side of the box, placed opposite to one another, two tubes *R* and *U* pass. The former is in communication with the pump by means of an indiarubber tube. In the latter there is another tube *V*, the end of which is covered with a plane-ground plate *p*, having a slit 0.1 mm. wide and furnished with edges of gold. Between this slit and the crystal a screen *I* of lead is placed, with a slit *s* of 2 mm. width. In the tube *V* is a cylindrical shield of lead 1.5 cm. thick, with an opening of about 1 mm. The X-ray tube (of the construction previously* described) is attached to the spectrograph in such a way, that the extended part of it is put into the tube *U*, and then an air-tight elastic connexion obtained by means of picein.

The photographic plate is placed in the casket *G*. This is attached to the sledge *L* by screws passing through the two feet, which are furnished with longilateral openings by means of which the casket can be turned or removed. Through this arrangement the plate (in position 0) can be adjusted perpendicularly to the line first slit—crystal rotation axis, and at the same distance as the slit from the crystal. The sledge is supported by three steel bullets (comp. Moseley), two of which rest in a circular groove, the third in one of the radial cavities. In this way the sledge can take up perfectly fixed positions, the number of which is thirteen, situated at about the same angular distance from each other. These angles have been very carefully determined. In the investigations described in the next paper, only five positions have been used, which are denoted by 0, I–IV. The aperture angle of the casket is about 37° , and thus the plates in two adjacent positions have an angle of 4° – 5° common.

The X-ray tube is evacuated with a molecular air-pump. The fore-pump is employed simultaneously for the molecular air-pump and the spectrograph. If the latter is separated from the X-ray tube through a foil placed on the greased plate *p*, a suitable vacuum can be obtained in a few minutes.

Physical Laboratory, University of Lund,
June 1916.

* *Phil. Mag.* vol. xxxii. p. 39 (1916).

LIII. *On the High-Frequency Spectra (L-Series) of the Elements Lutetium-Zinc.* By EINAR FRIMAN, *Dr. phil.**

[Plate XI.]

IN a preceding paper † Dr. Siegbahn and the author have given the results of an investigation of the high-frequency spectra (L-series) of the elements tantalum-uranium. In this domain we found at least 11 line-groups. With the vacuum spectrograph described in the previous paper (p. 494) I have been able to continue these examinations down to zinc (34 elements). Here I have reached a wave-length of $12\cdot346\cdot10^{-8}$ cm. ($L\alpha_1$ -line of zinc), while the greatest wave-length measured by Moseley was $8\cdot364\cdot10^{-8}$ cm. ($K\alpha$ -line of aluminium).

Most of the elements examined as compounds were finely pulverized and rubbed on the anticathode, which was scratched beforehand with a file, or still better with a knife. By this method, also used in the investigation above-mentioned, a small quantity of the substance (some mgr.) is enough for an experiment. In some cases, however, the elements in metallic form were soldered on the anticathode (Cd, Ag, Pd, Zn) or pressed in tongs of copper soldered on it (In, Ru).

With the elements lutetium-silver very sharp photogrammes ‡ were obtained with a rocksalt crystal as reflector and an exposure time of an hour. In the later experiments a gypsum crystal was used. This also gave several rather sharp lines, but the lines of weak intensity did not appear. As window between the spectrograph and the X-ray tube in the beginning a thin foil of aluminium (0·001 mm.) was used. From tellurium it was replaced by a thin foil of gold-beater's skin that very well transmits the soft rays (Moseley). It has a disadvantage, however, in also transmitting the luminescence light from the anticathode, which often was very strong and blackened the plate. The spectral lines generally were referred to the $K\alpha_1$ -line of copper, the wave-length of which has been very carefully determined by Siegbahn and Stenström §. The reflexion angles calculated have an uncertainty of about 0·3 per cent. As grating-constant of rocksalt the value $2\cdot814\cdot10^{-8}$ cm., given by Moseley, was

* Communicated by the Author.

† *Phil. Mag.* vol. xxxii. p. 39 (1916).

‡ Some of these photogrammes are reproduced in my doctor-dissertation, *Lunds Univ. Arsskr.* N. F. Avd. ii. Bd. xii. Nr. 9 (1916). Here a more detailed account of my investigations is given.

§ *Phys. Zeitschr.* xvii. p. 48 (1916).

TABLE I.

 $\lambda, 10^8 \text{ cm.}$

Element.	l	a_2	a_1	a_3	η	β_4	β_1	β_2	β_3	β_5	γ_1	γ_2	γ_3	γ_4
30 Zn	12346
33 As	9701	9449
35 Br	8391	8360	8141
37 Rb	7335	7305	7091
38 Sr	6879	6639
39 Y	6440	6227
40 Zr	6083	6057	5851	5386
41 Nb	...	5731	5724	5709	5493
42 Mo	...	5410	5403	5381	5175
44 Ru	...	4853	4845	4823	4630
45 Rh	4536	4577	4372
46 Pd	...	4374	4363	4352	4142
47 Ag	...	4156	4146	4133	3929	3720
48 Cd	3959	3733	3515
49 In	...	3774	3766	3550	3381
50 Sn	...	3604	3594	3381	3160
51 Sb	...	3443	3434	3222	2989
52 Te	...	3299	3290	3074	2712
53 I	...	3135	3146	2934	2583
55 Os	...	2899	2891	2684	2350
56 Ba	...	2786	2776	2558	2245
57 La	...	2674	2665	2453	2146
58 Ce	...	2573	2563	2359	2052
59 Pr	...	2472	2462	2259	1937
60 Nd	...	2379	2369	2167	1875
62 Sm	...	2210	2200	2000	1725
63 Eu	...	2131	2121	1918	1662
64 Gd	...	2054	2043	1844	1597
65 Tb	...	1983	1973	...	1935	...	1775	1531	1437
66 Dy	...	1916	1907	1709	1470
67 Ho	...	1854	1843	1646	1415
68 Er	...	1794	1783	1586	1367
70 Yb	...	1681	1670	...	1618	...	1474	1267
71 Lu	...	1629	1619	1421	1224

used. With gypsum this constant was found to be $7.621 \cdot 10^{-8}$ cm., being the mean of two determinations only differing by 0.16 per cent.

The values of the wave-lengths are put together in Table I., and in Plate XI. a graphical representation of the results (from zirconium to uranium) after the relation of Moseley is given. As seen, 14 line-groups seem to be sure in the L-series. Of these the *l*-group was discovered by Siegbahn*, who has examined these lines in the elements tungsten-uranium. The graphical representation shows further, that only in some cases ($\alpha_2, \alpha_1, \beta_2$) a Moseley relation holds good for the whole range, while the other groups generally form curves slightly convex to the X-axis. The order of the rare earths here found agrees with that generally adopted. The order Te-I found by Siegbahn† for the K-series is also verified here.

I have much pleasure in recording my thanks to Dr. M. Siegbahn for suggesting this work to me.

Physical Laboratory, University of Lund,
June 1916.

LIV. *Diffusion Cells in Ionized Gases.* By E. B. WOOD, M.A., O. A. DE LONG, B.S., and K. T. COMPTON, Ph.D., Assistant Professor of Physics, Princeton University‡.

IN a recent paper§ Mr. W. H. Jenkinson has shown very conclusively that an electromotive force is set up between two similar electrodes in a gas if the degree of ionization of the gas surrounding the two electrodes is unequal. The effect of this phenomenon on the relative potentials of an electrode and the surrounding gas had already been investigated by Professor John Zeleny||. The interpretation given by Mr. Jenkinson to his results, however, appears to us deserving of further examination.

Following the analogy of Nernst's treatment of electrolytic cells by the concept of solution tension and solution

* *Verh. d. Phys. Ges.* xviii. p. 150 (1916).

† *Verh. d. Phys. Ges.* xviii. p. 39 (1916).

‡ Communicated by the Authors.

§ "Concentration Cells in Ionized Gases," *Phil. Mag.* xxviii. p. 685 (1914).

|| *Phil. Mag.* xlvi. p. 134 (1898).

pressure, Mr. Jenkinson interpreted his results by the equation

$$E = \frac{RT}{e} (2x-1) \log \frac{p_1}{p_2}, \quad . \quad . \quad . \quad . \quad (1)$$

where p_1 and p_2 are the partial pressures of the ions of either kind just outside the electrodes, x is the proportion of the total current carried by the positive ions, and E is the electromotive force resulting from the unequal ionic concentrations at the two electrodes. According to this equation the sign of the electromotive force should depend on the value of x , *i. e.* on the kind of ion, positive or negative, which effects the greater transfer of electricity.

From the sign of the electromotive force observed in his experiments (the electrode exposed to the ionizing radiations always acquiring a positive charge), Mr. Jenkinson concluded that it must be the positive ions which determine the potential step between a metal and the surrounding gas. He takes from his experiments support of the view that a layer of occluded gas at the surface of a metal is essential to the production of a contact potential effect, and that positive ions only pass from this layer to the surrounding gas.

In view of the increasing importance of discovering the conditions which exist at the surface of a metal, we feel justified in pointing out a much more probable interpretation of Mr. Jenkinson's experiments which involves neither the formation of an occluded gas layer nor the passage of electricity to and from a metal by the sole agency of positive ions.

Theory.—Consider the simplest case, that of two infinite parallel metal plates and a source of ionizing radiations which ionizes the intervening gas more strongly near one plate than the other. If the two plates are originally at the same potential, the ions will move by diffusion from the regions of greater to those of less concentration; but this process will never produce uniform ionic concentration in all parts of the gas because of recombinations which take place simultaneously with diffusion. In this respect the present case differs from the similar case of an electrolyte.

If the positive and negative ions diffuse at the same rate the plates will obviously remain at equal potentials; but if the negative ions diffuse at a more rapid rate than the positive ions, as is usually the case, the plate in the region of less ionic concentration will acquire a negative charge and a negative potential with respect to the other plate. If the plates are insulated, this process will continue until the

electric field is just sufficient to cause the positive and negative ions to drift at equal average rates toward the negative plate, numbers of them recombining as they drift.

Such an explanation of the electromotive force of concentration-cells in ionized gases accounts for the sign of the electromotive force observed by Mr. Jenkinson without introducing any hypothesis regarding the phenomena at the surface of the metal. That it is the correct explanation is indicated by the following experiments.

The process just described may be put in a form for quantitative experimental test by applying Prof. J. J. Thomson's well-known equations for the time rate of increase of the number n of ions per unit volume at any point in the gas :

$$\frac{dn_1}{dt} = q - \alpha n_1 n_2 + D_1 \frac{d^2 n_1}{dx^2} - u_1 \frac{d}{dx} (X n_1),$$

$$\frac{dn_2}{dt} = q - \alpha n_1 n_2 + D_2 \frac{d^2 n_2}{dx^2} + u_2 \frac{d}{dx} (X n_2);$$

where subscripts 1 and 2 refer to the positive and negative ions, respectively. q represents the rate of production of ions by the ionizing agent; α and D are the coefficients of recombination and diffusion, respectively; u is the ionic velocity in a field of unit intensity; X is the electric intensity, and x is the coordinate normal to the plates.

We may eliminate the terms involving q and α by subtraction and integrate the resulting equation. The integration constant vanishes, since the current through the gas vanishes in the final steady state. By Poisson's equation it follows that in the steady state $n_1 = n_2$ at any point in the gas. Making this substitution and integrating again, we find

$$Xd = \frac{D_1 - D_2}{u_1 + u_2} \log \frac{n_a}{n_0},$$

where d is the distance between the plates and n_0 and n_a are the ionic concentrations in the immediate vicinities of the plates. $Xd = E$, where E is the electromotive force, and $D = RTu/e^*$. Thus we have

$$E = \frac{RT}{e} \frac{u_1 - u_2}{u_1 + u_2} \log \frac{n_a}{n_0}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation is given in Winkelmann's *Handbuch der Physik*, 2 ed. vol. iv. p. 616, being there derived by a consideration of partial pressures as in the case of an electrolyte.

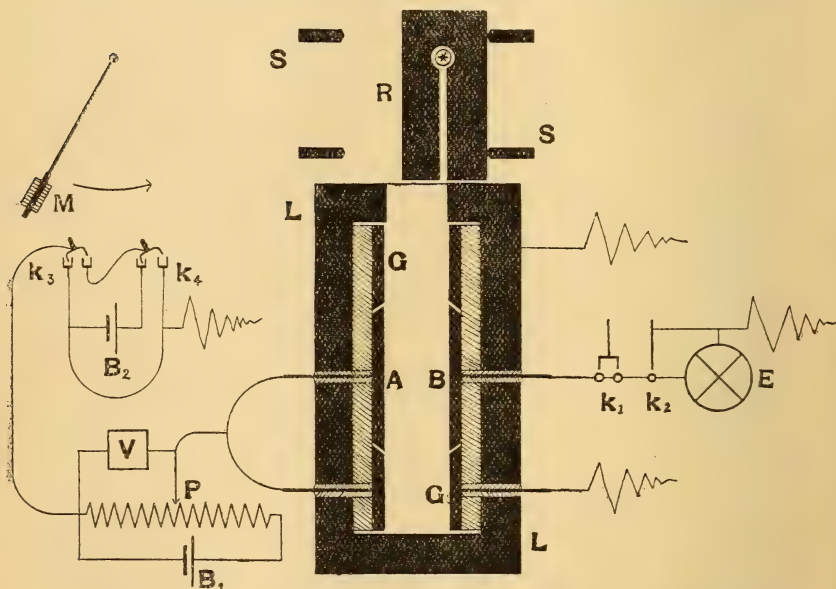
* J. J. Thomson, 'Conduction of Electricity through Gases,' p. 43.

When a current flows through the gas

$$\frac{u_1}{u_1 + u_2} \quad \text{and} \quad \frac{u_2}{u_1 + u_2}$$

give the proportions of the electricity carried by the positive and negative ions, respectively. Furthermore, the partial ionic pressures p are proportional to the ionic concentrations n . Thus we may write equation (2) in the form of equation (1) used by Mr. Jenkinson. The interpretation of x here suggested, however, is typical of a diffusion-cell rather than an ordinary concentration-cell such as Mr. Jenkinson considered.

Experimental Tests.—We tested the validity of equation (2) by an apparatus whose essential features are shown in the diagram. Brass plates A and B were surrounded by guard-rings G, mounted on ebonite blocks and enclosed in the



thick-walled lead box L L, which was lined with brass to avoid disturbances due to contact difference of potential. A tube of radium at R was also enclosed in a lead box provided with a fan-shaped opening. This box could be set against stops S so as to expose the gas near either plate to the ionizing γ rays. The potential divider P enabled the potential of the plate A to be adjusted until the electrometer E connected with the plate B showed no deflexion. The difference

between the voltmeter readings with the radium in the two positions gave twice the electromotive force of the diffusion-cell. An insulated switch, not shown in the figure, enabled the connexions to the two sets of plates to be interchanged.

By means of two switches k_3 and k_4 , actuated by an electromagnet pendulum-bob M, a field of two volts from the battery B_2 could be suddenly applied to the plate A for a definite small interval of time, usually about 0.05 second. During this interval ions of one sign within a fraction of a millimetre of the plate B were swept into the plate, producing a deflexion of the electrometer. This could be repeated with the ionizing source acting near the other plate, or with the connexions to A and B reversed. The ratio of these deflexions gave the ratio n_a/n_0 . All other quantities in equation (2) being known, these measurements enabled a quantitative test of the equation to be made.

Unfortunately our tube of radium did not produce sufficiently strong ionization to permit an accurate measure of the ratio n_a/n_0 , the electrometer deflexions being too small in comparison with a somewhat irregular shift due to an unbalanced induction effect while the field was changed by the switches k_3 and k_4 . We were consequently forced to use an X-ray tube as the ionizing source in this part of the experiment, and the measurements were subject to the somewhat erratic variations in its action. However, with this arrangement, we obtained results of which the following are examples. n_0 and n_a are in terms of the electrometer deflexions noted just after the fall of the pendulum M. The

TABLE I.

n_0 .	n_a .	E (calc.).	E (obs.).
24	7	0.00497	0.0042
18.5	5	0.00528	0.0034
24	5	0.00633	0.0067

electrometer sensitiveness, about 5000 divisions per volt, was such as to make the apparatus very sensitive to disturbing influences, so that we feel that our measurements may be taken as a fair support of the theory. Doubtless, by taking greater precautions to insure steady strong ionization and by increasing the distance between the plates A and B so as to increase the amount of recombination, and therefore the ratio n_a/n_0 , the experimental test could have been made more accurate.

We tested equation (2) in another manner, by substituting dry carbon dioxide gas for air between the plates. In this case, the difference between the ionic velocities u_1 and u_2 is much less than in air, from which we should expect a smaller electromotive force. In order to avoid large changes in the strength of the ionizing source during the substitution of one gas for the other, we used the radium for the ionizing source in this test and increased the electrometer sensitiveness to about 15,000 divisions per volt. The averages of a number of consistent measurements gave $E=0.00090$ volt for air and $E=0.00035$ volt for carbon dioxide. This difference is in the right direction and of the general size predicted by equation (2), though an exact prediction cannot be made because of the difference in the degrees of ionization and recombination in the two gases.

Finally we tried the effect of filling the apparatus with moist carbon dioxide, in which case the sign of the electromotive force should be reversed, owing to the fact that the positive ions diffuse more rapidly than the negative ions in this gas. This test, however, was unsuccessful because the insulation broke down before the measurements could be made.

In conclusion we might mention an attempt to test equation (2) without the necessity for measuring the ionic concentrations at the plates by the aid of the equation

$$\frac{d^2n}{dx^2} + A \frac{dn}{dx} + Bn^2 = 0,$$

which follows from Thomson's equations given earlier in the paper if we put $A=u_2X/D_2$ and $B=-\alpha/D_2$. This equation is of the type of the equation of motion of a particle moving, subject to friction, under the action of a force proportional to the square of the distance from a fixed point, and has not been solved. Approximate solutions indicated that the ratio n_a/n_0 is not independent of n_0 , as we might have anticipated. Thus even with an exact solution of this differential equation it would still be necessary to make some sort of measurement of the ionic concentration, so that this method was not pursued further.

We are glad to acknowledge our indebtedness to Professor E. P. Adams for suggesting approximate solutions of the differential equation mentioned above, and to Professor H. L. Cooke for suggesting the method which we employed for measuring the relative values of the ionic concentrations at the plates.

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LV. *Results of Crystal Analysis.*—III.
By L. VEGARD, *Dr. phil., University of Christiania* *.

[Plate XII.]

§ 1. **A**S an addition to the results of Crystal Analysis given in two previous papers †, I shall give an account of the determination of the structure of the following crystals:

Xenotime (YPO_4),

Anatase (TiO_2).

Both crystals belong to the ditetragonal bipyramidal class of the tetragonal system.

Xenotime.

§ 2. This mineral has a chemical formula analogous to that of zircon (ZrSiO_4); but although the formula may have a similar appearance, there are probably considerable differences with regard to chemical constitution. Thus the two elements associated with oxygen in the case of zircon have both a valency of four, while in the case of xenotime the one element (Y) has a valency of three, the other one (P) a valency of five.

Now, in spite of their chemical differences, the two substances have crystals which, both as regards form and physical properties, are very similar. Both belong to the ditetragonal bipyramidal class, and the ratios of the axes are very nearly the same.

For Zircon $a : c = 1 : 0.6391$,

„ Xenotime $a : c = 1 : 0.6177$.

In a previous communication it was found that in zircon the Zr and Si atoms took up a similar position in the lattice as regards their relation to the oxygen atoms. The arrangement corresponds to a formula ZrO_2SiO_2 and not $\text{Zr}(\text{SiO}_4)$. In view of the crystallographic similarity, the question arises whether also the atoms in xenotime might be arranged in a zircon lattice, which would mean that Y and P—in spite of their difference of valency—would have the same relation to

* Communicated by the Author.

† L. Vegard, *Phil. Mag.* January 1916, Paper I.; *ibid.* July 1916, Paper II.

Phil. Mag. S. 6. Vol. 32. No. 191. Nov. 1916. 2 M

the oxygen atoms. Thus the determination of the structure of xenotime is of considerable importance for the question of the part played by the valency in the constitution of solids.

The experimental determination of the X-ray spectra of the different faces is somewhat difficult on account of the fact, that most crystals are more or less transformed into amorphous mass, are partly "metamict," and as a consequence, it was impossible to get reflexion from a number of specimens tested.

At last I got from the Geological Laboratory some fresh and pure samples which gave fairly good reflexion for some of the faces.

The spectra of the faces (100) and (110) were obtained from a sample from the Langesundsfjord, Norway; those of the faces (111) and (101) from a crystal from Raade, Norway.

Several attempts to get reflexion from the base (001) were not successful.

The glancing-angles (θ_n) and intensities (I_n) of the observed spectra for the four faces are given in Table I. and in fig. 1.

TABLE I.

YPO₄.

<i>n.</i>	(100).		(110).		(101).		(111).	
	θ_n .	I_n .	θ_n .	I_n .	θ_n .	I_n .	θ_n .	I_n .
1.....	7° 16'	1.4	5° 9'	6.0	6° 58'	3.5	...	0
2.....	14 39	1.0	10 21	3.5	14 2	0.6	...	0
3.....	15 37	1.5	...	0	...	0
4.....	15° 42'	1.5

In the elementary cell $d_{100}^2 d_{001} = d_{100}^2 \frac{c}{a}$, we find from observations a number of molecules equal to 0.127 or $\frac{1}{8}$, which is the same as was found for zircon.

In a similar way, as in the case of zircon, we are led to a similar space-lattice for the Y and P atoms, as found for the Zr and Si atoms in zircon. If so, both the P and Y atoms are arranged in a lattice of the diamond type.

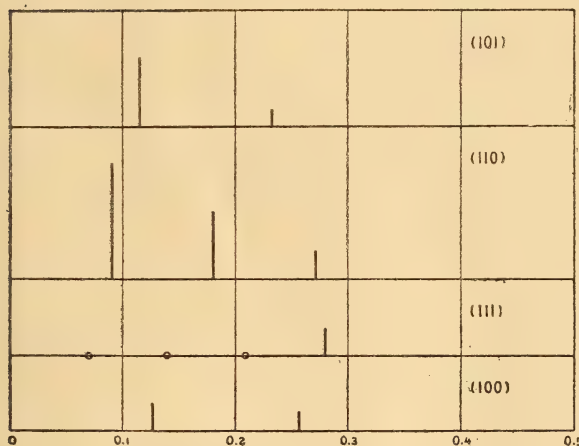
The lattice of P can be made to cover that of Y by a translation along the tetragonal axis a distance $c/2$.

The spacings of the four faces are in the relation :

$$d_{100} : d_{110} : d_{101} : d_{111} = 1 : 1.409 : 1.043 : 0.468,$$

Fig. 1.

Xenotime



or approximately

$$= 1 : \sqrt{2} : \frac{2}{\sqrt{1 + \left(\frac{a}{c}\right)^2}} : \frac{1}{\sqrt{2 + \left(\frac{a}{c}\right)^2}} \dots (1a)$$

If for a moment we do not take into consideration the oxygen atoms, but only the Y and P atoms, the lattice would give the following relation :

$$d_{100} : d_{110} : d_{101} : d_{111} = 1 : \sqrt{2} : \frac{2}{\sqrt{1 + \left(\frac{a}{c}\right)^2}} : \frac{4}{\sqrt{2 + \left(\frac{a}{c}\right)^2}} \dots (1b)$$

We see that the relations (1a) and (1b) are the same except for the face (111), where the spacing observed is only $\frac{1}{4}$ of the value to be expected from the lattice.

The question is now, whether the oxygen atoms might be arranged in such a way as to make the three first maxima of the (111) face disappear or become vanishingly small.

Arrangement of Oxygen Atoms.

§ 3. As far as the Y and P atoms are concerned, we have assumed a lattice of the Zircon type. The first question we shall put is: Can such an arrangement of the oxygen atoms as that found for zircon explain the spectra of xenotime?

The spectra of the faces (100), (110), (101), might be accounted for by a proper choice of the parameters, and the test will be whether the zircon lattice can explain the peculiar spectrum of the face (111), in other words, explain the fact that the first three spectra are too weak to be detected. With regard to the determination of the intensities I shall only refer to my previous paper (II. p. 84).

The functions $f_1(n)$ and $f_2(n)$, which determine the strength of the spectrum of order n , are for the (111) face of an arbitrary lattice of the Zircon type given by the expressions (II. eq. 12)

$$\begin{aligned} f_1(n) = & N_1 + (-1)N_2 + (N_1 + N_2 + 4N_3) \cos n \frac{\pi}{2} \\ & + 2N_3 (\cos n\alpha_1 + (-1)^n \cos n\alpha_2), \\ f_2(n) = & (N_1 - N_2) \sin n \frac{\pi}{2}. \end{aligned}$$

N_3 is the atomic number of oxygen; N_1, N_2 those of the two other elements. In our case we have to put $N_1=39$, $N_2=15$, $N_3=8$.

We can, however, prove that there is no value of the parameters which can give a distribution of intensities in accordance with observations. Thus it is impossible to make the first order spectrum sufficiently small as compared with the fourth.

Remembering that the amplitude $A_n = \sqrt{f_1(n)^2 + f_2(n)^2}$, we shall find that the smallest value of A_1 is 24, and the largest value of A_4 is 172. This would mean that the first order spectrum should be at last half as strong as that of the fourth order, and the spectra of the second and third orders would not be sufficiently weak at the same time.

Now the maximum observed was not very strong, but we should easily have detected a peak equal to $\frac{1}{6}$ of that observed for the fourth order spectrum. *We can then safely conclude that the oxygen atoms in xenotime cannot be arranged in a lattice of the Zircon type.*

If in the lattice proposed for the Y and P atoms of xenotime, both sorts of atoms were replaced by atoms of the same reflecting power, the first three orders of the face (111) would vanish. Then we must arrange the oxygen atoms so

as to increase the weight of the P planes so that the group of P and O atoms can balance the heavier Y atoms, and we are led to try to arrange the oxygen atoms tetragonally round the P atoms.

There are mainly two different arrangements possible :—

(1) The oxygen atoms are arranged on a line through the P atoms parallel to the tetragonal axis. This would make the spectra of the faces (110) and (100) normal; but in the case of the face (100) the second order spectrum is nearly as strong as that of the first order, consequently such an arrangement is excluded.

(2) Four oxygen atoms are arranged in a plane through the P atoms perpendicular to the tetragonal axis.

In order to keep up the highest symmetry class there will only be two arrangements of this description possible :

(a) The oxygen atoms are placed on lines through the P atoms which are parallel to the sides of the base ; (b) they are placed on lines through the P atoms which are parallel to the diagonals of the square base of the lattice.

Let us first consider the first possibility (a), and calculate the intensities for the faces (111) and (110).

After some reduction we find for the amplitudes,

$$\left. \begin{aligned} (111) \quad A_n &= (0.47 + (-1)^n 1.22 + \cos n\alpha) \cos \frac{\pi n}{4}, \\ (110) \quad A_n &= 1.69 + \cos 2n\alpha, \quad . \quad . \quad . \quad . \quad . \end{aligned} \right\} \quad (2)$$

where α is a parameter determining the distance between P and O.

In order to explain the disappearance of the three first spectra of (111) we should have to choose a value of α smaller than 25° , because a greater value would make the third order spectrum too strong not to be detected ; and even for $\alpha < 25^\circ$ the first order spectrum would probably be too strong to escape detection. Further, we cannot assume a very small value of α , because it would bring the centres of the P and O atoms of the group too close together.

Finally, a value of $\alpha < 25^\circ$ would produce a distribution of intensities of the face (110) which is not in accordance with observations. The observations give for the intensities of the first three orders the relative intensities 100, 58, 25, showing that the intensities of the second and third orders are larger than in the normal spectrum. In other words, the oxygen atoms must be arranged in such a way that

$$A_1 < A_2 < A_3,$$

while for $\alpha < 25^\circ$ the arrangement considered would give

$$A_1 > A_2 > A_3.$$

calculated and observed values for the other three faces, we put $\alpha = 65^\circ$.

In the following Table II. are given the calculated intensities corresponding to this value of α , as also the observed intensities.

TABLE II.

$$\alpha = 65^\circ.$$

(111).		(110).		(100).		(101).	
I _{cal.}	I _{obs.}	I _{cal.}	I _{obs.}	I _{cal.}	I _{obs.}	I _{cal.}	I _{obs.}
0.4	<20	100	100	100	100	100	100
0	<20	45	58	74	70	9	17
12	<20	20	25	1	0
100	100

The agreement between calculated and observed values is quite satisfactory.

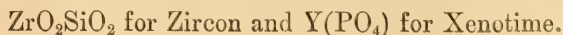
§ 4. Comparing the lattices found for the xenotime with that of zircon, we see that the lattice at the same time accounts for the crystallographic similarity and the difference in chemical constitution.

Both lattices have the symmetry properties which are necessary to give a crystal of the ditetragonal bipyramidal class.

If we would imagine the oxygen atoms removed the two lattices would be exactly of the same type.

In both lattices the same relative number of oxygen atoms are arranged with tetragonal symmetry in a plane perpendicular to the tetragonal axis, which accounts for the fact that the ratio c/a is almost the same for both minerals.

The difference found for the arrangement of the oxygen atoms accounts for the different chemical constitution to be expected for the two minerals. In zircon both Zr and Si take up essentially the same position towards the oxygen atoms; in xenotime, however, the oxygen atoms are attached in a singular way to the P atoms and we get groups (radicles) of the composition PO_4 . Thus the Röntgen-ray analysis of the space-lattice gives as the constitution formulæ:



Anatase (TiO₂).

§ 5. This mineral is isomeric with rutile, but although both crystals belong to the ditetragonal bipyramidal class of the tetragonal system they are not isomorphous. For rutile the ratio $c:a$ is equal to 0.644, for anatase it is equal to 1.777.

Also, in the case of anatase, we had some difficulty in obtaining reflexion from a sufficient number of faces; but finally we succeeded in finding a number of maxima for the following five: (100), (001), (110), (111), and (112).

The positions and intensities of the observed maxima are given in Table III. and in fig. 3.

TABLE III.

n.	(001).		(100).		(110).		(111).		(112).	
	θ.	I.	θ.	I.	θ.	I.	θ.	I.	θ.	I.
1.....	7° 30'	10	13° 19'	5.8	9° 21'	3.2	5° 0'	4.6	11° 48'	1.5
2.....	15 8	0.6	27 26	1.4	18 58	0.8	10 2 0	0
3.....	23 3	0.15	15 9 0.3
4.....	32 50	0.14	20 24 0.5

If we would write the formula $(\text{TiO}_2)_2$ as in the case of rutile, we should find that only $\frac{1}{16}$ of a molecule is associated with the elementary cell $d_{100}^2 d_{001}$, or just half the number found for the rutile cell.

For the ratios of the spacings we get:

$$d_{100} : d_{110} : d_{111} : d_{001} : d_{112} = 1 : 1.418 : 2.643 : 1.765 : 1.126,$$

or approximately

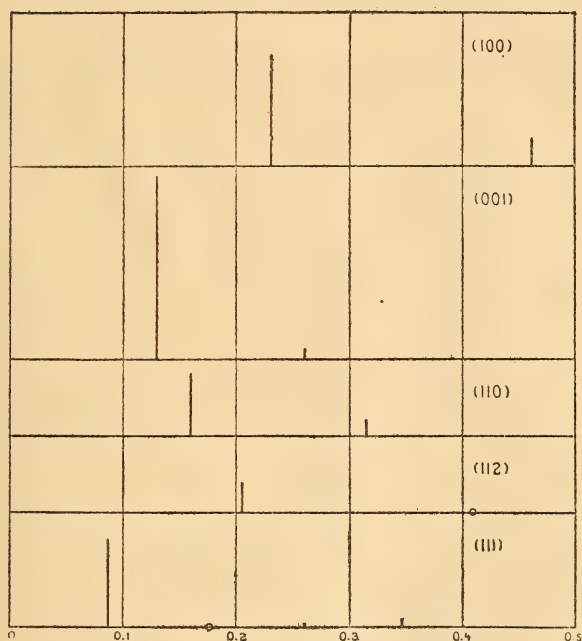
$$= 1 : \sqrt{2} : \frac{4}{\sqrt{2 + \left(\frac{a}{c}\right)^2}} : \frac{c}{a} : \frac{1}{\sqrt{\left(\frac{a}{c}\right)^2 + \frac{1}{2}}}.$$

These are exactly the ratios which correspond to a lattice of the diamond type which is drawn out in the direction of the tetragonal axis. For the sake of comparison we remember that in the case of the isomeric substance, rutile, the lattice of the Ti atoms is composed of two lattices of the diamond type put inside one another in such a way that the elementary cell is not altered. The lattices give the

right number of atoms in the elementary cell and explain the fact that the numbers in anatase are just half the number found for rutile. We should probably state this fact in the

Fig. 3.

Anatase



best way by writing the molecular formula of anatase TiO_2 , and that of rutile $(\text{TiO}_2)_2$, then we should have in both crystals $\frac{1}{2}$ molecule associated with the elementary cell.

The Position of the Oxygen Atoms.

§ 6. A fact of the first importance for the determination of the positions of the oxygen atoms is, that the faces (100) and (001) give spectra with very different relative intensities. This fact excludes any pseudocubic arrangement, or the tetragonal structure is also in the present case to be ascribed to a tetragonal arrangement of the oxygen atoms.

The oxygen atoms cannot be arranged with tetragonal symmetry in a plane perpendicular to the c -axis, as in the case of zircon and xenotime. Such an arrangement would

make the (001) spectrum normal, contrary to the observations, which give an abnormally weak second order spectrum.

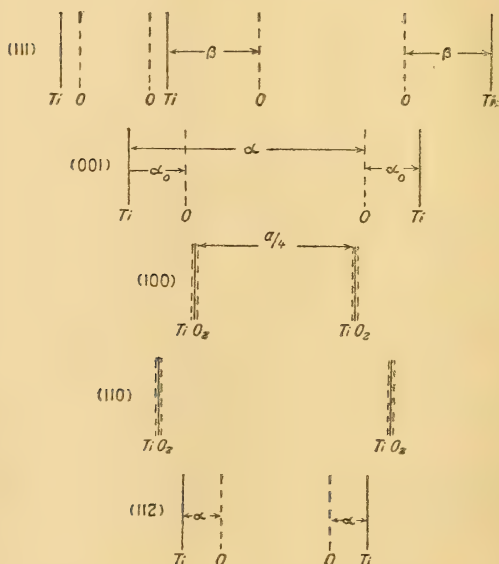
We are then naturally led to assume the following arrangement :

Two oxygen atoms are associated with each Ti atom with the latter in a central position forming a kind of molecular element. In the case of anatase all the molecular axes are parallel to the c -axis.

The spacings corresponding to this arrangement are shown in fig. 4.

Fig. 4.

Anatase



We see that the faces (100) and (110) should give a normal spectrum, and that the relative intensities of the faces (001) and (112) should be equal. Both consequences are in agreement with observations; it is only to be remarked that in the case of the face (112) the reflexion is so weak that the very weak spectra of second and third order could not be detected at all.

The arrangement has one parameter (the length of the molecular axis), and it remains to be seen whether a proper

choice of the parameter can give the right intensities for the spectra (001) and (111).

The amplitudes are determined by the following expressions:

$$\left. \begin{aligned} (001) \quad A_n &= 1.375 + \cos n\alpha, \quad . \quad . \quad . \\ (111) \quad A_n &= (1.375 + \cos n\beta) \cos \frac{n\pi}{4}, \\ \beta &= \frac{\alpha}{4} = 2\pi \frac{l}{c}, \quad . \quad . \quad . \end{aligned} \right\} \quad . \quad . \quad (4)$$

where l is the distance from the Ti atom to one of its oxygen atoms.

It follows from the lattice that $l/c < \frac{1}{2}$ or $\beta < \pi$, and $\alpha < 4\pi$. Let the smallest value of α which gives the right intensities of (001) be α_0 , then the amplitudes of (001) will be unaltered if we interchange α_0 by the values:

$$2\pi - \alpha_0, \quad 2\pi + \alpha_0, \quad 4\pi - \alpha_0.$$

Which of these is the right one must be determined from the (111) spectrum.

The right value of α_0 is very nearly equal to 60° , and from (111) we find $\alpha = 2\pi - \alpha_0 = 300^\circ$ and $\beta = 75^\circ$.

Table IV. gives the calculated and observed intensities for the two faces.

TABLE IV.

$$\beta = 75^\circ.$$

(001).		(111).	
I _{cal.}	I _{obs.}	I _{cal.}	I _{obs.}
100	100	100	100
7.6	6	0	0
0.5	1.5	2.3	6.6
0.8	1.4	9	11

The calculated and observed values agree as well as can be expected, and we have no doubt found the right lattice for anatase with the right symmetry properties.

Also in the case of anatase the tetragonal structure is produced by the oxygen atoms.

The molecular element must require much more space in the direction of its axis than in a direction perpendicular to it. As all molecular axes are parallel to the tetragonal axis,

the lattice must be drawn out in this direction, and also in this case the position of the oxygen atoms accounts for the value of c/a , which in this case is greater than unity.

To make clear the relation between the lattices of anatase and rutile, we imagine that in the latter substance we remove one of its two Ti lattices of the diamond type and the oxygen atoms associated with the Ti atom. The lattice left with the molecular axes in tetragonal arrangement in planes perpendicular to the tetragonal axis does not seem to be stable. If, however, all molecular elements are turned through an angle of 90° , so as to have their axes all parallel with the tetragonal axis, the configuration becomes stable and forms the mineral anatase.

Photographs of models of the lattices of xenotime and anatase are shown on Plate XII.

§ 7. The absolute dimensions of the lattices are given in Table V. For the sake of comparison I have also given the dimensions of the Zircon group.

TABLE V.

Substance.	a .	c .	c/a .	V .	l .
ZrO_2SiO_2	10^{-8} cm. 9.20	10^{-8} cm. 5.87	0.639	10^{-22} cm. ³ 4.97	10^{-8} cm. (Zr) 2.71 (Si) 1.08
$(SnO_2)_2$	9.35	6.29	0.673	5.50	2.08
$(TiO_2)_2$ (Rutile)	9.05	5.83	0.644	4.77	1.99
TiO_2 (Anatase).	5.27	9.37	1.777	2.60	1.95
YPO_4 ..	9.60	5.94	0.618	5.49	1.23

V is the volume of the elementary lattice, l is the distance from an oxygen atom to the central atom to which it belongs.

We see that also with regard to the absolute dimensions of the lattice xenotime comes very close to the Zircon group.

The minerals of the Zircon group show a small, but regular increase of dimensions with increase of atomic number of the central atom of the molecular group. Comparing the dimensions of rutile and anatase, we notice that in the direction which is perpendicular to all molecular axes the linear dimensions are nearly equal, and the value of a of rutile is nearly equal to the value of c of anatase. This may be due to the fact that in both cases there is the same number of molecular elements inside the lattice which have their molecular axis directed along the axis considered.

We may notice that the absolute dimensions are calculated

from the measured glancing-angle and the known wavelength. If V' is the volume of the elementary cell, M and ρ molecular weight and density, n the number of molecules in the cell, N the number of molecules in a gramme-molecule (61.5×10^{22}), we have the relation :

$$V'\rho = n \frac{M}{N}.$$

If we knew ρ and n and had measured V' , we might find M . As the glancing-angle can be found with a very great accuracy, the reflexion of Röntgen rays from crystals might furnish us with a valuable and accurate method of determining the atomic weight of an element which had entered into combination with elements of known atomic weight.

If we only want ordinary relative values of the atomic weight, the accuracy mainly depends on the accuracy of the glancing-angle, and not on the number N . To fix the idea, let us suppose that we have two cubic crystals, one of which we take as a standard. Let the glancing-angle of the cube faces be θ_0 and θ , then

$$\frac{M}{\rho} = \frac{M_0}{\rho_0} \frac{n_0}{n} \left(\frac{\sin \theta_0}{\sin \theta} \right)^3.$$

Summary.

(1) The crystalline structure of xenotime $Y(PO_4)$ and anatase (TiO_2) has been determined.

(2) The lattice of xenotime is not of the zircon type ; the difference, however, is only due to a different arrangement of the oxygen atoms. In xenotime they are arranged in groups of four round each P atom, while in zircon both Zr and Si have essentially the same relation to the oxygen atoms. The lattice gives the constitution formulæ $Y(PO_4)$ and ZrO_2SiO_2 for xenotime and zircon respectively.

(3) In spite of the different arrangement of the oxygen atoms, which accounts for the chemical differences, the lattice also accounts for the crystallographic similarity of the two substances. Both possess the right symmetry, and in both cases the same relative number of oxygen atoms are arranged in planes perpendicular to the tetragonal axis, and will expand the lattice perpendicular to this axis to the same extent and make the ratio c/a smaller than unity and almost equal for both crystals.

(4) We can imagine the lattice of anatase to be derived

from the zircon lattice by removing the Zr atoms and the oxygen atoms associated with them, and substitute the Si with Ti atoms. The oxygen atoms seem no longer to be able to remain in the position they had in zircon, the molecular axes all turn through an angle of 90° so as to become parallel to the tetragonal axis, and so form the stable lattice of anatase.

(5) Also, in the case of anatase, the deformation of the lattice from the cubic form is due to the oxygen atoms. The molecular axes being all parallel, the lattice most expands in the direction of the tetragonal axis to make room for the oxygen atoms, and thus make the ratio c/a greater than unity.

(6) The absolute dimensions of the lattices of zircon and xenotime are very nearly the same. The volume of the lattices of the Zircon group shows a small but regular increase with increasing atomic number.

Also, the distance from a central atom to one of the oxygen atoms associated with it increases with increase of atomic number of elements belonging to the same family. Generally the distance is smaller for a greater affinity to oxygen.

In conclusion, I wish to express my indebtedness to Professor W. C. Brögger, for supplying me with the crystals necessary for the research, and to Mr. H. Schjelderup for valuable assistance in making the observations.

LVI. *A Critical Test of the Crystallographic Law of Valency Volumes; a Note on the Crystalline Structure of the Alkali Sulphates.* By A. OGG, M.A., Professor of Physics, University College, Grahamstown, South Africa, and F. LLOYD HOPWOOD, M.Sc. (Lond.), A.R.C.Sc., Assistant in Physics, University of London, University College*.

THE complete investigation of the structure of crystalline substances involves two distinct inquiries. The first is concerned with the various ways in which the structural units may be arranged to form a homogeneous structure, and the second with the nature of these units.

While it is generally believed that the geometrical theory of crystal structure is now complete, the laws governing the relations between crystalline structure and chemical

* Communicated by Prof. W. H. Bragg, F.R.S.

constitution are but imperfectly understood. One of the generalizations in this connexion which has gained wide support, and is commonly accepted, is the theory due to Pope and Barlow, known as the Law of Valency Volumes. According to this theory*, the entire space occupied by a crystal can be regarded as a close-packed assemblage of approximately spherical cells of various sizes, representing by their relative volumes the spheres of influence of the component *atoms* of any particular crystalline structure. The volumes of the cellular domains allotted as above are further supposed to be approximately in the ratio of the integers which respectively express the fundamental chemical valencies of their contained *atoms*.

There is a certain amount of indirect evidence† against the truth of this theory; but the application of the X-ray method of investigating crystal structure now furnishes us, for the first time, with incontrovertible evidence of its failure in a typical case.

In the present paper the results of an investigation of the crystalline structure of the sulphates of potassium, rubidium, ammonium, and cæsium are set forth and discussed in reference to the Law of Valency Volumes.

These crystals all belong to the orthorhombic system, and form one of the best-known examples of an isomorphous series.

The crystals were examined in a Bragg X-ray spectrometer, using an X-ray bulb with a palladium anticathode.

It was soon evident that the elementary cell, or unit rhomb, of each of these crystals contained four molecules.

Thus it was found that the glancing angle in the second-order spectrum reflected from the (100) face of potassium sulphate was $5^{\circ} 51'$. Substituting this value, and that of the appropriate wave-length of the X-radiation used, in the usual formula

$$n\lambda = 2d \sin \theta,$$

we find

$$d_{100} = 5.73 \times 10^{-8} \text{ cm.}$$

Now the molecular weight of K_2SO_4 is 173.04, its density is 2.666 gms./cm.³, and its axial ratios

$$a:b:c = 0.5727:1:0.7418 \ddagger.$$

* *Vide* 'Annual Reports on the Progress of Chemistry,' vol. v. (1908) pp. 268 *et seqq.*; also 'Mineralogical Magazine,' vol. xvii. (April, 1916) pp. 314-323.

† Tutton, 'Crystalline Structure and Chemical Constitution' (Macmillan & Co., 1910), pp. 123 *et seqq.*

‡ Tutton, 'Crystalline Structure and Chemical Constitution,' p. 119.

Taking the mass of the hydrogen atom as 1.64×10^{-24} gram., the mass of unit rhomb, if it contains 4 molecules, is

$$4 \times 173.04 \times 1.64 \times 10^{-24} \text{ or } 113.51 \times 10^{-25} \text{ gram.},$$

$$\text{and } abc \times 2.666 = 113.51 \times 10^{-25};$$

and hence, from the known ratios of $a:b:c$, we find

$$a = 5.731 \times 10^{-8} \text{ cm.},$$

$$b = 10.008 \times 10^{-8} \text{ cm.},$$

$$c = 7.424 \times 10^{-8} \text{ cm.}$$

The equality in the values of d_{100} and a shows that the assumption made in our calculation, that there are 4 molecules in the elementary cell, is correct.

TABLE I.

Crystal.	Molecular weight.	Density at $20^{\circ}/4^{\circ}$.	Molecular volume.	Axial ratios. $a:b:c$.
K_2SO_4	173.04	2.666	64.91	0.5727 : 1 : 0.7418
$(NH_4)_2SO_4$	131.20	1.772	74.04	0.5635 : 1 : 0.7319
Rb_2SO_4	265.14	3.615	73.34	0.5723 : 1 : 0.7485
Cs_2SO_4	359.14	4.246	84.58	0.5712 : 1 : 0.7531

Crystal.	Length of sides of unit rhomb.			Volume of unit rhomb.
	a .	b .	c .	
K_2SO_4	cms. 5.731×10^{-8}	cms. 10.008×10^{-8}	cms. 7.424×10^{-8}	cm. ³ 425.78×10^{-24}
$(NH_4)_2SO_4$	5.951×10^{-8}	10.560×10^{-8}	7.729×10^{-8}	485.71×10^{-24}
Rb_2SO_4	5.949×10^{-8}	10.394×10^{-8}	7.780×10^{-8}	481.14×10^{-24}
Cs_2SO_4	6.218×10^{-8}	10.884×10^{-8}	8.198×10^{-8}	554.88×10^{-24}

In Table I. the volumes of the elementary cells, and the relative and actual values of the lengths of their sides, are shown for the whole series of crystals.

The most striking fact disclosed by the above table is the very close agreement, almost amounting to identity, in the volumes of the unit rhombs of rubidium sulphate and

ammonium sulphate*. This is of fundamental importance. It shows us that the replacement in the elementary cell of *eight* atoms of potassium by *forty* atoms of the four radicle groups NH_4 , causes no more distension of structure than if eight atoms of rubidium had replaced the eight atoms of potassium. Furthermore, the substitution of *eight* atoms of caesium for *eight* atoms of potassium causes *double* the distension of structure that the substitution of *forty* atoms of the four radicle groups NH_4 does.

To the authors this appears to be conclusive evidence against the general truth of the theory of crystal structure based on the closest packing of the constituent *atoms* or their spheres of influence.

Further evidence in support of the fact that the elementary cells of each of the above-mentioned crystals contain four molecules, and that the space-lattices of ammonium sulphate and rubidium sulphate are strictly comparable with each other, is exhibited in Table II.

This table shows the agreement between the values of the glancing angles calculated on the above assumption, and the observed values, for the whole series. The wave-length of the particular line in the X-ray spectrum of palladium used was 584×10^{-8} cm.†, except in the case of the rubidium and caesium crystals. The reflexions from some of the faces of these two crystals were rather faint, and were taken with slits too wide to entirely cut out the effects due to $\lambda = 516 \times 10^{-8}$ cm.

In general, there was no difficulty in separating these two lines.

The complete structure of this series of crystals has not yet been worked out. With the object of doing this, the authors contemplated investigating the related series of crystals—the alkali selenates; but this work has been interrupted. Some information, however, may be deduced from Table II.

The absence of the first- and third-order spectra from the (100) planes shows that the spacings are equal to one-half of the sides of the elementary cell. For the (010) planes the fourth order is strongest, the second being found only in the case of ammonium sulphate. The fourth order is also

* Since the sum of the fundamental valencies of $(\text{NH}_4)_2\text{SO}_4$ and Rb_2SO_4 are respectively 24 and 12, the volume of the elementary cell of the former should, according to Pope and Barlow, be double that of the latter crystal.

† Really the two lines $\lambda = 5828 \times 10^{-8}$ and $\lambda = 5872 \times 10^{-8}$.

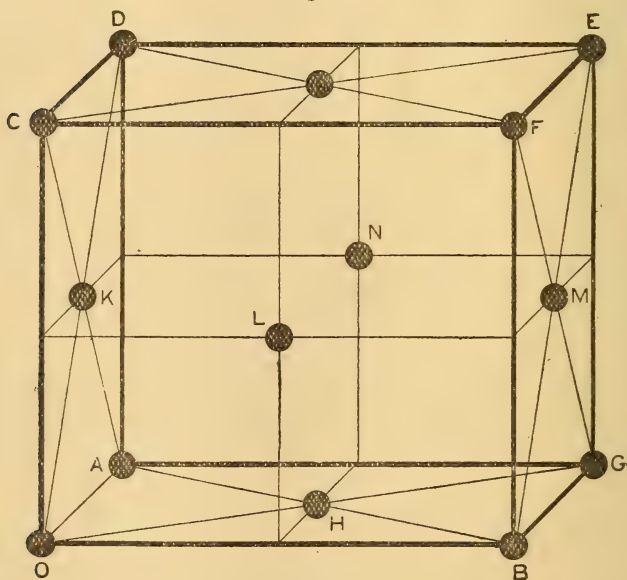
TABLE II. (continued).

Crystal.	Face.	Glancing angle.		Face.	Intensity observed.	Glancing angle.		Face.	Intensity observed.	Glancing angle.		Face.	Intensity observed.
		Calc. for $\lambda = 0.584 \text{ A.U.}$	Observed.			Calc. for $\lambda = 0.584 \text{ A.U.}$	Observed.			Calc. for $\lambda = 0.584 \text{ A.U.}$	Observed.		
K_2SO_4	111	4° 3'	4° 7'	110	75	3° 22'	6° 45'	021	0	4° 2'	4° 15'		80
		8 7	8 15		210	6 45	10 15		90	8 5	8 17		
		12 15			0	10 9			215	12 11			
		16 24			0	13 35			0	16 21			
		20 41			0	17 4			0	20 36			
$(\text{NH}_4)_2\text{SO}_4$...	111	25 5		110	0	20 38		021	0	24 59			370
		3° 53'	3° 53'		420	3° 14'	3° 12'		105	3° 50'	3° 59'		
		7 48	7 42		160	6 28	6 18		170	7 42	7 54		
		11 44			0	9 43	9 28		90	11 36			
		15 44	15 30		30	13 1			0	15 32			
Rb_2SO_4	111	19 49		110	0	16 21		021	0	19 34		130	128
		24 0			0	19 45			0	23 42			
		3° 53'	8° 0'		0	3° 14'			0	3° 52'	5° 40'		
		7 48			67	6 29			0	7 46	11 15		
		11 45			0	9 46	9° 50'		16	11 41	17 1		
Cs_2SO_4 ...	111	15 45		110	0	13 5		021	0	15 41		130	40
		19 50			0	16 25			0	19 45			
		24 2			0	19 50			0	23 55			
		3° 43'	7° 20'		0	3° 6'			No reflexion.	3° 41'	5° 21'		
		7 26			30	6 12				7 24	10 44		
Cs_2SO_4 ...	111	11 12		110	0	12 30		021		11 8	16 14	130	80
		15 0			0	9 20				14 55	10 57		
		18 53			0	15 41				18 47	21 53		
		22 52			0	18 56				22 44			

strongest in the reflexions from the (001) planes, except in the case of ammonium sulphate, which gives a stronger second order. These general considerations lead us to think that there are planes of sulphur atoms with spacings equal to half the side of the unit rhomb, with planes of metal atoms lying between them. The presence of a very strong first-order spectrum from the (111) planes of ammonium sulphate would seem to indicate a face-centred lattice for the sulphur atoms.

It appears, then, that a lattice with representative points at the corners and at the middle points of the faces (see fig. 1) would explain the spectra and give four molecules per unit rhomb.

Fig. 1.



The dimensions of the crystals are such that the triangles OHA and BHG are nearly equilateral. It has therefore been suggested by von Federow that the space-lattice of the rhombic sulphates is pseudo-hexagonal. Tutton considers the evidence so strong for such a structure that he has adopted it.

The arrangement we have suggested would give the representative points in the (001) planes at the corners of hexagonal prisms. But the plane KLMN is derived from

OAGB by sliding, and therefore the lattice would not be hexagonal. To make the system hexagonal, the representative points on the plane KLMN would have to be at the middle points of the sides and the centre of the rhomb. The middle plane must, however, be different in some way from the end planes. This might, of course, be brought about by the oxygen atoms.

Without entering further into a detailed analysis of the conditions which will satisfy the symmetry and explain the intensities of the various orders of the spectra, we at this stage merely wish to indicate the probability of the sulphur atoms being located at the corners and centres of the faces of the unit rhomb. This would give a pseudo-hexagonal structure for the sulphur atoms in planes parallel to the (001) face.

The metals are also probably arranged in hexagons.

Summary.

An account is given of an X-ray investigation into the crystalline structure of the Isomorphous Alkali Sulphates. It is shown that for each of these crystals there are four molecules in the elementary cell. It follows from this that the dimensions of the crystal units of ammonium sulphate and rubidium sulphate are almost identical. This furnishes a critical test of the Valency Volume Theory of Pope and Barlow, which is shown to break down in this case. Further evidence in support of this contention is provided by the agreement between the observed and calculated values of the angles of reflexion from these crystals. Although the complete structure of the series is not solved, the probability of the hexagonal arrangement of the metal and sulphur atoms is pointed out.

Most of the crystals used in this investigation were lent by Dr. A. E. H. Tutton, F.R.S. They had been prepared with optically worked faces and used by him for his classical work on the thermal expansion of crystals. We have to thank him for his very great kindness in this matter, and also for the keen interest which he took in the work.

It is also a pleasure to record our thanks to Professor W. H. Bragg for the valuable assistance and advice which he was ever ready to give during the progress of the measurements.

University College, London,
April 1916.

LVII. *Notices respecting New Books.**A Theory of Time and Space.* By ALFRED A. ROBB.

1914. Cambridge University Press.

MINKOWSKI'S mathematical treatment of the theory of Relativity reduces time to imaginary space, and presents the inter-relations of space and time in terms of the properties of a four-dimensional space. Mr. Robb in this truly remarkable book approaches a similar outlook by an altogether different route. The presentation is geometrical and of the approved modern type. That is to say, there are laid down certain postulates from which, with the aid of a series of closely connected theorems, there is constructed a geometry in which every element is determined by four coordinates. The argument is difficult to follow; and the full bearing of the conclusions reached is not easy to appreciate. The author is meanwhile content with having established a consistent geometry, and leaves developments and applications for a further volume. The fundamental notion which is the basis of the whole is the recognition of what Mr. Robb calls conical order. According to this conception, an event which is neither before nor after another is not necessarily simultaneous with it. The only events which are really simultaneous are those which occur also at the same place. When events occur at different places we may be able to say that one is neither before nor after the other, but we cannot say that they are simultaneous. In the four-dimensional space which is thus imagined a point represents a state of a particle at a given time, and "the theory of space becomes absorbed in the theory of time."

LVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 176.]

June 28th, 1916.—Dr. Alfred Harker, F.R.S., President,
in the Chair.

THE following communications were read:—

1. 'On a New Species of *Edestus* from the Upper Carboniferous of Yorkshire.' By A. Smith Woodward, LL.D. F.R.S., V.P.G.S. With a Geological Appendix by John Pringle, F.G.S.

2. 'The Tertiary Volcanic Rocks of Mozambique.' By Arthur Holmes, B.Sc., A.R.C.Sc., D.I.C., F.G.S.

Until recently, the district of Mozambique—geographically as well as geologically—was one of the least known of the East African coast-lands. During the seasons 1910–11, a prospecting

expedition was organized by the Memba Minerals Ltd., and during the second season, Mr. E. J. Wayland, Mr. D. A. Wray, and the author visited the country as geologists to the Company.

With the exception of a coastal belt of Cretaceous and Tertiary sediments, flanked on the west by later Tertiary volcanic rocks, the whole territory consists of a complex of gneisses and other foliated rocks, intruded upon by granites belonging to at least two different periods. From Fernão Velloso Harbour to Mokambo Bay the junction of the sedimentary formations with the crystalline complex is a faulted one, and the volcanic rocks, now greatly dissected by erosion, are distributed on each side of the fault. The lavas are of post-Oligocene age, and are clearly the result of fissure-eruptions, the feeding channels being exposed as numerous small dykes that penetrate the underlying rocks.

Throughout the area the prevailing lavas are amygdaloidal basalts, in which the chief amygdale minerals are chlorite, heulandite, and forms of silica. An andesite dyke of later date occurs near the Monapo River. In the north, near the Sanhuti River, picrite-basalt, basalt, phonolite, and sölvbergite have been found, and related lavas occurring elsewhere in the area are tephritic pumice and ægirine-trachyte. Thus, within the limits of a small volcanic field, series of both 'alkali' and 'calc-alkali' types of lava occur together. The 'alkali' series can be closely matched by the lavas of Abyssinia, British East Africa, Réunion, and Tenerife. The amygdaloidal basalts of the 'calc-alkali' series are similar to those of the Deccan, Arabia, and East Africa, and also to those (of late Karroo age) occurring in South Africa and Central Africa.

In all, ten analyses have been made, and the variation-diagrams constructed from them support the view that each of the series was evolved by a process of differentiation acting on a parent magma. From the composition of the amygdale minerals (which are referred to the closing phase of lava-consolidation), it is deduced that the parent magma of the 'alkali' series was rich in carbon dioxide, and undersaturated in silica; whereas that of the 'calc-alkali' series was rich in water, and oversaturated in silica. The radioactivity of the lavas indicates that the depth from which the parent magma came was probably between 33 and 44 miles from the earth's surface. The boundary-fault along which the lavas are aligned seems to mark a zone where pressure was relieved to an extent and depth sufficient to promote fusion.

Dr. A. STRAHAN, F.R.S., exhibited cores from borings in Kent, showing pebbles of coal embedded in Coal-Measure sandstones. With the coal-pebbles occurred a few partly-rounded fragments of chert, and in one of these radiolaria had been identified by Dr. G. J. Hinde. The chert resembled that which had been described from Lower Carboniferous rocks elsewhere. Its occurrence suggested that the sequence of strata had been similar in South Wales and Kent, and, taken in connexion

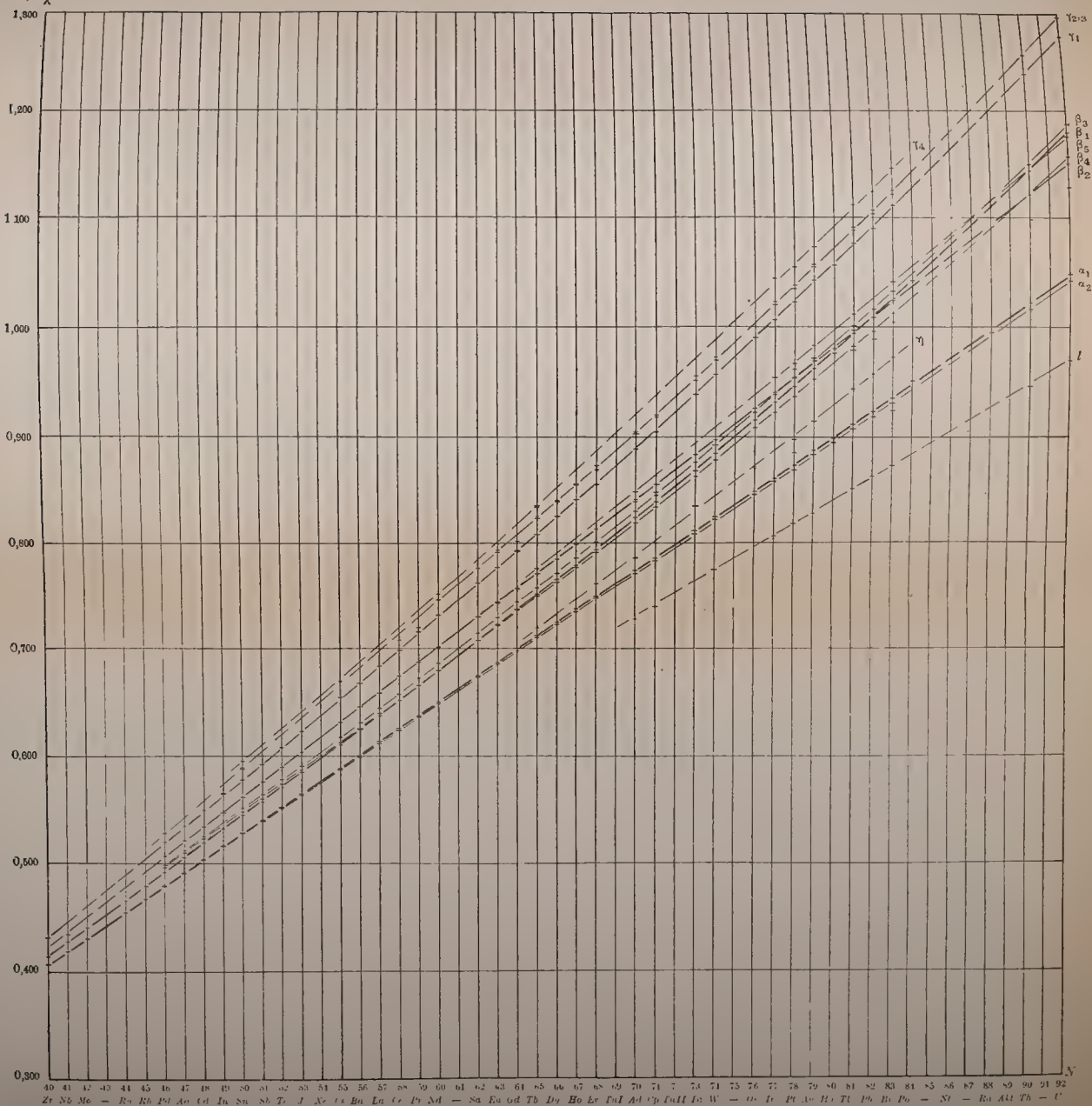
with the piping of the limestone-surface at Ebbsfleet and the absence of Millstone Grit in Kent, tended to confirm the view that there is unconformity between the Coal Measures and the Carboniferous Limestone in that county.

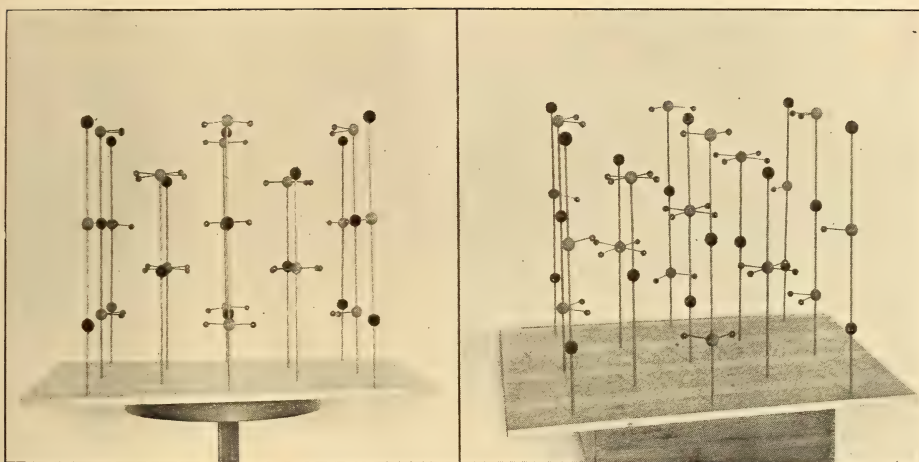
Mr. F. P. MENNELL exhibited a geological sketch-map of the northern margin of Dartmoor.

He said that the central part of Devon was to a great extent a *terra incognita*; but, as regarded the fringe of altered Carboniferous rocks along the northern border of the Dartmoor granite, he had been led, in the course of observations originally concerned with the petrology alone, to the conclusion that it might prove possible to establish a definite order of succession. This was rendered feasible by the occurrence of some well-characterized bands of rock, especially limestones and tuffs, which were exposed in every good river-section. It was true that almost everywhere overfolds, sometimes accompanied by thrusts, were to be detected, and tended to make the observer somewhat doubtful of his ground. Nevertheless, it seemed impossible to escape the conclusion that, as one approached the granite from the north, continuously older rocks were met with, and the extremely continuous character of some of the beds seemed to show that, despite all minor disturbances, the general sequence could be trusted. The comparison of the different lines of section leading up to Dartmoor showed them to be strikingly similar. The granite was, moreover, intruded all along at precisely the same horizon, and its direct offshoots never reached into the lower of the two important bands of limestone, but were confined to the altered shales at the bottom of the series, which afforded, where fresh, good examples of andalusite-hornfels. The series, which extends from south of Sourton to Drewsteignton and perhaps right round to Doddiscombeleigh, appears clearly older than the shales which have been so carefully searched for fossils in the Exeter region by Mr. F. J. Collins. These last are considered to be of Pendleside age, and nowhere contain any traces of limestone. The probability is thus indicated that the distinctly calcareous series under consideration may represent part of the Carboniferous Limestone.

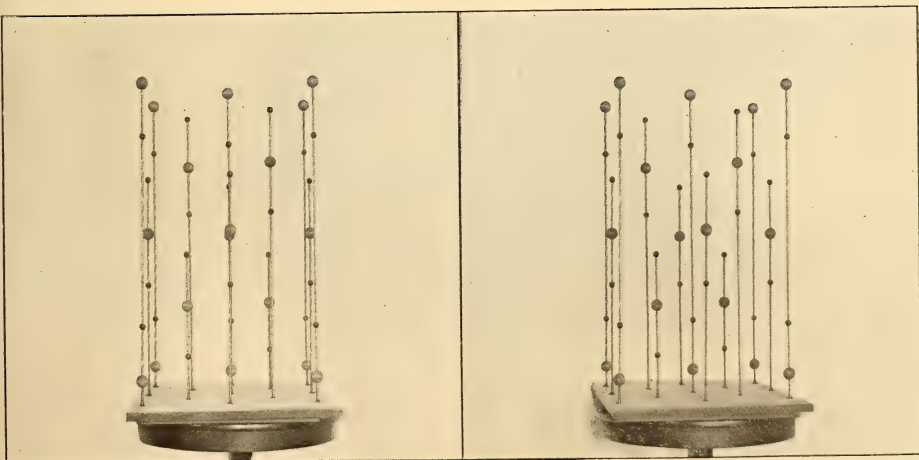
It may be noted that, although a number of bands of epidiorite representing intrusions of dolerite occur roughly parallel to the strike of the sediments, the contemporaneous rocks are never of such basic character. The main band of tuff stretches from Lake, near Bridestowe, to beyond Sticklepath, and, of the numerous well-preserved rock-fragments that it contains, most are of rhyolitic or trachytic character, with some which represent altered andesites.

$$\sqrt{\frac{1}{\lambda}} \cdot 10^{-4}$$





a



b

- a.* Xenotime.
b. Anatase.

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[SIXTH SERIES]

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LIX. *On Convection Currents in a Horizontal Layer of Fluid, when the Higher Temperature is on the Under Side.*
By Lord RAYLEIGH, O.M., F.R.S.*

THE present is an attempt to examine how far the interesting results obtained by Bénard † in his careful and skilful experiments can be explained theoretically. Bénard worked with very thin layers, only about 1 mm. deep, standing on a levelled metallic plate which was maintained at a uniform temperature. The upper surface was usually free, and being in contact with the air was at a lower temperature. Various liquids were employed—some, indeed, which would be solids under ordinary conditions.

The layer rapidly resolves itself into a number of *cells*, the motion being an ascension in the middle of a cell and a descension at the common boundary between a cell and its neighbours. Two phases are distinguished, of unequal duration, the first being relatively very short. The limit of the first phase is described as the “semi-regular cellular regime”; in this state all the cells have already acquired surfaces *nearly* identical, their forms being nearly regular convex polygons of, in general, 4 to 7 sides. The boundaries

* Communicated by the Author.

† *Revue générale des Sciences*, vol. xii. pp. 1261, 1309 (1900); *Ann. d. Chimie et de Physique*, t. xxiii. p. 62 (1901). M. Bénard does not appear to be acquainted with James Thomson's paper “On a Changing Tessellated Structure in certain Liquids” (Proc. Glasgow Phil. Soc. 1881-2), where a like structure is described in much thicker layers of soapy water cooling from the surface.

are vertical, and the circulation in each cell approximates to that already indicated. This phase is brief (1 or 2 seconds) for the less viscous liquids (alcohol, benzine, &c.) at ordinary temperatures. Even for paraffin or spermacetti, melted at 100° C., 10 seconds suffice; but in the case of very viscous liquids (oils, &c.), if the flux of heat is small, the deformations are extremely slow and the first phase may last several minutes or more.

The second phase has for its limit a permanent regime of regular hexagons. During this period the cells become equal and regular and align themselves. It is extremely protracted, if the limit is regarded as the complete attainment of regular hexagons. And, indeed, such perfection is barely attainable even with the most careful arrangements. The tendency, however, seems sufficiently established.

The theoretical consideration of the problem here arising is of interest for more than one reason. In general, when a system falls away from unstable equilibrium it may do so in several principal modes, in each of which the departure at time t is proportional to the small displacement or velocity supposed to be present initially, and to an exponential factor e^{qt} , where q is positive. If the initial disturbances are small enough, that mode (or modes) of falling away will become predominant for which q is a maximum. The simplest example for which the number of degrees of freedom is infinite is presented by a cylindrical rod of elastic material under a longitudinal compression sufficient to overbalance its stiffness. But perhaps the most interesting hitherto treated is that of a cylinder of fluid disintegrating under the operation of capillary force as in the beautiful experiments of Savart and Plateau upon jets. In this case the surface remains one of revolution about the original axis, but it becomes *varicose*, and the question is to compare the effects of different wave-lengths of varicosity, for upon this depends the number of detached masses into which the column is eventually resolved. It was proved by Plateau that there is no instability if the wave-length be less than the circumference of the column. For all wave-lengths greater than this there is instability, and the corresponding modes of disintegration may establish themselves if the initial disturbances are suitable. But if the general disturbance is very small, those components only will have opportunity to develop themselves for which the wave-length lies near to that of maximum instability.

It has been shown * that the wave-length of maximum

* Proc. Lond. Math. Soc. vol. x. p. 4 (1879); Scientific Papers, vol. i. p. 361. Also 'Theory of Sound,' 2nd ed. §§ 357, &c.

instability is 4.508 times the diameter of the jet, exceeding the wave-length at which instability first enters in the ratio of about 3 : 2. Accordingly this is the sort of disintegration to be expected when the jet is shielded as far as possible from external disturbance.

It will be observed that there is nothing in this theory which could fix the *phase* of the predominant disturbance, or the particular particles of the fluid which will ultimately form the centres of the detached drops. There remains a certain indeterminateness, and this is connected with the circumstance that absolute regularity is not to be expected. In addition to the wave-length of maximum instability we must include all those which lie sufficiently near to it, and the superposition of the corresponding modes will allow of a slow variation of phase as we pass along the column. The phase in any particular region depends upon the initial circumstances in and near that region, and these are supposed to be matters of chance*. The superposition of infinite trains of waves whose wave-lengths cluster round a given value raises the same questions as we are concerned with in considering the character of approximately homogeneous light.

In the present problem the case is much more complicated, unless we arbitrarily limit it to two dimensions. The cells of Bénard are then reduced to infinitely long strips, and when there is instability we may ask for what wave-length (width of strip) the instability is greatest. The answer can be given under certain restrictions, and the manner in which equilibrium breaks down is then approximately determined. So long as the two-dimensional character is retained, there seems to be no reason to expect the wave-length to alter afterwards. But even if we assume a natural disposition to a two-dimensional motion, the direction of the length of the cells as well as the phase could only be determined by initial circumstances, and could not be expected to be uniform over the whole of the infinite plane.

According to the observations of Bénard, something of this sort actually occurs when the layer of liquid has a general motion in its own plane at the moment when instability commences, the length of the cellular strips being parallel to the general velocity. But a little later, when the general motion has decayed, division-lines running in the perpendicular direction present themselves.

* When a jet of liquid is acted on by an external vibrator, the resolution into drops may be regularized in a much higher degree.

In general, it is easy to recognize that the question is much more complex. By Fourier's theorem the motion in its earlier stages may be analysed into components, each of which corresponds to rectangular cells whose sides are parallel to fixed axes arbitrarily chosen. The solution for maximum instability yields one relation between the sides of the rectangle, but no indication of their ratio. It covers the two-dimensional case of infinitely long rectangles already referred to, and the contrasted case of squares for which the length of the side is thus determined. I do not see that any plausible hypothesis as to the origin of the initial disturbances leads us to expect one particular ratio of sides in preference to another.

On a more general view it appears that the function expressing the disturbance which develops most rapidly may be assimilated to that which represents the free vibration of an infinite stretched membrane vibrating with given frequency.

The calculations which follow are based upon equations given by Boussinesq, who has applied them to one or two particular problems. The special limitation which characterizes them is the neglect of variations of density, *except in so far as they modify the action of gravity*. Of course, such neglect can be justified only under certain conditions, which Boussinesq has discussed. They are not so restrictive as to exclude the approximate treatment of many problems of interest.

When the fluid is inviscid and the higher temperature is below, all modes of disturbance are unstable, even when we include the conduction of heat during the disturbance. But there is one class of disturbances for which the instability is a maximum.

When viscosity is included as well as conduction, the problem is more complicated, and we have to consider boundary conditions. Those have been chosen which are simplest from the mathematical point of view, and they deviate from those obtaining in Bénard's experiments, where, indeed, the conditions are different at the two boundaries. It appears, a little unexpectedly, that the equilibrium may be thoroughly stable (with higher temperature below), if the coefficients of conductivity and viscosity are not too small. As the temperature gradient increases, instability enters, and at first only for a particular kind of disturbance.

The second phase of Bénard, where a tendency reveals itself for a slow transformation into regular hexagons, is not

touched. It would seem to demand the inclusion of the squares of quantities here treated as small. But the size of the hexagons (under the boundary conditions postulated) is determinate, at any rate when they assert themselves early enough.

An appendix deals with a related analytical problem having various physical interpretations, such as the symmetrical vibration in two dimensions of a layer of air enclosed by a nearly circular wall.

The general Eulerian equations of fluid motion are in the usual notation :—

$$\frac{Du}{Dt} = X - \frac{1}{\rho} \frac{dp}{dx}, \quad \frac{Dv}{Dt} = Y - \frac{1}{\rho} \frac{dp}{dy}, \quad \frac{Dw}{Dt} = Z - \frac{1}{\rho} \frac{dp}{dz}, \quad . \quad (1)$$

where

$$\frac{D}{Dt} = \frac{d}{dt} + u \frac{d}{dx} + v \frac{d}{dy} + w \frac{d}{dz}, \quad . \quad . \quad . \quad (2)$$

and X, Y, Z are the components of extraneous force reckoned per unit of mass. If, neglecting viscosity, we suppose that gravity is the only impressed force,

$$X=0, \quad Y=0, \quad Z=-g, \quad . \quad . \quad . \quad (3)$$

z being measured upwards. In equations (1) ρ is variable in consequence of variable temperature and variable pressure. But, as Boussinesq* has shown, in the class of problems under consideration the influence of pressure is unimportant and even the variation with temperature may be disregarded except in so far as it modifies the operation of gravity. If we write $\rho = \rho_0 + \delta\rho$, we have

$$g\rho = g\rho_0(1 + \delta\rho/\rho_0) = g\rho_0 - g\rho_0\alpha\theta,$$

where θ is the temperature reckoned from the point where $\rho = \rho_0$ and α is the coefficient of expansion. We may now identify ρ in (1) with ρ_0 , and our equations become

$$\frac{Du}{Dt} = -\frac{1}{\rho} \frac{dP}{dx}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{dP}{dy}, \quad \frac{Dw}{Dt} = -\frac{1}{\rho} \frac{dP}{dz} + \gamma\theta, \quad . \quad (4)$$

where ρ is a constant, γ is written for $g\alpha$, and P for $p + g\rho z$.

* *Théorie Analytique de la Chaleur*, t. ii. p. 172 (1903).

Also, since the fluid is now treated as incompressible,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \quad . \quad . \quad . \quad . \quad (5)$$

The equation for the conduction of heat is,

$$\frac{D\theta}{Dt} = \kappa \left(\frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2} + \frac{d^2\theta}{dz^2} \right), \quad . \quad . \quad . \quad . \quad (6)$$

in which κ is the diffusibility for temperature. These are the equations employed by Boussinesq.

In the particular problems to which we proceed the fluid is supposed to be bounded by two infinite fixed planes at $z=0$ and $z=\zeta$, where also the temperatures are maintained constant. In the equilibrium condition u, v, w vanish and θ being a function of z only is subject to $d^2\theta/dz^2=0$, or $d\theta/dz=\beta$, where β is a constant representing the temperature gradient. If the equilibrium is stable, β is positive; and if unstable with the higher temperature below, β is negative. It will be convenient, however, to reckon θ as the departure from the equilibrium temperature Θ . The only change required in equations (4) is to write ϖ for P , where

$$\varpi = P - \rho\gamma \int \Theta dz. \quad . \quad . \quad . \quad . \quad (7)$$

In equation (6) $D\theta/Dt$ is to be replaced by $D\theta/Dt + w\beta$.

The question with which we are principally concerned is the effect of a small departure from the condition of equilibrium, whether stable or unstable. For this purpose it suffices to suppose u, v, w , and θ to be small. When we neglect the squares of the small quantities, D/Dt identifies itself with d/dt and we get

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{d\varpi}{dx}, \quad \frac{dv}{dt} = -\frac{1}{\rho} \frac{d\varpi}{dy}, \quad \frac{dw}{dt} = -\frac{1}{\rho} \frac{d\varpi}{dz} + \gamma\theta, \quad . \quad (8)$$

$$\frac{d\theta}{dt} + \beta w = \kappa \left(\frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2} + \frac{d^2\theta}{dz^2} \right), \quad . \quad . \quad . \quad (9)$$

which with (5) and the initial and boundary conditions suffice for the solution of the problem. The boundary conditions are that $w=0, \theta=0$, when $z=0$ or ζ .

We now assume in the usual manner that the small quantities are proportional to

$$e^{ilx} e^{imy} e^{nt}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

so that (8), (5), (9) become

$$nu = -\frac{il\varpi}{\rho}, \quad nv = -\frac{im\varpi}{\rho}, \quad nw = -\frac{1}{\rho} \frac{d\varpi}{dz} + \gamma\theta, \quad (11)$$

$$ilu + imv + dw/dz = 0, \quad . \quad . \quad . \quad (12)$$

$$n\theta + \beta w = \kappa(d^2/dz^2 - l^2 - m^2)\theta, \quad . \quad . \quad . \quad (13)$$

from which by elimination of u, v, w , we derive

$$\frac{n}{l^2 + m^2} \frac{d^2 w}{dz^2} = nw - \gamma\theta. \quad . \quad . \quad . \quad (14)$$

Having regard to the boundary conditions to be satisfied by w and θ , we now assume that these quantities are proportional to $\sin sz$, where $s = q\pi/\zeta$, and q is an integer. Hence

$$\beta w + \{n + \kappa(l^2 + m^2 + s^2)\}\theta = 0, \quad . \quad . \quad . \quad (15)$$

$$n(l^2 + m^2 + s^2)w - \gamma(l^2 + m^2)\theta = 0, \quad . \quad . \quad (16)$$

and the equation determining n is the quadratic

$$n^2(l^2 + m^2 + s^2) + n\kappa(l^2 + m^2 + s^2)^2 + \beta\gamma(l^2 + m^2) = 0. \quad (17)$$

When $\kappa = 0$, there is no conduction, so that each element of the fluid retains its temperature and density. If β be positive, the equilibrium is stable, and

$$n = \frac{\pm i\sqrt{\{\beta\gamma(l^2 + m^2)\}}}{\sqrt{\{l^2 + m^2 + s^2\}}}, \quad . \quad . \quad . \quad (18)$$

indicating vibrations about the condition of equilibrium. If, on the other hand, β be negative, say $-\beta'$,

$$n = \frac{\pm \sqrt{\{\beta'\gamma(l^2 + m^2)\}}}{\sqrt{\{l^2 + m^2 + s^2\}}}. \quad . \quad . \quad . \quad (19)$$

When n has the positive value, the corresponding disturbance increases exponentially with the time.

For a given value of $l^2 + m^2$, the numerical values of n diminish without limit as s increases—that is, the more subdivisions there are along z . The greatest value corresponds with $q = 1$ or $s = \pi/\zeta$. On the other hand, if s be given, $|n|$ increases from zero as $l^2 + m^2$ increases from zero (great wave-lengths along x and y) up to a finite limit when $l^2 + m^2$ is large (small wave-lengths along x and y). This case of

no conductivity falls within the scope of a former investigation where the fluid was supposed from the beginning to be incompressible but of variable density*.

Returning to the consideration of a finite conductivity, we have again to distinguish the cases where β is positive and negative. When β is negative (higher temperature below) both values of n in (17) are real and one is positive. The equilibrium is unstable for all values of $l^2 + m^2$ and of s . If β be positive, n may be real or complex. In either case the real part of n is negative, so that the equilibrium is stable whatever $l^2 + m^2$ and s may be.

When β is negative ($-\beta'$), it is important to inquire for what values of $l^2 + m^2$ the instability is greatest, for these are the modes which more and more assert themselves as time elapses, even though initially they may be quite subordinate. That the positive value of n must have a maximum appears when we observe it tends to vanish both when $l^2 + m^2$ is small and also when $l^2 + m^2$ is large. Setting for shortness $l^2 + m^2 + s^2 = \sigma$, we may write (17)

$$n^2\sigma + n\kappa\sigma^2 - \beta'\gamma(\sigma - s^2) = 0, \quad \dots \dots (20)$$

and the question is to find the value of σ for which n is greatest, s being supposed given. Making $dn/d\sigma = 0$, we get on differentiation

$$n^2 + 2n\kappa\sigma - \beta'\gamma = 0; \quad \dots \dots (21)$$

and on elimination of n^2 between (20), (21)

$$n = \frac{\beta'\gamma s^2}{\kappa\sigma^2}. \quad \dots \dots (22)$$

Using this value of n in (21), we find as the equation for σ

$$\frac{2s^2}{\sigma} = 1 - \frac{\beta'\gamma s^4}{\kappa^2\sigma^4}. \quad \dots \dots (23)$$

When κ is relatively great, $\sigma = 2s^2$, or

$$l^2 + m^2 = s^2. \quad \dots \dots (24)$$

A second approximation gives

$$l^2 + m^2 = s^2 + \frac{\beta'\gamma}{8\kappa^2 s^2}. \quad \dots \dots (25)$$

The corresponding value of n is

$$n = \frac{\beta'\gamma}{4\kappa s^2} \left\{ 1 - \frac{\beta'\gamma}{8\kappa^2 s^4} \right\}. \quad \dots \dots (26)$$

* Proc. Lond. Math. Soc. vol. xiv. p. 170 (1883); Scientific Papers, vol. ii. p. 200.

The modes of greatest instability are those for which s is smallest, that is equal to π/ζ , and

$$l^2 + m^2 = \frac{\pi^2}{\zeta^2} + \frac{\beta' \gamma}{8\kappa^2 \pi^2 / \zeta^2} \cdot \cdot \cdot \cdot (27)$$

For a two-dimensional disturbance we may make $m=0$ and $l=2\pi/\lambda$, where λ is the wave-length along x . The λ of maximum instability is thus approximately

$$\lambda = 2\zeta \cdot \cdot \cdot \cdot (28)$$

Again, if $l=m=2\pi/\lambda$, as for square cells,

$$\lambda = 2\sqrt{2} \cdot \zeta \cdot \cdot \cdot \cdot (29)$$

greater than before in the ratio $\sqrt{2} : 1$.

We have considered especially the cases where κ is relatively small and relatively large. Intermediate cases would need to be dealt with by a numerical solution of (23).

When w is known in the form

$$w = W e^{ilx} e^{imy} \sin sz \cdot e^{nt}, \cdot \cdot \cdot \cdot (30)$$

n being now a known function of l, m, s, u and v are at once derived by means of (11) and (12). Thus

$$u = \frac{il}{l^2 + m^2} \frac{dw}{dz}, \quad v = \frac{im}{l^2 + m^2} \frac{dw}{dz} \cdot \cdot \cdot \cdot (31)$$

The connexion between w and θ is given by (15) or (16). When β is negative and n positive, θ and w are of the same sign.

As an example in two dimensions of (30), (31), we might have in real form

$$w = W \cos x \cdot \sin z \cdot e^{nt}, \cdot \cdot \cdot \cdot (32)$$

$$u = -W \sin x \cdot \cos z \cdot e^{nt}, \quad v = 0 \cdot \cdot \cdot \cdot (33)$$

Hitherto we have supposed the fluid to be destitute of viscosity. When we include viscosity, we must add $\nu(\nabla^2 u, \nabla^2 v, \nabla^2 w)$ on the right of equations (1), (8), and (11), ν being the kinematic coefficient. Equations (12) and (13) remain unaffected. And in (11)

$$\nabla^2 = d^2/dz^2 - l^2 - m^2 \cdot \cdot \cdot \cdot (34)$$

We have also to reconsider the boundary conditions at $z=0$ and $z=\zeta$. We may still suppose $\theta=0$ and $w=0$; but for a further condition we should probably prefer $dw/dz=0$, corresponding to a fixed solid wall. But this entails much complication, and we may content ourselves with the

supposition $d^2w/dz^2=0$, which (with $w=0$) is satisfied by taking as before w proportional to $\sin sz$ with $s=q\pi/\xi$. This is equivalent to the annulment of lateral forces at the wall. For (Lamb's 'Hydrodynamics,' §§ 323, 326) these forces are expressed in general by

$$p_{xz} = \frac{dw}{dx} + \frac{du}{dz}, \quad p_{yz} = \frac{dw}{dy} + \frac{dv}{dz}, \quad \dots \quad (35)$$

while here $w=0$ at the boundaries requires also $dw/dx=0$, $dw/dy=0$. Hence, at the boundaries, $d^2u/dx dz$, $d^2v/dy dz$ vanish, and therefore by (5), d^2w/dz^2 .

Equation (15) remains unaltered:—

$$\beta w + \{n + \kappa(l^2 + m^2 + s^2)\}\theta = 0, \quad \dots \quad (15)$$

and (16) becomes

$$\{n + \nu(l^2 + m^2 + s^2)\}(l^2 + m^2 + s^2)w - \gamma(l^2 + m^2)\theta = 0. \quad (36)$$

Writing as before $\sigma = l^2 + m^2 + s^2$, we get the equation in n

$$(n + \kappa\sigma)(n + \nu\sigma) + \beta\gamma(l^2 + m^2) = 0, \quad \dots \quad (37)$$

which takes the place of (17).

If $\gamma=0$ (no expansion with heat) the equations degrade, and we have two simple alternatives. In the first $n + \kappa\sigma=0$ with $w=0$, signifying conduction of heat with no motion. In the second $n + \nu\sigma=0$, when the relation between w and θ becomes

$$\beta w + \sigma(\kappa - \nu)\theta = 0. \quad \dots \quad (38)$$

In both cases, since n is real and negative, the disturbance is stable.

If we neglect κ in (37), the equation takes the same form (20) as that already considered when $\nu=0$. Hence the results expressed in (22), (23), (24), (25), (26), (27) are applicable with simple substitution of ν for κ .

In the general equation (37) if β be positive, as γ is supposed always to be, the values of n may be real or complex. If real they are both negative, and if complex the real part is negative. In either case the disturbance dies down. As was to be expected, when the temperature is higher above, the equilibrium is stable.

In the contrary case when β is negative ($-\beta'$) the roots of the quadratic are always real, and one at least is negative. There is a positive root only when

$$\beta'\gamma(l^2 + m^2) > \kappa\nu\sigma^3. \quad \dots \quad (39)$$

If κ , or ν , vanish there is instability; but if κ and ν are finite and large enough, the equilibrium for this disturbance is stable, although the higher temperature is underneath.

Inequality (39) gives the condition of instability for the particular disturbance (l, m, s). It is of interest to inquire at what point the equilibrium becomes unstable when there is no restriction upon the value of $l^2 + m^2$. In the equation

$$\beta'\gamma(l^2 + m^2) - \kappa\nu\sigma^3 = \beta'\gamma(\sigma - s^2) - \kappa\nu\sigma^3 = 0, \quad (40)$$

we see that the left-hand member is negative when $l^2 + m^2$ is small and also when it is large. When the conditions are such that the equation can only just be satisfied with some value of $l^2 + m^2$, or σ , the derived equation

$$\beta'\gamma - 3\kappa\nu\sigma^2 = 0 \quad (41)$$

must also hold good, so that

$$\sigma = 3s^2/2, \quad l^2 + m^2 = \frac{1}{2}s^2, \quad (42)$$

and

$$\beta'\gamma = 27\kappa\nu s^4/4. \quad (43)$$

Unless $\beta'\gamma$ exceeds the value given in (43) there is no instability, however l and m are chosen. But the equation still contains s , which may be as large as we please. The smallest value of s is π/ζ . The condition of instability when l, m , and s are all unrestricted is accordingly

$$\beta'\gamma > \frac{27\pi^4\kappa\nu}{4\zeta^4} \quad (44)$$

If $\beta'\gamma$ falls below this amount, the equilibrium is altogether stable. I am not aware that the possibility of complete stability under such circumstances has been contemplated.

To interpret (44) more conveniently, we may replace β' by $(\Theta_2 - \Theta_1)/\zeta$ and γ by $g(\rho_2 - \rho_1)/\rho_1(\Theta_2 - \Theta_1)$, so that

$$\beta'\gamma = \frac{g}{\zeta} \frac{\rho_2 - \rho_1}{\rho_1}, \quad (45)$$

where $\Theta_2, \Theta_1, \rho_2$, and ρ_1 are the extreme temperatures and densities in equilibrium. Thus (44) becomes

$$\frac{\rho_2 - \rho_1}{\rho_1} > \frac{27\pi^4\kappa\nu}{4g\zeta^3} \quad (46)$$

In the case of air at atmospheric conditions we may take in C.G.S. measure

$$\nu = .14, \quad \text{and} \quad \kappa = \frac{5}{2}\nu \text{ (Maxwell's Theory).}$$

Also $g=980$, and thus

$$\frac{\rho_2 - \rho_1}{\rho_1} = \frac{0.033}{\zeta^3} \dots \dots \dots (47)$$

For example, if $\zeta=1$ cm., instability requires that the density at the top exceed that at the bottom by one-thirtieth part, corresponding to about 9° C. of temperature. We should not forget that our method postulates a small value of $(\rho_2 - \rho_1)/\rho_1$. Thus if $\kappa\nu$ be given, the application of (46) may cease to be legitimate unless ζ be large enough.

It may be remarked that the influence of viscosity would be increased were we to suppose the horizontal velocities (instead of the horizontal forces) to be annulled at the boundaries.

The problem of determining for what value of $l^2 + m^2$, or σ , the instability, when finite, is a maximum is more complicated. The differentiation of (37) with respect to σ gives

$$n^2 + 2n\sigma(\kappa + \nu) + 3\kappa\nu\sigma^2 - \beta'\gamma = 0, \dots \dots (48)$$

whence

$$n = \frac{\beta'\gamma s^2 - 2\kappa\nu\sigma^3}{\sigma^2(\kappa + \nu)}, \dots \dots \dots (49)$$

expressing n in terms of σ . To find σ we have to eliminate n between (44) and (45). The result is

$$\sigma^6 \kappa \nu (\kappa - \nu)^2 + \sigma^4 \beta' \gamma (\kappa + \nu)^2 - \sigma^3 \cdot 2\beta' \gamma s^2 (\kappa^2 + \nu^2) - \beta'^2 \gamma^2 s^4 = 0, \dots \dots (50)$$

from which, in particular cases, σ could be found by numerical computation. From (50) we fall back on (23) by supposing $\nu=0$, and again on a similar equation if we suppose $\kappa=0$.

But the case of a nearly evanescent n is probably the more practical. In an experiment the temperature gradient could not be established all at once and we may suppose the progress to be very slow. In the earlier stages the equilibrium would be stable, so that no disturbance of importance would occur until n passed through zero to the positive side, corresponding to (44) or (46). The breakdown thus occurs for $s=\pi/\zeta$, and by (42) $l^2 + m^2 = \pi^2/2\zeta^2$. And since the evanescence of n is equivalent to the omission of d/dt in the original equations, the motion thus determined has the character of a *steady* motion. The constant multiplier is, however, arbitrary; and there is nothing to determine it so long as the squares of u, v, w, θ are neglected.

In a particular solution where w as a function of x and y has the simplest form, say

$$w = 2 \cos x \cdot \cos y, \quad . \quad . \quad . \quad (51)$$

the particular coefficients of x and y which enter have relation to the particular axes of reference employed. If we rotate these axes through an angle ϕ , we have

$$\begin{aligned} w &= 2 \cos \{x' \cos \phi - y' \sin \phi\} \cdot \cos \{x' \sin \phi + y' \cos \phi\} \\ &= \cos \{x'(\cos \phi - \sin \phi)\} \cdot \cos \{y'(\cos \phi + \sin \phi)\} \\ &\quad + \sin \{x'(\cos \phi - \sin \phi)\} \cdot \sin \{y'(\cos \phi + \sin \phi)\} \\ &\quad + \cos \{x'(\cos \phi + \sin \phi)\} \cdot \cos \{y'(\cos \phi - \sin \phi)\} \\ &\quad - \sin \{x'(\cos \phi + \sin \phi)\} \cdot \sin \{y'(\cos \phi - \sin \phi)\}. \end{aligned} \quad (52)$$

For example, if $\phi = \frac{1}{4}\pi$, (52) becomes

$$w = \cos (y' \sqrt{2}) + \cos (x' \sqrt{2}). \quad . \quad . \quad . \quad (53)$$

It is to be observed that with the general value of ϕ , if we call the coefficients of x' , y' , l and m respectively, we have in every part $l^2 + m^2 = 2$, unaltered from the original value in (51).

The character of w , under the condition that all the elementary terms of which it is composed are subject to $l^2 + m^2 = \text{constant } (k^2)$, is the same as for the transverse displacement of an infinite stretched membrane, vibrating with one definite frequency. The limitation upon w is, in fact, merely that it satisfies

$$(d^2/dx^2 + d^2/dy^2 + k^2)w = 0. \quad . \quad . \quad . \quad (54)$$

The character of w in particular solutions of the membrane problem is naturally associated with the nodal system ($w=0$), where the membrane may be regarded as held fast; and we may suppose the nodal system to divide the plane into similar parts or cells, such as squares, equilateral triangles, or regular hexagons. But in the present problem it is perhaps more appropriate to consider divisions of the plane with respect to which w is symmetrical, so that dw/dn is zero on the straight lines forming the divisions of the cells. The more natural analogy is then with the two-dimensional vibration of air, where w represents velocity-potential and the divisions may be regarded as fixed walls.

The simplest case is, of course, that in which the cells are

squares. If the sides of the squares be 2π , we may take with axes parallel to the sides and origin at centre

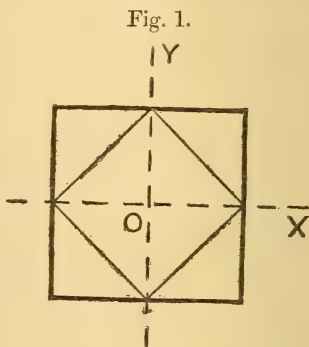
$$w = \cos x + \cos y, \quad . \quad . \quad . \quad . \quad (55)$$

being thus composed by superposition of two parts for each of which $k^2=1$. This makes $dw/dx = -\sin x$, vanishing when $x = \pm\pi$. Similarly, dw/dy vanishes when $y = \pm\pi$, so that the sides of the square behave as fixed walls. To find the places where w changes sign, we write it in the form

$$w = 2 \cos \frac{x+y}{2} \cdot \cos \frac{x-y}{2}, \quad . \quad . \quad . \quad . \quad (56)$$

giving $x+y = \pm\pi$, $x-y = \pm\pi$, lines which constitute the inscribed square (fig. 1). Within this square w has one sign (say +) and in the four right-angled triangles left over the - sign. When the whole plane is considered, there is no want of symmetry between the + and the - regions.

The principle is the same when the elementary cells are equilateral triangles or hexagons; but I am not aware that an analytical solution has been obtained for these cases. An experimental determination of k^2 might be made by observing



the time of vibration under gravity of water contained in a trough with vertical sides and of corresponding section, which depends upon the same differential equation and boundary conditions*. The particular vibration in question is not the slowest possible, but that where there is a simultaneous rise at the centre and fall at the walls all round, with but one curve of zero elevation between.

In the case of the hexagon, we may regard it as deviating comparatively little from the circular form and employ the approximate methods then applicable. By an argument analogous to that formerly developed † for the boundary condition $w=0$, we may convince ourselves that the value of k^2 for the hexagon cannot differ much from that appropriate to a circle of the same area. Thus if a be the radius

* See Phil. Mag. vol. i. p. 257 (1876); Scientific Papers, vol. i. pp. 265, 271.

† Theory of Sound, § 209; compare also § 317. See Appendix.

of this circle, k is given by $J_0'(ka)=0$, J_0 being the Bessel's function of zero order, or $ka=3.832$. If b be the side of the hexagon, $a^2=3\sqrt{3} \cdot b^2/2\pi$.

APPENDIX.

On the nearly symmetrical solution for a nearly circular area, when w satisfies $(d^2/dx^2 + d^2/dy^2 + k^2)w=0$ and makes $dw/dn=0$ on the boundary.

Starting with the true circle of radius a , we have w a function of r (the radius vector) only, and the solution is $w=J_0(kr)$ with the condition $J_0'(ka)=0$, yielding $ka=3.832$, which determines k if a be given, or a if k be given. In the problem proposed the boundary is only approximately circular, so that we write $r=a+\rho$, where a is the mean value and

$$\rho = \alpha_1 \cos \theta + \beta_1 \sin \theta + \dots + \alpha_n \cos n\theta + \beta_n \sin n\theta. \quad (57)$$

In (57) θ is the vectorial angle and α_1 &c. are quantities small relatively to a . The general solution of the differential equation being

$$w = A_0 J_0(kr) + J_1(kr) \{ A_1 \cos \theta + B_1 \sin \theta \} \\ + \dots + J_n(kr) \{ A_n \cos n\theta + B_n \sin n\theta \}, \quad (58)$$

we are to suppose now that A_1 , &c., are small relatively to A_0 . It remains to consider the boundary condition.

If ϕ denote the small angle between r and the normal dn measured outwards,

$$\frac{dw}{dn} = \frac{dw}{dr} \cos \phi - \frac{dw}{r d\theta} \sin \phi, \quad (59)$$

and

$$\tan \phi = \frac{dr}{r d\theta} = \frac{d\rho}{a d\theta} = \frac{n}{a} (-\alpha_n \sin n\theta + \beta_n \cos n\theta) \quad (60)$$

with sufficient approximation, only the general term being written. In formulating the boundary condition $dw/dn=0$ correct to the second order of small quantities, we require dw/dr to the second order, but $dw/d\theta$ to the first order only. We have

$$\frac{1}{k} \frac{dw}{dr} = A_0 \{ J_0'(ka) + k\rho J_0''(ka) + \frac{1}{2} k^2 \rho^2 J_0'''(ka) \} \\ + \{ J_n'(ka) + k\rho J_n''(ka) \} \{ A_n \cos n\theta + B_n \sin n\theta \}, \\ \frac{dw}{a d\theta} = \frac{n}{a} J_n(ka) \{ -A_n \sin n\theta + B_n \cos n\theta \}$$

and for the boundary condition, setting $ka = z$ and omitting the argument in the Bessel's functions,

$$\begin{aligned} & A_0 \{ J_0' \cdot \cos \phi + k\rho J_0'' + \frac{1}{2}k^2\rho^2 J_0''' \} \\ & + \{ J_n' + k\rho J_n'' \} \{ A_n \cos n\theta + B_n \sin n\theta \} \\ & - \frac{n^2}{az} J_n \{ -A_n \sin n\theta + B_n \cos n\theta \} \{ -\alpha_n \sin n\theta \\ & \qquad \qquad \qquad + \beta_n \cos n\theta \} = 0. \quad (61) \end{aligned}$$

If for the moment we omit the terms of the second order, we have

$$\begin{aligned} & A_0 J_0' + k A_0 J_0'' \{ \alpha_n \cos n\theta + \beta_n \sin n\theta \} \\ & + J_n' \{ A_n \cos n\theta + B_n \sin n\theta \} = 0; \quad (62) \end{aligned}$$

so that $J_0'(z) = 0$, and

$$k A_0 J_0'' \cdot \alpha_n + J_n' \cdot A_n = 0, \quad k A_0 J_0'' \cdot \beta_n + J_n' \cdot B_n = 0. \quad (63)$$

To this order of approximation $z = ka$, has the same value as when $\rho = 0$; that is to say, the equivalent radius is equal to the mean radius, or (as we may also express it) k may be regarded as dependent upon the *area* only. Equations (63) determine A_n, B_n in terms of the known quantities α_n, β_n .

Since J_0' is a small quantity, $\cos \phi$ in (61) may now be omitted. To obtain a corrected evaluation of z , it suffices to take the mean of (61) for all values of θ . Thus

$$\begin{aligned} & A_0 \{ 2J_0' + \frac{1}{2}k^2 J_0''' (\alpha_n^2 + \beta_n^2) \} \\ & + \{ kJ_n'' - n^2 J_n / az \} \{ \alpha_n A_n + \beta_n B_n \} = 0, \end{aligned}$$

or on substitution of the approximate values of A_n, B_n from (63),

$$J_0' = \frac{1}{2}k^2 (\alpha_n^2 + \beta_n^2) \left\{ \frac{J_0''}{J_n'} \left(J_n'' - \frac{n^2 J_n}{z^2} \right) - \frac{J_0'''}{z} \right\}. \quad (64)$$

This expression may, however, be much simplified. In virtue of the general equation for J_n ,

$$J_n'' - \frac{n^2}{z^2} J_n = -\frac{J_n'}{z} - J_n;$$

and since here $J_0' = 0$ approximately,

$$J_0'' = -J_0, \quad J_0''' = -z^{-1} J_0'' = z^{-1} J_0.$$

Thus

$$J_0'(z) = \frac{1}{2}k^2 J_0 \cdot \Sigma (\alpha_n^2 + \beta_n^2) \left\{ \frac{J_n}{J_n'} + \frac{1}{2z} \right\}, \quad (65)$$

the sign of summation with respect to n being introduced.

Let us now suppose that $a + da$ is the equivalent radius, so that $J_0'(ka + kda) = 0$, that is the radius of the exact circle which corresponds to the value of k appropriate to the approximate circle. Then

$$J_0'(z) + kda J_0''(z) = 0,$$

and

$$da = -\frac{J_0'}{kJ_0''} = k \Sigma (\alpha_n^2 + \beta_n^2) \left\{ \frac{J_n}{2J_n'} + \frac{1}{4z} \right\}. \quad (66)$$

Again, if $a + da'$ be the radius of the true circle which has the same area as the approximate circle

$$da' = \frac{1}{4a} \Sigma (\alpha_n^2 + \beta_n^2), \quad . \quad . \quad . \quad (67)$$

and

$$da' - da = -\Sigma \frac{\alpha_n^2 + \beta_n^2}{2a} \frac{z J_n(z)}{2J_n'(z)}, \quad . \quad . \quad . \quad (68)$$

where z is the first root (after zero) of $J_0'(z) = 0$, viz. 3.832.

The question with which we are mainly concerned is the sign of $da' - da$ for the various values of n . When $n=1$, $J_1(z) = -J_0'(z) = 0$, so that $da = da'$, a result which was to be expected, since the terms in α_1, β_1 represent approximately a displacement merely of the circle, without alteration of size or shape. We will now examine the sign of J_n/J_n' when $n=2$, and 3.

For this purpose we may employ the sequence equations

$$J_{n+1} = \frac{2n}{z} J_n - J_{n-1}, \quad J_n' = \frac{1}{2} J_{n-1} - \frac{1}{2} J_{n+1},$$

which allow J_n and J_n' to be expressed in terms of J_1 and J_0 , of which the former is here zero. We find

$$\begin{aligned} J_2 &= -J_0, & J_3 &= -4z^{-1}J_0, & J_4 &= (1 - 24z^{-2})J_0; \\ J_1' &= J_0, & J_2' &= 2z^{-1}J_0, & J_3' &= (12z^{-2} - 1)J_0. \end{aligned}$$

Thus

$$\frac{J_1}{J_1'} = 0, \quad \frac{J_2}{J_2'} = -\frac{z}{2}, \quad \frac{J_3}{J_3'} = \frac{4z}{z^2 - 12};$$

whence on introduction of the actual value of z , viz. 3.832, we see that J_2/J_2' is negative, and that J_3/J_3' is positive.

When $n > z$, it is a general proposition that $J_n(z)$ and $J_n'(z)$ are both positive*. Hence for $n=4$ and onwards, J_n/J_n' is positive when $z=3.832$. We thus arrive at the

* See, for example, Theory of Sound, § 210.

curious conclusion that when $n=2$, $da' > da$, as happens for all values of n (exceeding unity) when the boundary condition is $w=0$, but that when $n>2$, $da' < da$. The existence of the exceptional case $n=2$ precludes a completely general statement of the effect of a departure from the truly circular form; but if the terms for which $n=2$ are absent, as they would be in the case of any regular polygon with an even number of sides, regarded as a deformed circle, we may say that $da' < da$. In the physical problems the effect of a departure from the circular form is then to depress the pitch when the area is maintained constant ($da'=0$). But for an elliptic deformation the reverse is the case.

At first sight it may appear strange that an elliptic deformation should be capable of raising the pitch. But we must remember that we are here dealing with a vibration such that the phase at both ends of the minor axis is the opposite of that at the centre. A parallel case which admits of complete calculation is that of the rectangle regarded as a deformed square, and vibrating in the gravest *symmetrical* mode*. It is easily shown that a departure from the square form raises the pitch. Of course, the one-dimensional vibration parallel to the longer side has its pitch depressed.

LX. *On some Investigations of the Spectra of Carbon and Hydrocarbon.* By CHARLES W. RAFFETY, F.R.A.S.†

[Plate XIII.]

THE spectrum of the Bunsen flame (the "Swan" spectrum) has been made the object of a vast amount of experimental work directed towards the determination of its chemical origin, a *résumé* of which is given by Dr. Marshall Watts in a paper in the 'Philosophical Magazine' of July 1914‡.

The present communication gives the results of a photographic study of the "Swan" spectrum obtained from different sources, and is intended to direct attention to certain features which appear to be worthy of further research—namely, the physical significance of the changes produced in the spectrum by a change in the conditions

* Theory of Sound, § 267 ($p=q=2$).

† Communicated by the Author.

‡ "On the Spectra given by Carbon and some of its Compounds; and, in particular, the 'Swan' Spectrum," W. Marshall Watts, Phil. Mag. July 1914.

of the source, and the origin of certain new lines and bands which have been recorded photographically.

The "Swan" spectrum, characteristic of the flame of hydrocarbons, consists of the well-known groups of flutings in the yellow, green, and blue attributed to the element carbon, as well as some bands of more complicated structure in the violet which are ascribed to hydrocarbon. This spectrum is given brightly and with purity by the base of the flame of a Mecker gas-burner (coal-gas burning in air).

Other flutings of similar structure and obviously belonging to the same system of bands can be distinguished in the red-orange region of the flame spectrum; but these are very faint and difficult to photograph. In the spectrum of the electric arc between carbon poles in air, the yellow, green, and blue "Swan" bands are reproduced brightly with the addition of the intense fluted bands in the violet and ultra-violet which are ascribed to cyanogen since nitrogen must be present for their production. In the absence of hydrogen, however, the hydrocarbon bands do not appear.

In the red and orange regions of the carbon arc spectrum we find very complex series of lines in which cyanogen is represented by a number of flutings*, but the bands of carbon, faintly visible in the flame spectrum, are nevertheless obscured and inconspicuous, their heads being traceable only with some difficulty.

In the "Swan" spectrum, carbon is represented by the following groups of flutings † :—

	Red-orange Group.	Yellow Group.	Green Group.	Blue Group.	Violet Group.
λλ.	6191·1	5635·4	5165·3	4737·2	4381·9
	6121·8	5585·5	5129·4	4715·3	4371·3
	6059·9	5540·9	5101·0	4697·6	4365·0
	6005·1	5501·9	5081·9	4684·9	("the three")
	5958·2	5472·7			

An attempt was made to apply the electrical conditions of the arc discharge to the products of the flame in which carbon was in the process of changing its chemical combination.

* "The Less Refrangible Spectrum of Cyanogen, and its Occurrence in the Carbon Arc," Fowler and Shaw, *Proc. Roy. Soc. A*, vol. lxxxvi. (1912).

† See also 'Index of Spectra,' W. Marshall Watts, Appendix V., pp. 67, 68.

The arc between soft-cored carbon rods 8 mm. in diameter, connected to an alternating current supply of about 200 volts, was struck within a flame of coal-gas in air, the current being regulated by an inductance and a variable resistance in circuit. It was thus hoped to intensify the carbon bands and also to discover whether any modifications occurred in the hydrocarbon bands which are produced under these conditions.

An image of a horizontal arc was projected with a lens on the slit of the spectrograph in the usual way, the arc having been struck in a flame of coal-gas which completely enveloped it. After careful adjustment it was found that the normal blue region at the centre of the arc became replaced by one of a pronounced green colour, and at the same time the arc emitted a characteristic sound by which the change could always be recognized.

A visual examination of the spectrum revealed the carbon flutings with especial brilliancy, the series of lines forming them appearing much extended so as to fill the spaces separating the groups. At the same time the red-orange bands of the normal flame spectrum emerged from their obscurity, being exalted in brightness until their "heads" became quite conspicuous. The red line of hydrogen ($H\alpha$, λ 6563.0) also appeared.

In the more refrangible part of the spectrum the cyanogen bands still possessed great brilliancy, there being, apparently, sufficient access for atmospheric nitrogen in the flame; but together with them, the hydrocarbon bands were well developed.

Fig. 1, Pl. XIII., is an enlargement from a spectrogram of the normal carbon arc and the arc-in-flame, taken with a grating spectrograph, showing the yellow and green bands of carbon.

The difference in intensity between the two spectra, however, is actually much greater than is here indicated as, in order to produce clear photographs and avoid extreme contrast, longer exposure was given to the one of the normal arc.

Fig. 2, Pl. XIII., shows the corresponding intensification of the red-orange bands as compared with the normal arc.

From experiment it appeared that the flame most favourable for the production of the carbon bands was the ordinary luminous jet rather than the air-fed Bunsen or Mecker flames; and the arc was moved about from one part of the flame to another, but was finally located about the centre of the luminous region.

Probably the chief interest of the arc-in-flame spectrum lies in the region about $\lambda 4382$ to $\lambda 4364$, and in the hydrocarbon bands. In the former region are the three lines which appear to form part of the "Swan" spectrum, and to be attributable to carbon. These three lines occur in the flame spectrum, where they are superposed on the first portion of the hydrocarbon "*f*" group, but are frequently so faint as to be hardly visible amongst the lines forming the band. Their brightness, however, changes considerably with the conditions of combustion. In a Mecker flame with a rapid flow of gas, the region immediately above the grid is blue and the flame roars, but when the flow is sufficiently reduced it becomes almost silent and the colour changes to yellowish green.

The spectrum then shows an increase in brilliancy of the "Swan" bands, and simultaneously "the three" appear clearly, standing out on the background of the hydrocarbon band.

In the normal carbon arc in air these lines are not conspicuous, but are recorded with suitable conditions and exposure.

In the spectrum of the arc-in-flame we find "the three" present with very considerable intensity, and their sympathetic variation with the carbon bands lends support to the conclusion that they have the same origin.

The sources from which the carbon and hydrocarbon spectra were obtained were as follows :—

1. Flame of coal-gas in air from a Mecker burner.
2. Carbon arc in air.
3. Carbon arc in flame of coal-gas in air.
4. High-tension transformer spark in air between carbon electrodes (capacity and self-induction in circuit).
5. Transformer spark between carbon electrodes in an atmosphere of coal-gas (capacity and self-induction in circuit).
6. The same as No. 5, but in a flame of coal-gas in air.
7. Transformer spark in absolute alcohol, and also in glycerine (no capacity or self-inductance in circuit).

Visual observations of the spectra were made with various spectroscopes, and the enlargements which accompany this

paper are reproduced from negatives obtained with the following instruments :—

1. Grating spectrograph of 22-inches focus.
2. Grating spectrograph of 6-feet focus.
3. Prismatic spectrograph with compound prism and camera of about 9-inches focus.

Fig. 3, Pl. XIII., shows the hydrocarbon bands as they appear in the spectrum of the Mecker gas-flame, the comparison spectrum being that of iron in the carbon arc. The line-series are seen to be complex, and the whole system of bands is both beautiful and intricate.

The region of "the three," and the less refrangible part of the hydrocarbon spectrum as given by the arc-in-flame, is shown in fig. 4, Pl. XIII. The former are strong and widened, and a close examination indicates a multiple structure.

In the case of the hydrocarbon bands the most striking features, apart from the question of variation of relative intensities, appear to be the new companion lines to the isolated line of the flame spectrum about $\lambda 4324$, which appears single with small dispersion.

The intricacy of structure of the bands, however, especially in the "head" ($\lambda 4314$), requires instruments of considerable power, and only a partial resolution is shown in these spectrograms.

The ionization within the flame is so great, that to maintain the disruptive discharge of the spark a source not only of sufficient potential, but also of considerable output is required to work with a condenser of the requisite capacity, and a high-tension transformer was employed for this purpose.

In addition to an inductance in the primary circuit of the transformer, another high-tension inductance was inserted in the secondary circuit to control the current and prevent arcing at the electrodes within the flame.

In the oscillatory discharge circuit formed by the condenser and spark-gap, a variable inductance was introduced so as to control the oscillation period of the spark. With the spark between carbon electrodes in air, with capacity but without self-induction, the *line* spectra of nitrogen and carbon are seen. In the case of the same spark within a flame of coal-gas there appear, in addition, the "Swan" bands of carbon, the flutings of cyanogen, and, less strongly, the hydrocarbon bands.

The two spectra are thus superposed; but their relative

intensities can be changed considerably by an alteration of the length of the spark, and by other variations of the conditions of discharge.

When sufficient self-induction is introduced into the oscillatory circuit, the line spectra of nitrogen and carbon disappear, leaving a bright "Swan" spectrum with "the three," the hydrocarbon bands, and those of cyanogen.

The spectrum then becomes practically identical with that of the arc-in-flame.

A spectrum very closely similar is given also by the spark with self-induction in an atmosphere of coal-gas, the survival of the cyanogen bands being attributable, presumably, to an admixture of some nitrogen with the gas.

The spectrum was obtained from the transformer spark between carbon electrodes in a glass cylinder through which a gentle stream of coal-gas was maintained, the gas being brought into the sparking cylinder through the electrodes themselves, which were drilled so as to form tubes.

Here again, with the introduction of self-induction into the discharge circuit, the line spectrum of carbon gives place to the "Swan" bands. The spectrum from about λ 4550 to λ 3800 is reproduced in fig. 5, Pl. XIII.

The modification of these spectra from that given by the Mecker flame suggested the investigation of the spark under alcohol, in order to ascertain whether or not the same new lines were present in this case. The transformer discharge was accordingly taken between electrodes of platinum under absolute alcohol. With condenser in circuit the discharge was intermittent and violently disruptive, giving a bright continuous spectrum; but when the capacity was disconnected, the spark became much easier to maintain and appeared of a pronounced greenish colour.

An examination of the spectrum revealed the "Swan" bands brightly. Owing to the rapid liberation of solid carbon in the liquid, photographic exposures were attended with considerable difficulty, as the alcohol very soon became turbid and semi-opaque. By using a fairly large quantity, however, and by bringing the electrodes close up against the side of the glass containing vessel in order to reduce the absorption by the liquid, it became possible to give the long exposures necessary.

The spark needed constant adjustment which added very considerably to the difficulty of obtaining good spectrograms. The image on the slit of the spectrograph was apt to wander, and required to be brought back on the same spot repeatedly.

Fig. 6, Pl. XIII., is reproduced from a spectrogram enlarged with a cylindrical lens.

Examination reveals a close agreement with the spectra of the arc and the spark in the coal-gas flame, and with the spark in an atmosphere of coal-gas.

In addition, certain conspicuous new bands occur, some of which appear to be identical with members of Deslandres' negative groups*.

Their chemical origin seems as yet to be unknown. The approximate wave-lengths of the heads of the bands shown in the spectrogram are as follows:—

$\lambda\lambda$	4102.5
	4068.0
	3854.0
	3827.5 (?)

The bands of cyanogen are completely absent.

[In connexion with the spectrum of the arc-in-flame, there is a very interesting point to be noticed with regard to the behaviour of lines due to impurities in the carbon rods. Various elements occur as impurities: calcium, iron, sodium, aluminium, silicon, and magnesium are common, and often their spectra are strongly represented. The action of the flame surrounding the arc is to quench out these lines more or less completely, and a good illustration of this is given in fig. 7, Pl. XIII.]

The two spectra here shown were obtained from the same carbon rods—the upper one being that of the normal open arc, and the lower one that of the arc in a coal-gas flame. In the first, the lines of calcium— $\lambda\lambda$ 4226.9, 3968.6, 3933.8—are very strong; lines of silicon, aluminium, &c. also appear. In the spectrum of the arc-in-flame these lines are either absent or very faint, the most striking case being the practically complete disappearance of the strongest calcium lines, H and K.]

It was realized that to study in detail the modifications of the hydrocarbon bands, and the other features of these spectra, greater instrumental power was desirable; for although the optical performance of the smaller grating spectrograph was excellent, an instrument of greater focal length would have given better resolution on the photographic film.

The spectra were accordingly photographed with a grating spectrograph of 6-feet focus. The mounting of this instrument leaves much to be desired when long exposures are concerned, as its situation is subject to very considerable

* See 'Index of Spectra,' W. Marshall Watts, Appendix V., p. 77; Deslandres and D'Azambuja, *C. R.* 1905, cxl. p. 917.

vibration with corresponding detriment to the best definition, and it is also liable to temperature changes to some extent. Under these disadvantageous conditions the recording of the Mecker flame spectrum, necessitating exposures of many hours' duration, was attended with considerable difficulty; and even greater difficulties were experienced in the case of the spark under alcohol.

The results obtained, however, were fairly satisfactory, especially in the case of the arc-in-flame, where the intensity of the light rendered short exposures possible, and the detrimental effects of intermittent vibration and temperature changes could be avoided.

The spectrum of iron in the carbon arc was employed as a reference spectrum for wave-length determinations.

The Mecker burner was placed in a horizontal position as before, the luminous region immediately above the grid being viewed from one side. The image thus formed on the slit by the condensing lens was a narrow line of light, and fairly uniform illumination was obtained over the entire width of the spectrum.

Fig. 8, Pl. XIII., is reproduced from a spectrogram taken with the larger instrument, and shows the main portion of the hydrocarbon spectrum as given by the Mecker coal-gas flame, with Fe : comparison. The very fine and complicated structure of the head of the main band (about λ 4314) is here revealed, although the densest region is still beyond the resolving-power of the instrument.

It is to be noted that in this case "the three" ($\lambda\lambda$ 4382, 4371, 4365) are inconspicuous.

The same region of the spectrum of the arc-in-flame is shown in fig. 9, Pl. XIII. Seen on the scale given by the increased instrumental power, the wealth of detail is remarkable and the differences from the flame spectrum become the more apparent.

The greatly increased intensity of "the three" is seen to be due in part to the appearance of new components; while the flame-line of λ 4324 has now become a group with a new isolated line on the more refrangible side. Photographs of this region were obtained also in the 2nd order spectrum of the grating; one enlargement is reproduced in fig. 10, Pl. XIII.

Examination reveals that, in addition to the appearance of new lines in the arc-in-flame spectrum, there are differences in relative intensities as compared with the flame spectrum.

The flame shows two strong lines— $\lambda\lambda$ 4292.2, 4291.3; but in the spectrum of the arc-in-flame, the less refrangible

line is much fainter, while a third line— $\lambda 4293\cdot26$ —is strong. This line appears to coincide with a faint line in the flame spectrum, but its intensification in the case of the arc-in-flame is as remarkable as is the reduction of intensity of the line $\lambda 4292\cdot2$.

The recent announcement by Prof. Newall of the identification of the hydrocarbon lines with those of the "G" group in the solar spectrum has given a greatly increased interest to the hydrocarbon spectrum; and a further study of the variations mentioned in this paper may yield important information for the interpretation of the solar absorption lines. If it should be possible to trace variations in the lines of the "G" group depending on the particular region of the sun from which the light is received, a comparison with different laboratory spectra such as those herein described might lead to definite conclusions as to temperature and the probable location of the absorption.

In the flame spectrum there are no lines between the head of the band (about $\lambda 4315$) and the isolated line $\lambda 4324$; but the arc-in-flame shows many lines in this region.

The author is indebted to Dr. W. Marshall Watts for his kindness in making some preliminary measurements of the spectrograms. The wave-lengths of the components of "the three," as they occur in the normal arc, the arc-in-flame, and the flame, and of the group of lines about $\lambda 4324$ typical of the arc-in-flame, are as follows:—

	Normal Arc.	Arc-in-flame.	Flame.
3_1	4381·65	4381·70	
		80·67	4380·57
3_2	4370·74	4372·34	4372·33
		70·74	70·77
		69·09	
3_3	4364·84	4364·84	4364·03
		4324·03	
		23·30	
		23·19	
		23·08	
		22·20	
		4321·6	

(Wave-lengths on the International scale.)

The spectrograms of the spark in alcohol obtained with the larger spectrograph were insufficiently exposed for successful reproduction; but the negatives show clearly a close agreement of the main features with those of the arc-in-flame, although the fainter lines are not recorded.

The normal carbon arc shows only faint lines in the region between about $\lambda 4380$ and the head of the cyanogen band $\lambda 4216$, and a comparatively long exposure is required to photograph them.

For this reason lines due to impurities in the carbon rods are very strongly recorded on the plate, even though their actual intensities are not great. The region in question is shown in fig. 11, Pl. XIII.

*An Investigation of the Fainter Regions of the
Hydrocarbon Band.*

In some spectrograms of the hydrocarbon bands obtained with the smaller grating instrument, faint traces of a new series of lines were detected in the region between $\lambda 4107$ and $\lambda 4025$. The discovery of radiations in this apparently dark interval at the centre, between the main head and tail series, was considered to be of sufficient interest to justify an attempt to explore this region as thoroughly as possible in the hope of connecting the two main parts of the hydrocarbon spectrum.

The first negatives with the grating spectrograph showed that the main difficulty was the presence of a faint continuous spectrum extending throughout the band, which tended to mask the lines. The latter were so very little brighter than the continuous background on which they appeared that increasing the length of the exposure beyond a certain point did not serve to increase their visibility, but merely made the whole image darker. Fortunately, however, the intensity of the continuous spectrum seemed to be variable with the precise state of the gas-flame, becoming fainter as the supply of gas was reduced and the flame became more silent.

In order to obtain a spectrum of greater luminosity, and decrease the length of the exposures, it was decided to employ a prismatic spectrograph. The instrument used was provided with a compound prism of high dispersion and a camera of about 9-inches focus. The effective working aperture was fairly large, and the angular dispersion of the prism for the region in question was very considerable.

Several successful spectrograms were obtained, which showed the new series more distinctly.

The flame of the Mecker burner was adjusted and carefully

focussed upon the slit of the spectrograph, and a visual examination made with an eyepiece. The flow of gas was varied and the precise region of the flame image selected, until the hydrocarbon band appeared at its brightest. Only the main head at about $\lambda 4314$ was at all distinct; but the visibility of the fainter doublets on the more refrangible side was taken as an approximate guide to the best conditions.

A reproduction of one of the spectrograms is given in fig. 12, Pl. XIII. The main series of lines forming the "tail" is well shown; for not only are the lines comparatively bright, but they are also free from the masking effect of the continuous spectrum, since the latter does not extend with appreciable intensity quite so far.

The faint new series can be discerned converging to a point situated between the first two (least refrangible) members of the main "tail" series.

In the case of lines so faint showing against a veil of continuous light, measurements of wave-lengths are attended with considerable difficulty and some little uncertainty. The following approximate values were obtained. The first two columns are from photographs obtained with the grating instrument, on which the lines appear very faintly, and the third is from a spectrogram with the prism.

<i>Spectrogram 1.</i>	<i>Spectrogram 2.</i>	<i>Spectrogram 3.</i>
Grating.	Grating	Prism
λ .	(determined graphically). λ .	(determined graphically). λ .
4106.7		4108
4095.0	4094.8	4095.3
4084.2	4084.6	4084.8
4075.1	4074.0	4074.0
4065.9	4065.0	4066.0
4059.3	4059.0	4060.7
4052.6	4053.0	4053.0
4047.2		4047.8
4043.4	4044.0	4043.2
4040.1	4039.5	4040.0
4037.6		4037.0
4031.8		4031.3
4025.2		

The horizontal dotted lines in column 1 indicate that there occurred lines in these intervals too faint to be measured.

In the attempts to photograph these lines, the fainter members of the other series starting from the head of the main band were being simultaneously recorded on the plates. The enlargement reproduced reveals the pairs of lines proceeding with gradually increasing separation up to what appear to be two nebulous pairs or narrow bands. Between these and the new faint series only a short space intervenes, so that the seemingly void region has been greatly reduced.

In structure the faint series closely resembles the main series of the tail, but its scale is smaller.

As this paper is not intended to be a detailed communication, no discussion of these line series is attempted and their investigation awaits more exact data.

*Some Bands of Unknown Origin associated with the
"Swan" Spectrum.*

In the course of some work on the Mecker flame spectrum, some small-scale spectrograms were taken with a grating camera.

On close examination of these, it was noticed that certain faint bands were visible in the region between the green and the blue carbon flutings. These bands were too faintly recorded to enable anything more than rough wave-lengths to be obtained, nor was any structure visible in them; but during the investigation of the lines of the hydrocarbon bands with the prismatic spectrograph, the new bands appeared on the negatives with much greater clearness. It became important, then, to measure them more accurately and endeavour to ascertain something of their structure.

At best the bands are very feeble, being only just visible on the faint continuous spectrum which extends between the "Swan" bands. Ordinary plates were used, as the range of wave-length under investigation fell within their limits of sensitivity; but long exposures were necessary owing to the faintness of the light. As in the case of the hydrocarbon bands, a limit to successful photography was imposed by the fogging produced by the feeble continuous radiation; but in spite of this difficulty, some useful spectrograms were obtained.

For the purpose of wave-length determinations the spark spectrum of iron was employed, the discharge being passed between iron electrodes with capacity and self-induction in circuit. It fortunately happens that iron gives excellent reference lines in the region under investigation, and a

narrower Fe: spectrum was superposed on that of the Mecker flame.

Fig. 13, Pl. XIII., is an enlargement from a negative obtained with the prismatic spectrograph on which the unknown bands are recorded.

The least refrangible edge falls about midway between the two strongest iron lines and appears to consist of a double line. The second and stronger head is nearly coincident with the iron line λ 4891. A third head is also somewhat vaguely indicated, but the most remarkable feature is the ribbed or fluted structure of the band, the close and regular sequence of which can be traced nearly up to the carbon fluting (λ 4737.2) in the blue.

The faintness of the detail, however, makes the components somewhat difficult to recognize.

In addition to these bands, a well-marked single line occurs close to the blue carbon band. Owing to the long exposures necessary the "Swan" spectrum is greatly overexposed; and this accounts for the appearance of the bright green carbon band (λ 5165), for an ordinary plate has little sensitivity to this region.

The approximate wave-length measurements are as follows:—

Fe: Comparison Spectrum.	Unknown Bands.
λ .	λ .
4957.6''	(1) { 4942.0
4939.0	{ 4937.5
4924.1	
{ 4920.7	(2) 4890.0
{ 4919.1	
4891.3''	(3) 4853 (?)
4878.4	
4871.9''	(4) line 4743
4860.0	

In order to show the line λ 4743 well, the exposure must be suitable. It is seen more clearly in fig. 14, Pl. XIII., where it is recorded at the expense of the bands.

These features appear to be of constant occurrence in association with the "Swan" spectrum as given by the Mecker coal-gas flame. There does not seem to be any reason for attributing them to the presence of some chance impurity, and the inference is that they are probably due to carbon or one of its compounds. The fainter details are liable to variations and changes in relative intensities, but

until they are more fully investigated, no definite conclusions as to their origin and significance are possible.

Similarly, the variations in the hydrocarbon spectrum with the changes in the conditions of its source and the appearance of new lines with "the three" await further research.

The discovery of such variations and of new bands associated with the spectra of carbon and its compounds may be of importance in connexion with astrophysics, and particularly in the study of comets.

September 1, 1916.

LXI. *The Collapse of Short Thin Tubes.*

By ALBERT P. CARMAN *.

[Plate XIV.]

THE problem of the collapse of tubes is to find an equation to express the relation between the collapsing pressure and the dimensions and elastic properties of the tube. In this general form no solution has been found, and it is probable that no simple solution is possible. We have reached, however, from theory and from experiments, fairly simple equations (1) for long thin tubes, and (2) for long tubes of moderate thickness. The term "long" is used for a tube the length of which is beyond a certain "critical length," which is generally taken as about six diameters. The thickness of the tube is expressed in terms of the ratio of the wall thickness t to the mean diameter d . If t/d is not more than about .025, the tube is ordinarily called "thin." By "moderately thick" we mean tubes for which t/d has a value between .03 and .07. The equation for long thin tubes was deduced first by G. H. Bryan †, from theoretical considerations, and has the form $p=c(t/d)^3$, where c is a constant and equal theoretically to $\frac{2E}{1-s^2}$ (E is

Young's modulus, and s is Poisson's ratio for the material of the tube). The form of this formula has been confirmed by the experiments of Carman and Carr ‡ on seamless steel, lap-welded steel, and brass tubes, and by Stewart § for lap-welded tubes. The values of the constants found by these

* Communicated by the Author.

† Proc. Camb. Phil. Soc. (1888).

‡ Univ. of Illinois Eng. Experiment Station Bulletin (1906).

§ Trans. Am. Soc. Mech. Eng. (1906).

experiments are, however, about 25 per cent. less than those indicated by theory, probably in consequence of the material being beyond the elastic limits. The formula for moderately thick tubes has the form $p = a \frac{t}{d} - b$, where a and b are constants. This formula is purely empirical both in its form and its constants.

Carman concluded from his early experiments on small brass tubes (Phys. Rev. vol. xxi., 1905) "that there is a minimum length for each tube, beyond which the collapsing pressure is constant, and further, that this minimum length is quite definite. Again, for lengths less than this critical minimum length, the collapsing pressures rise rapidly. As definitely as can be determined from these small tubes, the collapsing pressure varies inversely as the length for lengths less than the critical length." This last law is stated by Gilbert Cook in what he has called "Carman's equation"

(Phil. Mag. July 1914, p. 53). $p' = \frac{L}{l} p$, where p is the collapsing pressure of an infinitely long tube, L is the critical length, l the length of the given tube, and p' the corresponding collapsing pressure. A curve drawn with lengths as abscissæ and collapsing pressures as ordinates, would thus consist, as R. V. Southwell has noted (Phil. Mag. Jan. 1915) of two discontinuous branches, a straight line parallel to the axis of abscissæ and a rectangular hyperbola intersecting the straight line at the point corresponding to the critical length. For both theoretical and practical reasons, the form of this pressure-length curve at and within the critical length has recently aroused much interest and discussion. The practical interest came first from the problem of spacing "collapse rings" in boiler-flues. Another practical problem comes from the collapse of steel flumes by atmospheric pressure, when accidents have suddenly let out the water and reduced the pressure almost to the zero on the inside. The theoretical interest comes from a formula deduced by R. V. Southwell in a very important paper on "Elastic Stability," read in 1912 before the Royal Society of London. In this paper Professor Southwell has deduced the formula

$$p = 2E \frac{t}{d} \frac{z}{k^4(k^2-1)} \frac{d^4}{l^4} + \frac{1}{3} \frac{m^2}{m^2-1} (k^2-1) \frac{t^2}{d^2},$$

where p is the collapsing pressure, E the Young's modulus, $\frac{1}{m}$ is Poisson's ratio, z is a constant depending upon the

end constraints, l , d , and t are the length, diameter, and wall thickness, and k represents the number of lobes into which the tube collapses. As is known, long tubes collapse into two lobes, shorter tubes into three lobes, and still shorter tubes into four lobes (see Phys. Rev. vol. xxi. p. 396, figs. 3, 4, 1905). This formula is represented by a family of curves, corresponding to the values of 2, 3, 4, etc., for k . Southwell points out that the envelope of this family of curves is very nearly a rectangular hyperbola. For longer tubes for which $k=2$, and $\frac{d^4}{l^4}$ is very small, Southwell's formula reduces

directly to Bryan's formula. Southwell at first used the term "critical lengths" to designate the lengths at which the tube may collapse into either 2 or 3 lobes, or into either 3 or 4 lobes, that is, the points of intersection of the branches of this curve. He has, however, also deduced theoretically an expression for the critical length in the sense in which Love and Carman used the term*. In this expression $L=k\sqrt{d^3/t}$, k is a constant. Mr. Cook, in the Phil. Mag. for July, 1914, gets a value of 1.74 for k from the discussion of a series of careful experiments made by him on short steel tubes of three inches mean diameter and different thicknesses. Cook concluded that the critical length is apparently "from about 13 to 18 times the diameter" instead of about 6 diameters as originally suggested by Carman. Unfortunately, Cook's apparatus limited him to tubes less than 13 inches in length; that is, to lengths of about 4 diameters, and so his curves do not reach the important bends or "critical" points. Mr. Cook himself says: "the tests cannot be regarded as sufficient in number or covering enough range of dimensions to confirm definitely the equation $L=k\sqrt{d^3/t}$."

The following experiments have been made to obtain more data on the collapsing pressures of tubes, particularly near and inside of the "critical" bend in the curve for the relation between pressure and length. The work to be described is part of work which is being done under the auspices of the Engineering Experiment Station of the University of Illinois, and with the help of Mr. S. Tanabe, Research Fellow. The apparatus was mostly that which was used by the author a number of years ago, and the methods of the experiments used were similar†. The tube

* Southwell, Phil. Mag. vol. xxix. p. 69.

† Bulletin No. 5, Engineering Experiment Station, University of Illinois.

to be collapsed was closed by end-plugs, and these were supported by an internal rod so as to eliminate end pressures on the tube when the tube was placed under external water pressure. Leakage to the inside of the tube was prevented by wrapping the joints with "friction" insulation tape, and covering the whole joint with asphaltum. End constraints were thus avoided. The experiments described here were on seamless-drawn steel tubes of 1, 2, and 3 inches diameter. The tubes were carefully machined to different thicknesses, so results have been obtained for several values of t/d for each kind of tube. The nickel-steel receptacle was 40 inches long, so that results well beyond the "critical length" could be obtained in most cases. As a matter of interest and of possible importance, experiments were made on the collapse of a series of small tubes of brass, aluminium, hard-rubber, and glass. These results are not given and discussed in this paper, but it may be stated that the curves obtained have similar forms to those for steel. The collapse of the glass tubing was striking, as the tube was reduced to a fairly fine powder upon collapse. The strain was apparently uniform throughout, and all parts seemed to give way at the same time.

The results of the experiments with steel tubes are given in Tables I., II., and III. These results are also shown by the curves of figs. 1, 2, and 3 (Pl. XIV.). The number of lobes of the collapsed tube is given in the fourth column of the Tables, and is also indicated by a small numeral at the point on the curve. This is made a matter of record on account of its importance in Southwell's theory, but it is not discussed in the present paper. The data given here are the results of the collapse of over 125 tubes. There were in practically each case two tubes of the given size, and thus an average could be made for each point. This also gave an immediate check on freak collapses. Very few freak collapses, however, occurred, and these were easily explained by irregularities that appeared upon the inspection of the collapsed tube. While the machine work on these tubes was done with great care by the mechanician of the department, it was impossible always to get the value of t/d with the desired exactness. It was thus necessary to make a correction in the observed collapsing pressure, so as to have sets of results for a curve between pressures and lengths with t/d constant. These corrections were made by interpolation, assuming that the collapsing pressure varies as $\left(\frac{t}{d}\right)^3$. Since the total variations of $\left(\frac{t}{d}\right)$ in these corrections were small, there was little assumption

TABLE I.

"One-inch" Steel Tubes. Inside diameter .942 inch.

Length in inches.	t/d .	Pressure of Collapse. lb. per sq. in.	Number of Lobes.
1.72	.0245	2490	4
1.74	.0190	1600	3 or 4
1.78	.0150	1030	3 or 4
1.78	.01	420	4
2.63	.0245	2060	3
2.63	.0190	1180	3
2.63	.0150	670	3
2.63	.010	200	3
3.69	.0245	1390	2 or 3
3.69	.0190	800	3
3.69	.0150	550	3
3.69	.010	120	3
3.72	.0225	1170	3
5.63	.0245	920	2
5.63	.0225	730	2
5.63	.0190	520	2
5.63	.0150	320	2
5.63	.010	115	2
5.5	.0227	675	3
7.5	.0245	900	2
7.59	.0225	710	2
7.56	.010	50	2
7.44	.015	210	2
7.62	.015	190	2
7.62	.019	420	2
9.6	.0245	900	2
9.56	.0225	625	2
9.6	.019	440	2
9.56	.015	150	2
11.6	.0245	750	2
11.63	.0225	600	2
11.63	.019	375	2
11.6	.015	145	2

(if any) in the above interpolation and corrections. We note:

(1) The curves for different thicknesses and diameters are all similar in shape. Thus the series of curves for the 1-inch tubes and that for the 2-inch tubes show the same characteristics. The very thin tubes show the greater variations, but this is to be expected owing to the difficulty in getting uniformity in thickness and material for thin tubes.

(2) By taking the point for the length of six diameters, and drawing an hyperbola through this point, using the equation $p' = p \frac{L}{l}$, and taking p as the collapsing pressure at the length of six diameters, there is, in most cases, a satisfactory

TABLE II.

"Two-inch" Steel Tubes. Inside diameter 1.873 inches.

Length in inches.	t/d .	Pressure of Collapse. lb. per sq. in.	Number of Lobes.
3.91	.0245	2870	3
3.88	.0197	1770	3
4	.015	920	3
3.88	.010	350	2
5.94	.0245	2050	3
5.94	.0197	1280	3
5.94	.015	540	3
5.94	.010	150	2
7.8	.0245	1630	2
7.63	.0197	980	2
7.63	.015	515	2
7.66	.010	135	2
11.34	.0245	1080	2
11.34	.0197	590	2
11.34	.015	260	2
11.34	.010	70	2
15.13	.0245	960	2
15.13	.0197	510	2
15.13	.0150	220	2
15.13	.0099	55	2
18.94	.0245	794	2
23	.0245	780	2
27.8	.0245	780	2

TABLE III.

"Three-inch" Steel Tubes. Inside diameter 2.87 inches.

Length in inches.	t/d .	Pressure of Collapse. lb. per sq. in.	Number of Lobes.
5.88	.0227	2640	4
9.12	.0227	1960	3 or 4
12.14	.0227	1390	2
12.17	.0227	1040	2
12.19	.0227	1010	2
15.06	.0227	873	2
15.06	.0227	910	2
18.13	.0227	756	2
18.28	.0227	895	2
24.13	.0227	684	2
24.25	.0227	770	2

agreement between the observed and calculated curves for lengths less than six diameters. In figs. 4, 5, and 6, we have the observed and calculated curves for lengths less than

six diameters. In fig. 6 we have the calculated hyperbola extended beyond the length of six diameters. The particular curves of figs. 6 and 7 are for 1-inch and 3-inch tubing of thicknesses of $t/d = .0245$ and $.0227$, but the tubes of other diameters and thicknesses show similar characteristics. It is seen that, for lengths greater than six diameters, the hyperbola thus calculated shows pressures increasingly less than those given by experiment. At this length of six diameters there thus appears to be a "critical length." The curve at this length bends rapidly towards the horizontal [especially and particularly so] in the case of the thick tubes. In the case of the thinnest tubes both for the 1-inch and the 2-inch tubes, that is, for tubes in which the ratio of t/d is $.001$, the agreement with the hyperbola is not so good and, indeed, the maximum bend seems to occur at a much shorter length. While the curves show much uniformity and the same characteristics are found in the 1-inch and the 2-inch tubes for this thickness, yet the percentages of error for such low pressures of collapse are necessarily greater.

(3) The experimental curves of this investigation are not in agreement with the Southwell formula $L = k \sqrt{d^3/t}$, which Cook has used. This is shown in typical cases in figs. 6 and 7, where the curves for $L = 1.75 \sqrt{d^3/t}$ are drawn for the 1-inch and 3-inch tubes having a ratio t/d of $.0245$ and $.0227$. On the same figures we have the rectangular hyperbolas drawn through the experimental point for a length of six diameters. If we write the formula $L = k \sqrt{a^3/t}$ in the form

$$L = \frac{kd}{\sqrt{t/d}},$$

we see that the critical length should increase directly as the diameter, and inversely as the square-root of t/d . It is evident that the length corresponding to the critical bend does vary as the diameter of the tube, but the curves do not show that the thinner tubes have the longer critical lengths. Indeed, the very thin tubes for both the 1-inch tubes and the 2-inch tubes rather indicate a shorter critical length than a length of six diameters. It will be remembered that Cook was unfortunately limited by his apparatus to lengths of less than four diameters, and so his curves could not show the above.

The experimental curves for seamless steel tubes, for which the ratio t/d is between $.0245$ and $.015$, thus show that there is a "critical" bend at the length of about six diameters, the part of the curve for the shorter lengths being

a rectangular hyperbola. Beyond this critical bend, the curve approaches more or less rapidly to a straight line parallel to the axis of abscissæ. The above, in connexion with previous results on the collapsing pressures of long tubes, thus should allow us to calculate with reasonable approximation the collapsing pressures of short steel tubes of these thicknesses.

Laboratory of Physics, University of Illinois,
Urbana, Illinois, May 1916.

LXII. *Experimental Determination of the Ionization Potential of Helium.* By CHARLES B. BAZZONI, Ph.D., Harrison Research Fellow, University of Pennsylvania*.

THE energy of impact necessary to remove an electron from simple atoms can be calculated directly from Bohr's theory. The application of Bohr's formula to helium indicates that an energy corresponding to that of an electron falling through 29.3 volts is necessary to remove one electron from the helium atom, and that an energy corresponding to a fall through 83.4 volts is necessary to remove both electrons. If the theory is correct, no assumed configuration of the electrons in the atom can give any other probable values but these, provided that impact ionization, so called, results directly and solely from impacts as is generally assumed. It follows that an experimental determination of the ionization potential of helium ought to serve as an excellent direct check on the validity of Bohr's theory itself.

The first determination of this potential was made by Franck and Hertz † using a very sensitive method for detecting small amounts of impact ionization. The value obtained was 20.5 volts. The only other determination has been made by Pawlow ‡, who used a method essentially the same as Franck and Hertz, and got a result, 20 volts, in excellent agreement with those investigators. Pawlow, however, found it unnecessary to apply any correction for the initial velocity of the impinging electrons, while Franck and Hertz's result involves such a correction. Recently K. Compton § has calculated the ionization potential, making use of a formula which he has developed and of various recorded data on helium ionization. He obtains 22.5 volts

* Communicated by Prof. O. W. Richardson, F.R.S.

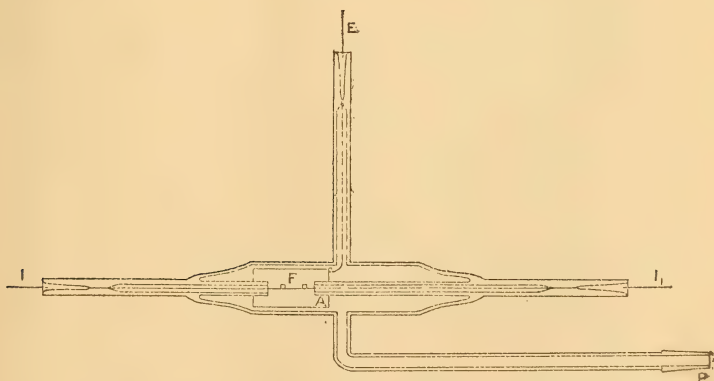
† *Ber. der D. Phys. Ges.* 1913, p. 34.

‡ *Proc. Roy. Soc.* vol. xc. p. 338 (1914).

§ *Phys. Rev.* 2nd ser. vol. vii. pp. 4 & 5.

as the most probable value. All of these values are too low to fit into Bohr's theory. Bohr has suggested that the effects observed in the experiments might have been due to ionization of impurities by radiation from the helium under 20.5-volt bombardment, or to the liberation of electrons from the metal parts of the apparatus from the same cause. The fact that the characteristic frequency of helium as calculated by Cuthbertson * from dispersion data (5.9×10^{15}) corresponds very closely to 20.5-volt impacts, lends some support to this explanation. Further, certain experiments of Rau's † in which he finds 30 volts necessary to excite the ordinary many-lined spectrum of helium and 80 volts necessary to excite $\lambda 4686$ have been taken, although without sufficient justification ‡, to imply that Bohr's calculated value is the correct one. In view of the remarkable agreement with fact of many of the deductions from this theory of Bohr's, it has seemed worth while to re-determine the ionization potential of helium, avoiding the suggested possibilities of error, and, further, in view of the extraordinary susceptibility of helium to disturbances due to minute traces of impurities, with an arrangement such that all traces of occluded gases could positively be got rid of in the first place and the gas subsequently be maintained absolutely without contamination. The apparatus used in this investigation has been designed with these ends in view.

Fig. 1.



The experimental tube, which is constructed of transparent quartz, is shown in fig. 1. The filament F, which is of

* Proc. Roy. Soc. 1910.

† Sitz. der Phys. Med. Ges. zu Würzburg, Feb. 1914.

‡ Richardson & Bazzoni, 'Nature,' Sept. 7, 1915.

tungsten about .8 cm. in length and .08 cm. in diameter, was surrounded by a coaxial cylinder of sheet copper (A), the lead (E) from which was connected with a quadrant electrometer. As the currents dealt with were sometimes of considerable magnitude, the electrometer was provided with an adjustable capacity. The sensibility of the electrometer with no capacity was roughly 600 mm. per volt. At P is a ground cone for connecting the tube with the air-pump and with the apparatus for producing and purifying the helium. The joint at this cone is preferably covered with sealing-wax without the use of tap-grease, but in these experiments it was actually covered with soft wax, which seemed a satisfactory arrangement. The heating current was taken from a 50-volt storage battery, and, since it was absolutely essential that the temperature of the filament should be maintained constant over relatively long periods, the filament was put in one arm of a Wheatstone bridge, which was kept accurately balanced by the use of three continuously adjustable resistances in parallel in the main circuit. This heating circuit was insulated from the ground. The negative end of F was connected to a sliding contact on a rheostat, which was in series with a 200-volt storage battery. The potentials were measured directly through a high-resistance Weston standard voltmeter between the negative end of F and the ground. This voltmeter was checked against a Clark cell and a standard resistance, and found correct within 0.1 volt over the range used.

Connected to P there was first a discharge-tube for examining the purity of the gas with a spectroscope; next a U tube to be surrounded with liquid air to get rid of mercury; next a mercury cut-off; then a U tube with charcoal; then another mercury cut-off; then a discharge-tube containing phosphorus pentoxide. There was next a stopcock beyond which was a quartz tube containing cleveite, with a lateral tube containing potassium permanganate crystals and a final connexion with the pump. Since the ionization potential of helium is higher than that of any of the substances with which it may be expected to be contaminated, it is of the highest importance to get the gas absolutely pure. In this arrangement the major portion of the hydrogen was removed by sparking for a long time with oxygen over phosphorus pentoxide. The residual hydrogen and other impurities were then taken out by the liquid air and charcoal. The gas finally obtained in the experimental tube was of a high degree of purity, and in the work that follows showed no trace, spectroscopically, of contamination

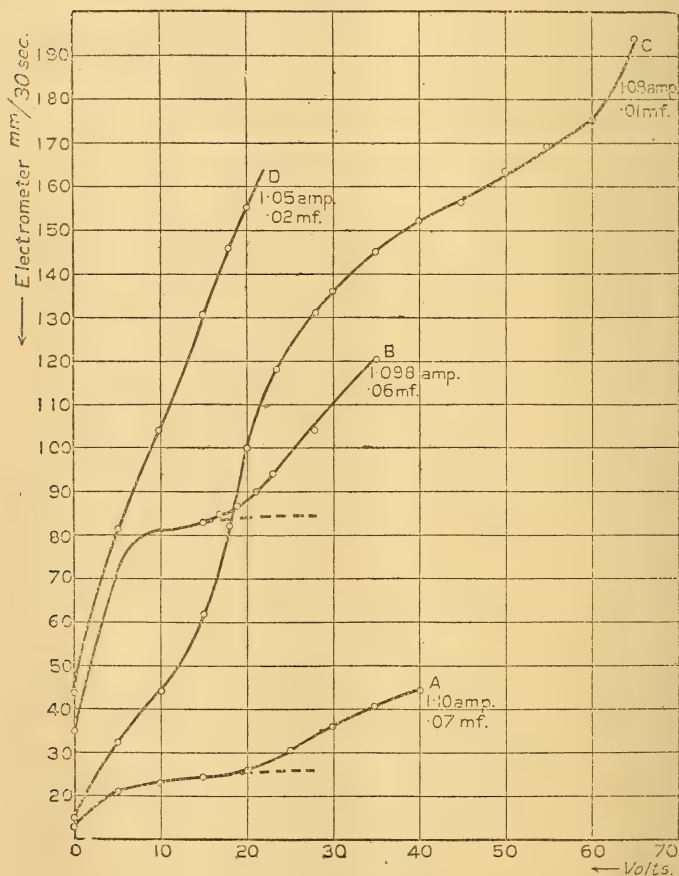
with anything excepting where it is specifically stated otherwise. As might be expected, the main difficulty was in getting the occluded gas out of the cylinder A and in properly ageing the filament. The occluded gas was finally eliminated by running the pump steadily while maintaining the entire apparatus (excepting the seals I, I₁, and E) at a bright red heat for two hours by the use of a blowpipe. Spurious effects due to the filament surface were got rid of by glowing the filament out at a very high temperature during several hours. The only impurity which appeared subsequent to this treatment was mercury, which could, of course, be kept down only by the constant use of liquid air.

In addition to the points mentioned above, there is a difference between this apparatus and the arrangements used by Franck and Hertz and by Pawlow which might have been of considerable importance. These authors made use of a device by which a large number of electrons were subjected in a space A to an accelerating field, after which they passed into a retarding field in a space B where impact ionization occurred. The amount of impact ionization was measured by the number of positive ions collected by an electrode bounding space B, all of the negative electrons being returned by the field into the space A. Such an arrangement is very sensitive for detecting the potentials at which ionization sets in, but it has the disadvantage that it affords no opportunity for an accurate comparison between the extra current due to impact ionization and the electron current which is the cause of it. For this reason the results of such experiments may always be open to the objection that the impact ionization results observed are possibly due either to the presence of small amounts of some more easily ionizable impurity, or to the occurrence of multiple impacts under low voltages, which, although giving rise to some ionization, may only form an insignificant part of the impact ionization in the same gas when higher voltages are available. The present arrangement, on the other hand, measures directly both the primary electron current and the extra current it gives rise to by impacts.

A series of observations made before the last traces of occluded gases had been expelled is of interest in showing the marked effect of minute quantities of impurities. In fig. 2 we have the steady deflexion of the electrometer after a 30-second charge plotted against the negative driving potential from F to A. The heating current and the capacity across the quadrants of the electrometer are shown

beside each curve. In A we have a fresh charge of pure gas—ionization by collisions is seen to set in at, roughly, 20 volts. In B, which was run with the same gas after the filament had been glowed for about an hour, the setting in

Fig. 2.

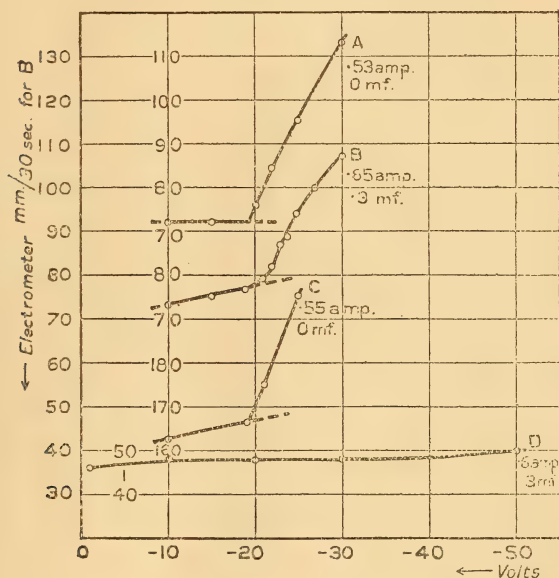


of ionization at 14 or 15 volts indicates the presence of hydrogen, although there was no certain spectroscopic evidence of this gas at this stage. In C we have the curve for the same gas when the spectroscope showed hydrogen faintly, and in D the curve when hydrogen was plain in the spectrum together with a faint indication of CO. In curve C there is evidence of the setting in of a second ionization above

40 volts. This, together with the value of the primary ionization potential taken from A, is a preliminary confirmation of the experiments of Franck and Hertz.

After the cylinder and filament had been properly cleaned up, results were obtained of which the curves in fig. 3 serve as specimens. With the apparatus completely evacuated,

Fig. 3.



washed out repeatedly with air, and then reduced to a liquid-air vacuum, the current-voltage curve has the form shown in D. Saturation, which is nearly reached at -1 volt, is complete at -10 volts, after which the current remains practically constant. In B the apparatus contained pure helium at a pressure of about $.85$ mm. Liquid air had been on the U tube for six hours, but the green line of mercury was still faintly visible in the spectrum. The electron currents here were of such magnitude, due to the relatively high temperature of the filament, that $.3$ microfarad was necessary across the quadrants of the electrometer. Under these conditions complete saturation was not attained. The curve, nevertheless, shows clearly that impact ionization sets in when the potential applied is 19.5 volts. Curves of which A is a specimen were obtained after liquid air had been on the U tube for 24 hours or more, and the spectrum contained

no trace of mercury or of any other impurity. The electron currents were also reduced by lowering the temperature of the filament until no capacity was necessary across the electrometer. Here saturation is complete at 10 volts, and ionization is seen to set in at 19.5 volts. Curve C was taken after the liquid air had evaporated, and a trace of mercury was present. The mercury, which presumably ionizes under 4.9 volt impacts, makes it impossible to get complete saturation, but, nevertheless, the amount present is too small to obscure the setting in of the helium ionization, which is again seen to start at 19.5 volts.

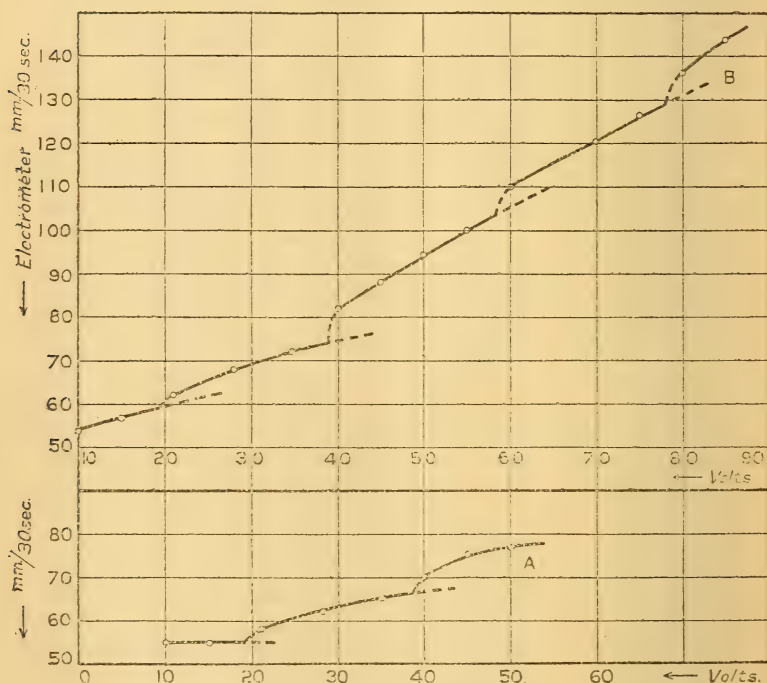
In order to deduce the correct ionization potential from these data, it is necessary to consider two disturbing factors—first, the drop of potential along the filament and, second, the initial velocity with which the electrons leave the wire. The first factor might be important because the filament is cool at the ends, so that no electrons are liberated exactly where the voltmeter is connected. The total drop along the filament with the currents used was found to be .75 volt. Allowing one-tenth of the length of the filament at each end as the part where the cooling effect would be appreciable—which, judging from a visual examination, was an ample allowance—the corresponding correction to be applied to the voltmeter readings is seen to be 0.1 volt. The distribution of velocities amongst the emitted electrons was studied by applying positive potentials to the filament, and it was found that the maximum velocity of emission at the temperatures used was one volt. It is seen, therefore, that when the voltmeter read 19.5 volts there were some electrons coming from the negative end of the filament with velocities corresponding to 20.4 volts. The results are consequently in general accordance with those of Franck and Hertz. If the true value is different from 20.4 volts, it is more likely to be somewhat below this value than above it, since very few of the electrons have initial velocities approaching one volt. The experimental arrangement, however, with a large anode completely surrounding the filament, and with the greater part of the drop of potential close to the filament, is a sensitive one, and ought to detect the ionization even when the number of electrons impinging with the necessary energy is small.

The phenomena of ionization in gases like helium and mercury vapour, involving, as they apparently do, elastic collisions below a certain velocity, and totally inelastic ones above that velocity, are worthy of close study. The hypothesis of a critical velocity for a totally inelastic impact was

introduced by Franck and Hertz to explain the results of certain of their experiments, and seems in general to be in accordance with the facts. The ordinary interpretation of this hypothesis leads to the statement that an electron cannot exist in mercury vapour with a velocity above 5 volts, or in helium with a velocity above about 20 volts, provided that the dimensions of the receptacle are sufficiently large compared with the mean free path of an electron at the given pressure. A check on this hypothesis can be obtained from a consideration of the successive current maxima at multiple values of the critical voltage. With an apparatus like that used by Franck and Hertz, in which the positive ions are dealt with, no conclusion can, however, be drawn from the successive values of the current on account of the retarding effect of the accumulated positive ions. In mercury the successive maxima appear to be, as a matter of fact, nearly equal one to the other. In studying this point with the apparatus used in the present investigation, the pressure and the filament temperature were necessarily much reduced in order to get that approach to saturation in each successive current which was needed to allow the current increase at the succeeding critical value to be apparent. The pressure worked with (.2 mm.) was unfortunately one for which the mean free path of the electrons is not sufficiently small, compared with the radius of the receiving cylinder, to render it safe to draw definite conclusions from the observations. The results obtained are illustrated by the curves of fig. 4, in which the electron current arriving at the copper cylinder is plotted against the accelerating voltage. A is a preliminary curve in which particular attention was paid to the points around 40 volts. In B the points are carried to 85 volts. It will be observed that a definite increase takes place at multiple values of the ionization potential, but that the successive currents are not related to the original current in a 1.2.4.8 ratio, as would be the case if every electron collided inelastically with an atom and there were no disturbances due to positive layers. The only effects which can be produced by the positive ions in this experiment are, first, their very marked effect on the distribution of potential—a matter not yet sufficiently investigated—and, second, their action in removing a certain number of free electrons by recombination. The current ratios taken from curve B are, roughly, 1.1.25.1.76.2.18. On the assumption that three tenths of the electrons undergo collisions at the specified ranges these numbers would be 1.1.30.1.69 and 2.20. This

proportion is about what would be expected from the dimensions of the apparatus and from the pressure of the helium used. It seems doubtful whether electron collisions with helium under the ionizing potential are perfectly elastic, but it is difficult to draw definite conclusions about this question from the present experiments.

Fig. 4.



The curves of fig. 4 also indicate strongly that the true ionization potential of helium is not more than 20 volts, rather than 20.5 or 21 volts. This follows from the fact that the jump in ionization is seen already to have taken place at 40, 60, and 80 volts, at which potentials the correction to allow for initial velocities is relatively unimportant.

In conclusion, it is to be observed that the results obtained in these experiments cannot be ascribed either to the ionization of impurities—for there were no impurities present in the critical experiments—or to the liberation of electrons from the metal parts of the apparatus, for this would have registered a positive charge on the electrometer. It would

seem, therefore, definitely established that ionization takes place in helium under an applied potential of about 20 volts. This fact can be reconciled with Bohr's theory only by making certain new assumptions as to the mechanism of impact ionization in helium. These new assumptions have not yet been subjected to experimental investigation.

I wish to express my obligation to Professor O. W. Richardson for valuable advice given throughout the course of these experiments. The quartz tube was constructed by Mr. Reynolds of the Silica Syndicate.

Wheatstone Laboratory,
King's College.
August 20, 1916.

LXIII. *The Compression of the Earth's Crust in Cooling.*
By HAROLD JEFFREYS, B.A., M.Sc., Fellow of St. John's
College, Cambridge*.

I. Introduction.

IT is generally agreed among geologists that the principal cause of the elevation of continents and mountains is that the crust of the earth must be in a state of horizontal compression, under which it frequently gives way, the strata being then folded into a shorter length in the neighbourhood of the point where the weakness has been shown. Such a compression appears to be the only mechanism that has been suggested that is qualitatively capable of producing the observed results. The cause of the compression itself is, however, very uncertain. The contraction hypothesis is the most satisfactory of those that have been offered, but grave doubts have frequently been expressed about its quantitative adequacy. According to this hypothesis, the earth was originally at a very high temperature, and different parts have since cooled by different amounts, changing in volume in consequence. A state of strain is thus set up in the crust. The mathematical aspect of the theory is due originally to Dr. C. Davison† and Sir G. H. Darwin‡; they showed that as the inner layers after a certain time are cooling more rapidly than the outer ones, and therefore contracting more rapidly, the outer layers will have to undergo compression

* Communicated by the Author.

† Phil. Trans. 178 A, pp. 231-242 (1887).

‡ Phil. Trans. 178 A, pp. 242-249 (1887); or Sci. Papers, vol. iv. p. 354.

in order to continue to fit the inner ones. At the same time they showed that the inner parts are being stretched. The boundary between the parts in process of extension and compression respectively is known as the "level of no strain." Initially it was at the surface, and has moved steadily downwards. The amount of compression geologically available is not, however, the integral amount since the time when the earth became solid, for if we consider a layer at a definite depth, in the earlier part of the interval stretching was taking place there, and compression has only taken place since the layer of no strain was at that depth. In calculating the available compression we must then take the amount that has occurred since that date, and ignore the stretching that took place at first. That this is true can be seen from the analogy of a football which has been inflated to such an extent that the outer case is just taut. If it is partially deflated, the size of the bladder decreases, and we have a condition resembling that of an outer layer too large to fit the inner parts. The case then crumples up and departs from the spherical form. On the other hand, if the ball is inflated further, the case only expands a little and remains spherical. Thus compression can produce folding, but tension cannot. In the case of the earth, the effect of long-continued extension is to produce permanent set, thus relieving the strain without departure from symmetry, and even a small subsequent compression can cause folding. Sir G. H. Darwin showed that the effect would be sufficient to shorten every great circle of the earth by a quantity of the order of 100 miles. This estimate was probably excessive for the form of the theory that he adopted, his initial temperature and his assumed coefficient of expansion being both too high. Nevertheless it appeared inadequate to satisfy the requirements of geology. A later paper* by Davison showed that when the increase of the coefficient of expansion with temperature was taken into account, a considerable improvement was obtained, but the compression was still insufficient.

Both of these estimates rested on the theory of the cooling of the earth given by Lord Kelvin, who treated the earth as a simple body, initially at a uniform temperature, and losing its heat from the surface. The recent discovery of radioactivity has, however, shown that so much heat is actually being generated within the crust of the earth, that if the same amount per unit volume were being produced throughout the mass, the temperature gradient at the surface would

* *Phil. Mag.* xli. pp. 133-138 (1896).

be 300 times its actual value*, and the earth would be getting hotter instead of colder. There are many objections to this view, and it seems that the amount of radioactive matter per unit volume must decrease so rapidly with depth that the total is insufficient to supply more than about $\frac{3}{4}$ of the present loss of heat from the surface†. At the same time the ratio of the amounts of uranium and lead in minerals gives valuable direct evidence concerning the age of the mineral, and hence a minimum estimate of the age of the earth‡, which is very much greater than that derived on Lord Kelvin's hypothesis. From a review of the evidence obtained by several different lines of investigation, Holmes has found that the observational data can be satisfied exceedingly well if the age of the earth be taken to be about 1600 million years, and if the rate of liberation of heat per unit volume decrease exponentially with the depth. The interval of time concerned is so much greater than that found by Lord Kelvin that his theory of cooling requires to be revised so as to take into account our most recently acquired knowledge, and at the same time the contraction theory, which depends on it, needs similar revision. This is the principal object of the present paper.

The level of no strain is found now to be at a considerably greater depth than the older determinations gave, and at the same time the amount of compression at the surface is much increased. For both reasons the volume of crumpled rock is increased. On the basis of the exponential distribution of radioactive matter the available compression is 133 kilometres; in other words, enough to shorten *every* great circle of the earth by this amount. Actually, folding is not uniformly distributed over the earth, but nearly confined to certain definite lines of weakness, as one would naturally expect. In some places, as in the case of the valleys of East and Central Africa, a tension is actually indicated. Some great circles thus show very little crumpling, and others are free to be folded to a greater extent than would be possible if the distribution of mountains were uniform. When a numerical estimate is made of the amount of compression required to produce the known mountain ranges, that found

* A. Holmes, 'The Age of the Earth,' 1913, p. 129.

† Holmes, "Radioactivity and the Earth's Thermal History," Part II, *Geol. Mag.* March 1915, p. 109.

‡ Holmes, 'The Age of the Earth,' p. 157; or "Radio-activity and the Measurement of Geological Time," *Proc. Geol. Assoc.* vol. xxvi. pp. 289-309 (1915).

to be available on the contraction hypothesis appears to be quite adequate.

II. *The effect on underground temperatures of a uniform distribution of radioactive matter through a horizontal layer.*

Let the number of calories generated per unit time per cubic centimetre be A .

Let the temperature at depth x at time t be V .

Let the depth of the radioactive layer be λ .

Then, when x is between 0 and λ , V satisfies the equation

$$\frac{\partial V}{\partial t} - h^2 \frac{\partial^2 V}{\partial x^2} = \frac{A}{c\rho}, \quad \dots \dots \dots (1)$$

and when x is greater than λ , we have

$$\frac{\partial V}{\partial t} - h^2 \frac{\partial^2 V}{\partial x^2} = 0 \quad \dots \dots \dots (2)$$

Here ρ is the density of the rocks and c their specific heat. If k is their conductivity,

$$h^2 = k/c\rho. \quad \dots \dots \dots (3)$$

The boundary conditions are that when $t=0$, $V=mx+S$ for all values of the depth, and when $x=0$, $V=0$ for all values of the time. Further, V and $\partial V/\partial x$ must both be continuous at $x=\lambda$.

Evidently $V = -A(x-\lambda)^2/2k$ satisfies the first equation. Substitute then

$$V = u + A\{\lambda^2 - (x-\lambda)^2\}/2k \text{ when } x \text{ is less than } \lambda, \dots (4)$$

and

$$V = u + A\lambda^2/2k \quad \text{when } x \text{ is greater than } \lambda. \quad (5)$$

Then

$$\frac{\partial u}{\partial t} - h^2 \frac{\partial^2 u}{\partial x^2} = 0 \text{ everywhere.} \quad \dots \dots \dots (6)$$

$u=0$ when $x=0$; u and $\partial u/\partial x$ are both continuous at $x=\lambda$.

When $t=0$,

$$\left. \begin{aligned} u &= mx + S - A\{\lambda^2 - (x-\lambda)^2\}/2k \text{ when } x \text{ is less than } \lambda, \\ u &= mx + S - A\lambda^2/2k \quad \text{when } x \text{ is greater than } \lambda. \end{aligned} \right\} \quad (7)$$

Now it has been shown by Fourier* that the solution of (6) that makes $u=f(x)$ when $t=0$, the value of $f(x)$ being specified for all values of x from $-\infty$ to $+\infty$, is

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dq e^{-q^2} f(x+2qh\sqrt{t}). \quad \dots \dots \dots (8)$$

* 'Analytical Theory of Heat,' p. 354.

In the case here considered, $f(x)$ is only specified from $x=0$ to $x=\infty$, but we have the condition that u is always 0 when $x=0$. This is satisfied automatically if we specify $f(x)$ for negative values of x , so that $f(x) = -f(-x)$. Thus u can be determined.

We find then

$$u = mx + S \operatorname{Erf} \frac{x}{2ht^{\frac{1}{2}}} + \frac{A}{2k} \left[(x^2 + 2h^2t) \operatorname{Erf} \frac{x}{2ht^{\frac{1}{2}}} - \frac{1}{2} \{ (x-\lambda)^2 + 2h^2t \} \operatorname{Erf} \frac{x-\lambda}{2ht^{\frac{1}{2}}} - \frac{1}{2} \{ (x+\lambda)^2 + 2h^2t \} \operatorname{Erf} \frac{x+\lambda}{2ht^{\frac{1}{2}}} + 2xht^{\frac{1}{2}}\pi^{-\frac{1}{2}}e^{-x^2/4h^2t} - ht^{\frac{1}{2}}\pi^{-\frac{1}{2}}(x-\lambda)e^{-(x-\lambda)^2/4h^2t} - ht^{\frac{1}{2}}\pi^{-\frac{1}{2}}(x+\lambda)e^{-(x+\lambda)^2/4h^2t} \right], \quad (9)$$

and V is at once obtained on substitution in (4) and (5). It can easily be verified that it satisfies all the conditions.

If we differentiate V and then put $x=0$, we find for the temperature gradient at the surface the value

$$\left(\frac{\partial V}{\partial x} \right)_{x=0} = m + \frac{S}{h\sqrt{\pi t}} + \frac{A}{k} \left[\lambda - \lambda \operatorname{Erf} \frac{\lambda}{2h\sqrt{t}} + h\sqrt{\left(\frac{t}{\pi} \right)} (1 - e^{-\lambda^2/4h^2t}) \right]. \quad (10)$$

In this substitute the values given by Holmes*, namely

$$\begin{aligned} m &= 0.00005 \text{ C. per cm.}, & S &= 1200^\circ \text{ C.}, \\ \left(\frac{\partial V}{\partial x} \right)_{x=0} &= 0.00038 \text{ C. per cm.}, \\ t &= 5.05 \times 10^{16} \text{ secs.} = 1.6 \times 10^9 \text{ years}, \\ A &= 63.9 \times 10^{-14} \text{ calorie per cm. cube per sec.}, \\ k &= 0.005 \text{ C.G.S. units}, & h &= 0.084. \end{aligned} \quad (11)$$

$$\text{Put} \quad \lambda/2ht^{\frac{1}{2}} = l. \quad (12)$$

* Geol. Mag. March 1915, p. 108.

† This value of m agrees well with that given by J. Johnston, Journ. Geol. xxiii. p. 736 (1915).

‡ This melting-point is approximately that of basalt, representing material just under the acid shell and intermediate in composition between the acid rocks and the deep-seated peridotites.

Then
$$l(1 - \text{Erf } l) + \frac{1}{2\sqrt{\pi}}(1 - e^{-l^2}) = .063.$$

This gives $l = .066$, whence

$$\lambda = 2.49 \times 10^6 \text{ cm.} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Thus the thickness of the radioactive layer on this hypothesis of a uniform distribution of active matter is 25 kilometres.

It may be remarked here that for any value of t such that $l(= \lambda/2ht^{\frac{1}{2}})$ is small, u can be expanded in powers of l . Put then

$$x/2ht^{\frac{1}{2}} = q. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Then

$$\text{Erf}(q+l) = \text{Erf } q + \frac{2l}{\sqrt{\pi}}e^{-q^2} - \frac{2l^2}{\sqrt{\pi}}qe^{-q^2}$$

to the second order in l .

$$\text{Also} \quad e^{-(q+l)^2} = e^{-q^2}(1 - 2ql + 2q^2l^2 - l^2)$$

to the same order.

Now substitute in u and work to the second order in l .

Then it is found that V reduces to the form

$$V = mx + \left(S - \frac{A\lambda^2}{2k}\right) \text{Erf} \frac{x}{2h\sqrt{t}} + \frac{A\lambda^2}{2k} - \mu, \quad . \quad (15)$$

$$\text{where} \quad \mu = 0 \text{ if } x > \lambda,$$

$$\text{and} \quad \mu = A(\lambda - x)^2/2k \text{ if } x < \lambda.$$

III. *The effect of other distributions of radioactive matter.*

So far the liberation of heat by radioactive matter has been assumed to be given by the special law that the distribution is uniform down to a particular depth, and zero below that depth. This restriction will now be removed.

Consider a distribution compounded of an infinite number of such distributions; the amount of heat liberated per unit volume per unit time by those extending to depths between λ and $\lambda + d\lambda$ is $\phi'(\lambda)d\lambda$ for depths less than λ and zero for depths greater than λ . If $\phi'(\lambda)$ is supposed specified for all values of λ from 0 to infinity, then the total rate of evolution of heat at depth λ is

$$\int_{\lambda}^{\infty} \phi'(\lambda)d\lambda = \phi(\infty) - \phi(\lambda).$$

Hence if we want to find the effect of a supply of heat according to the law $\phi(\infty) - \phi(\lambda)$, we must write $\phi'(\lambda)d\lambda$

for A in the previous solution and integrate with regard to λ from zero to infinity.

In particular, in the case of the exponential law used by Holmes*, $\phi(\infty) - \phi(\lambda)$ must be equal to $Ae^{-a\lambda}$; in this case $\phi'(\lambda) = Aae^{-a\lambda}$, and on substituting in V and integrating we find

$$V = mx + \left(S - \frac{A}{a^2 k} \right) \text{Erf} \frac{x}{2h\sqrt{t}} + \frac{A}{a^2 k} (1 - e^{-ax}). \quad (16)$$

It must be noted that equation (15) is only valid when t is so great that $\lambda/2ht^{\frac{1}{2}}$ is small, and similarly, (16) only holds when $1/2ah t^{\frac{1}{2}}$ is small. On the other hand, u is an even function of λ , and therefore contains no term in λ^3 . Hence (15) and (16) will hold provided the *fourth* power of λ can be neglected.

Differentiating (16) with regard to x and then putting x zero, we find that the known velocity gradient at the surface gives a quadratic equation to determine a . The positive root of this equation is $a = 4.14 \times 10^{-7}/1 \text{ cm.}$

IV. The straining of the crust in cooling.

If it be assumed that throughout the process of cooling the earth preserves a state of spherical symmetry, then it is evident that as the changes of temperature are not the same at all points, a state of strain must be set up, consequent on the variations of volume that take place.

Thus, consider a shell of internal radius r and external radius $r + \delta r$.

Let the coefficient of linear expansion with temperature be n . It is supposed to be a function of the temperature.

Let the initial density of the shell be ρ .

Also let the rise in temperature in a definite time be V , which is supposed small. Then V is a function of r . The density of the shell will change at the same time from ρ to $\rho(1 - 3nV)$. Let the radius at the same time change to $r(1 + \alpha)$.

The external radius will change to

$$r(1 + \alpha) + \delta r \left\{ 1 + \frac{\partial}{\partial r}(r\alpha) \right\}.$$

Hence the mass of the shell after the change of temperature is

$$4\pi r^2 \rho \delta r \left\{ 1 + 2\alpha + \frac{\partial}{\partial r}(r\alpha) - 3nV \right\}.$$

* Geol. Mag. March 1915, p. 108.

But the mass is unaltered. Hence we have the equation of continuity

$$2\alpha + \frac{\partial}{\partial r}(r\alpha) - 3nV = 0. \quad (1)$$

V being supposed known throughout the earth, this is a differential equation to determine α , subject to the boundary condition that α is zero at the centre of the earth. Thus for any shell α is determined by the changes of temperature within that shell. If the shell simply expanded independently of the interior, the radius would increase by nV instead of by $r\alpha$, so that the excess $r(\alpha - nV)$ is due to stretching.

Denote $\alpha - nV$ by k , and substitute for α in (1). (2)

Then the stretching is given by the equation

$$\frac{d}{dr}(kr^3) = -r^3 \frac{\partial(nV)}{\partial r}, \quad (3)$$

whence

$$k = -\frac{1}{r^3} \int_0^r r^3 \frac{\partial(nV)}{\partial r} dr. \quad (4)$$

Now consider the changes that must take place in a short time dt . If the integral stretching be K , then $k = \frac{\partial K}{\partial t} dt$, and for V we must write $\frac{\partial V}{\partial t} dt$. Hence

$$\frac{\partial K}{\partial t} = -\frac{1}{r^3} \int_0^r r^3 \frac{\partial}{\partial r} \left(n \frac{\partial V}{\partial t} \right) dr.$$

If now c denote the radius of the earth, and x the depth of a point below the surface, then $r = c - x$. As the temperature changes only extend downwards through a small fraction of the radius of the earth, we can neglect the square of x/c . Finally then

$$\frac{\partial K}{\partial t} = \left(1 + \frac{3x}{c}\right) \int_x^\infty \left(1 - \frac{3x}{c}\right) \frac{\partial}{\partial x} \left(n \frac{\partial V}{\partial t} \right) dx,$$

or, integrating by parts and again neglecting $(x/c)^2$,

$$\frac{\partial K}{\partial t} = -n \frac{\partial V}{\partial t} + \frac{3}{c} \int_x^\infty n \frac{\partial V}{\partial t} dx. \quad (5)$$

Let now $n = \epsilon + \epsilon'V$, where ϵ and ϵ' are two constants. Consider first the case of a distribution finite in depth.

Put $A\lambda^2/2k = \alpha$, $S - \alpha = \beta$. Then $\alpha = 398^\circ$, $\beta = 802^\circ$.

$$\text{Then} \quad V = mx + \beta \operatorname{Erf} (x/2ht^{\frac{1}{2}}) + \alpha - \mu,$$

$$\frac{\partial V}{\partial t} = - \frac{\beta}{\sqrt{\pi}} \frac{x}{2ht^{\frac{3}{2}}} e^{-x^2/4h^2t}.$$

Then performing the integration we find

$$\begin{aligned} \frac{\partial K}{\partial t} = & (\epsilon + \epsilon'\alpha + 2\epsilon'mht^{\frac{1}{2}}q + \epsilon'\beta \operatorname{Erf} q) \frac{\beta}{\sqrt{\pi}} \frac{q}{t} e^{-q^2} \\ & - \frac{3\beta h}{\sqrt{(\pi t)}c} \left[(\epsilon + \epsilon'\alpha) e^{-q^2} + \epsilon'mht^{\frac{1}{2}} \{ 2qe^{-q^2} + \pi^{\frac{1}{2}}(1 - \operatorname{Erf} q) \} \right. \\ & \left. + \epsilon'\beta \left\{ e^{-q^2} \operatorname{Erf} q + \frac{1}{\sqrt{2}}(1 - \operatorname{Erf} q \sqrt{2}) \right\} \right]. \quad (6) \end{aligned}$$

The terms depending on μ have been omitted.

When q is zero, this is negative, but when q is large enough, $\partial K/\partial t$ is positive. For one value of q , $\partial K/\partial t$ is zero, and the rock is being neither stretched nor crumpled; this value of q corresponds to the "level of no strain." Evidently if then we put $\partial K/\partial t = 0$, and regard (6) as an equation to find q , the solution will have $1/c$ as a factor. Hence in accordance with the approximations already made, q may be put zero inside the square brackets in making an approximation to a solution.

Then q has to be found from

$$\begin{aligned} 0 = & [\epsilon + \epsilon'\alpha + \epsilon'(mh\sqrt{\pi t} + \beta) 2q\pi^{-\frac{1}{2}}] \beta q / t\pi^{\frac{1}{2}} \\ & - \frac{3\beta h}{c\sqrt{(\pi t)}} [\epsilon + \epsilon'\alpha + \epsilon'mh\sqrt{(\pi t)} + \epsilon'\beta/\sqrt{2}]. \end{aligned}$$

With the data already given, this gives $q = 0.207$. In accordance with Fizeau's results, ϵ/ϵ' is taken to be 300°C .

Hence the depth of the level of no strain is 79 kilometres. This is greater than the depth of the radioactive layer, so that μ is zero, as has already been assumed.

In addition to the level of no strain, we require the amount of compression at the surface. This is obtained directly by putting $q = 0$ in $\partial K/\partial t$, and then integrating from $t = 0$ to the present time. The approximation is not good in the early part of the period of integration, but as the compression must in any case vanish with the time, that short interval may safely be neglected.

Then

$$K = - \frac{6\beta h}{c} \left(\frac{t}{\pi} \right)^{\frac{1}{2}} \{ \epsilon + \epsilon'\alpha + \epsilon'mh\sqrt{(\pi t)} + \epsilon'\beta/\sqrt{2} \}.$$

Fizeau's results indicate that average values of ϵ and ϵ' are given by

$$\epsilon = 7 \times 10^{-6} \div 1^\circ \text{C.}; \quad \epsilon' = 2.4 \times 10^{-3} \div (1^\circ \text{C.})^2.$$

Adopting these numbers, we find that at the surface

$$K = -5.68 \times 10^{-3};$$

from this the shortening of the circumference of the earth by compression is at once found to be 227 km.

A rough approximation to the average effect of the compression may be made by adapting the method of Rev. Osmond Fisher. The volume of rock crumpled up is $\int 8\pi c(-Kr).dx$, taken through the compressed layer. At the surface $K = -5.68 \times 10^{-3}$; at a depth of 79 km. it is zero, as also is $\partial K / \partial x$.

Hence, with sufficient accuracy for the present purpose we can interpolate for K by the formula

$$K = -5.68 \times 10^{-3} \left(1 - \frac{x}{79 \times 10^5} \right)^2.$$

If this crumpled rock were then spread in a uniform layer over the surface of the earth, the depth of it would be 300 metres.

Next, consider the exponential distribution of radioactive matter. The temperature is in this case given by

$$V = mx + \left(S - \frac{A}{a^2 k} \right) \text{Erf} \frac{x}{2h\sqrt{t}} + \frac{A}{a^2 k} (1 - e^{-ax}).$$

Put $A/a^2 k = \alpha$, $S - \alpha = \beta$;
then $\alpha = 746^\circ$, $\beta = 454^\circ$.

Then

$$\begin{aligned} \frac{\partial K}{\partial t} = & \left\{ \epsilon + 2\epsilon' m h t^{\frac{1}{2}} q + \epsilon' \beta \text{Erf } q + \epsilon' \alpha (1 - e^{-2ahqt^{\frac{1}{2}}}) \right\} \frac{\beta}{\sqrt{\pi}} \frac{q}{t} e^{-q^2} \\ & - \frac{3\beta h}{\sqrt{(\pi t)}c} \left[(\epsilon + \epsilon' \alpha) e^{-q^2} + \epsilon' m h t^{\frac{1}{2}} \{ 2q e^{-q^2} + \pi^{\frac{1}{2}} (1 - \text{Erf } q) \} \right. \\ & \quad \left. + \epsilon' \beta \left\{ e^{-q^2} \text{Erf } q + \frac{1}{\sqrt{2}} (1 - \text{Erf } q \sqrt{2}) \right\} \right. \\ & \quad \left. - \epsilon' \alpha \{ e^{-(q+\gamma)^2} - \gamma \sqrt{\pi} (1 - \text{Erf } (q+\gamma)) \} \right], \end{aligned}$$

where $\gamma = h a t^{\frac{1}{2}}$. Its present value is 7.82, so that the terms depending on it are small enough to be neglected.

Solving by successive approximation to find the level of no strain, we find that the value of q corresponding to it is .200, and hence the depth is 76 km. The depth is thus not much different from that found on the hypothesis of a homogeneous radioactive layer of uniform depth. The value of K at the surface is 3.32×10^{-3} , so that the compression is adequate to shorten the circumference by 133 kilometres. If the crumpled rock were spread uniformly over the surface, it would cover it to a depth of 170 metres.

V. The amount of compression required to produce existing mountains.

In the cases of certain particular ranges, geologists have succeeded in obtaining a direct quantitative estimate of the amount of compression that must have been necessary to produce the folding that is observed. In the Appalachians, for instance, the width of the rocks, measured perpendicularly to the chain, is estimated to have been shortened by about 40 kilometres. Similarly, the compression in the Rockies is 25 miles, in the Coast Range in California 10 miles, and in the Alps 74 miles*. The larger ranges, particularly in Asia, have not been so exhaustively treated; but a rough idea of their importance can be obtained by a comparison with the Rockies, or in the case of narrower ranges, the Coast Range, whose geological age is about the same. The Alps are probably abnormal, and have not been used as a standard. The elevation of a continent or a large tableland involves little crumpling within it, and no great amount at the coast as long as the slope is there gradual. Hence, in determining the amount of compression, we need consider only the steep slopes of mountains. The amount in any range is here supposed to be proportional to the mean height†, and in the following table the latter has been obtained roughly from the maps in Philips' Student's Atlas. From this estimated compression combined with the length of the range the area lost by folding is at once found. The amount of compression found is of course essentially provisional and must be revised when further geological evidence is available.

* These data are from Pirsson & Schuchert, 'Textbook of Geology,' p. 361.

† This would be exactly true if the strata were similarly folded in all mountains.

TABLE.—*The Great Mountain Ranges.*

Range.	Length (km.).	Mean Height (metres).	Com- pression (km.).	Area compressed (thousands of sq. km.).
Scandinavian	1400	1000	10	14
Alps	1000	3000	118 †	118
Carpathians	1300	1000	10	13
Apennines	900	700	10	9
Urals	2200	700	10	22
Caucasus and Armenia ...	3500	2000	40	140
Iran	1400	2000	40	56
Himalaya	4000	5000	100	400
Suleiman &c.....	1200	2000	30	36
Karakorum and Hindu } Kush	2400	1000 *	20	48
Kwen Lun.....	2300	1000 *	20	46
Tian Shan	2000	3000	60	120
Altai	1600	2500	50	80
Abyssinia	2000	2000	40	80
Drakensberg	1500	1000	20	30
Atlas	2000	1500	30	60
Coast Range &c.	4000	2000	16 †	40
Rockies	7000	2000	40 †	280
Appalachians	800	1000	60 †	48
Andes.....	7000	2000	40	280
Total				1920

* It is assumed that the folding needed to produce the plateau of Tibet was all at the margins, and is thus included in that found for the Himalayas and Tian Shan. Thus, for the mountains within the area it is only necessary to consider the height above the general level of the plateau.

† These data are those that have been obtained directly from the geological evidence.

It is thus found that in the formation of mountains the surface of the earth has been diminished by crumpling

by about 1.8 million square kilometres $= 1.8 \times 10^{16}$ square centimetres.

Now if K be the relative compression at the surface, the area crumpled is

$$4\pi c^2\{1-(1-K)^2\} = 8\pi c^2K.$$

By substituting, we see that the value of K requisite to account for all the mountain ranges of the earth is 1.8×10^{-3} , which is only about half of the amount (3.3×10^{-3}) that the contraction hypothesis has been shown able to produce. If we allow for possible folding in regions at present submerged and for old ranges, now almost denuded away, the agreement may be better, or the available compression may even be insufficient; but in any case the theoretical and the observed compressions are of the same order of magnitude, so that it seems highly probable that the contraction hypothesis is adequate to account for a very large fraction of the mountain-building that has taken place, and perhaps for the whole of it*.

VI. *The influence of denudation and thermal blanketing.*

Suppose that by some means not yet specified, a large area of the solid earth has been raised up to form a continent, and at the same time other parts have been lowered to form oceans. Then the influence of the atmosphere causes the continent to be denuded, and the solid outer layers of it come to be redeposited on the ocean bed. Then the uniform law of cooling already considered is disturbed in two ways. In the first place, it has been shown that the stretching or crumpling of rocks contains as a factor the quantity $S - \int \lambda^2 \phi'(\lambda) d\lambda$, and the second term of this is a large proper fraction of the first. If, however, the upper layers, with their radioactive constituents, are removed, then, instead of the limits of the integral being 0 and ∞ , they will be λ_0 and ∞ , where λ_0 is the depth of the rocks denuded away. Thus the integral is much diminished in value, and consequently denudation increases the factor as a whole. Hence a greater strain will be thrown on the crust in continental areas. Similarly, if the ocean bottom is considered, the compression is seen to be diminished there. In the second place, although convection currents in the ocean will cause

* I am much indebted to Dr. Bonney for calling my attention to the possibility of the comparison here made.

heat to pass outwards in time, yet the ocean will nevertheless act as a thermal blanket, reducing the cooling below it and hence the surface compression.

For both reasons, then, an oceanic area will tend to require round its margin a smaller horizontal pressure than the mean, if it is to remain spherical. Similarly, a continental one requires a larger pressure round its margin. These two statements are inconsistent, since by the Third Law of Motion the pressures should be equal. Evidently, then, the earth cannot remain spherical. The pressure is inadequate to compress the surface layers of the continental areas so as to keep both these and the ocean areas part of the same sphere, so that the outer layers on the continents must expand somewhat relative to the inner ones, the margins remaining fixed by cohesion. The only way in which this can happen is by a reduction of the radius of curvature of the crust—in other words, the continents will tend to rise. Similarly, the ocean bed will sink. The effect capable of being thus produced is very great. Thus, if the level of no strain were at a depth of 70 km., and a surface layer, 3000 km. in length, were 3 km. too long to continue to fit the inner sphere, the radius of curvature would be increased by as much as 500 km., and the elevation would be 15 km. in the centre. The estimate of the available excess of length is probably conservative. This differential compression is then probably a very important cause of the elevation and maintenance of continents and oceans. The adjustment of the figure of the earth to make the compression constant all over is not likely to be complete, however, as further stresses would be produced in the crust, which would cause fracture. The adjustment is not then likely to be much greater than would give isostatic compensation of the changes produced by denudation. The remaining part of the differential compression will be shown in extra folding in the continents, and diminished compression, or even, in extreme cases, a tension, in the ocean bed.

Mr. Holmes has called my attention to the importance of another factor. The crumpling here considered is that which has occurred at any depth since the level of no strain reached that depth. Before it had gone so far down, there would be a considerable tension in the interior of the continents, which might be capable of causing rifts and accelerating intrusions. Such rifts would occur very soon after the denudation. The flexure of the crust here indicated would, on the other hand, increase steadily with the time, and is essentially subsequent to the denudation.

VII. *On the causes of Isostasy.*

From a detailed consideration of the evidence for and against the theory of isostasy, Professor Joseph Barrell* has come to the conclusion that while the uppermost layers of the crust of the earth are probably very strong, the strength gradually diminishes with depth, so that at a depth of about 400 kilometres, in the middle of the layer of weakness, called the asthenosphere, the strength is only about $\frac{1}{25}$ of what it is at the surface. In virtue of this weakness a large area of excessive mass per unit area tends to sink, while lighter areas rise. The smaller the dense area is, the worse is the compensation, in consequence of the fact that the maximum shearing stress occurs at a smaller depth, and less stress is transmitted to the asthenosphere. Barrell considers that the adjustment takes place by progressive local melting under strain, with subsequent recrystallization when the tangential stresses have been removed. Section VI. of the present paper may give an additional important cause of the maintenance of isostasy.

A further problem of extreme difficulty is that of deciding how the continents and ocean basins ever came to be formed. If there were no isostatic compensation, they might be attributed to crumpling by compression, of a type not very different from that which produces mountain ranges. It appears, however, that continental areas, though varying enormously in height and shape, are fairly permanent in position, and that their height is really due to their being made of lighter materials than the ocean bed. The question is to decide how the lighter materials succeeded in being collected to certain points, leaving the denser rocks exposed in other parts. The most satisfactory explanation of any widespread inequality in the shape of the crust of the earth is the theory of Gravitational Instability, due to Jeans. This theory shows that if the earth be regarded as compressible, and any arbitrary displacements be given to its constituent particles, the alteration in the potential energy is of three parts:—

- (1) The elasticity of the materials causes it to be increased.
- (2) The normal displacements of matter at the boundary cause it to be increased.
- (3) The changes of density within the earth lead to changes in the gravitational potential which in general cause the potential energy to be diminished.

* 'Journal of Geology,' vol. xxii. (1914) pp. 28, 145, 209, 289, 441, 537, 655, 729; vol. xxiii. (1915) pp. 27, 425, 499; or a review by the present writer, 'Observatory,' April 1916.

If, then, the third part is able to counterbalance the other two, the spherical state of the earth is unstable and a departure from it will occur. It has been shown by Love that such instability will occur first for deformations expressed by spherical harmonics of the first order, in the case of a homogeneous sphere. If the outer layer of the sphere is of somewhat lower density than the rest, as in the actual earth, the problem is more complex; if the outer layer is, however, thin and of small strength, the effect of the change can be predicted. For the same displacements inside, the changes (1) and (3) in the energy are the same as before; but if displacements of mass occur over the outer surface to such an extent as to compensate isostatically the displacements below it, the change (2) attains its minimum value: this is seen at once from the fact that the isostatic state, being that in which the interior of the earth is in hydrostatic equilibrium, must be a state of minimum potential energy for a given thickness of the surface layers. Hence one of the causes making for stability is reduced if heterogeneity is allowed for and the inequalities are supposed compensated. Gravitational instability will then be first manifested in the case of those degrees of freedom that are consistent with the isostatic state. Thus if inequalities can be produced at all in this way, it is highly probable that they will be isostatically compensated from the start. The explanation of the low density of the continents and the high density of the matter below the oceans may then be connected with gravitational instability, accelerated by the mechanism of isostasy.

VIII. *The effect of changes in the rotation of the Earth.*

An essential feature of the present investigation is the hypothesis that the earth was formerly in a very heated, and probably fluid, condition. This leads directly to several other consequences. In this liquid state the coefficient of viscosity must have been practically zero; but at present it is exceedingly high, probably greater than 10^{17} c.g.s. units. Now, as the substances of which the earth is formed are not in general pure, they would pass through a pasty state in solidifying and the viscosity would thus increase *continuously*. Hence it would take in the process every value between its initial and final values. Now Sir G. H. Darwin showed that if the viscosity had a certain

variable value, depending on the distance of the moon from the earth, but always between 10^{14} and 10^{15} , the process of transference of angular momentum from the earth to the moon in consequence of tidal friction would take place at the greatest rate possible; and the changes would be complete in about 50 million years. This optimum viscosity must evidently have been attained at some time during the process of cooling, by what has been above stated, so that for a considerable part of the time the conditions must have been suitable for evolution to take place at the maximum rate possible. The truth of the theory of Tidal Evolution thus follows as a natural consequence of the present paper, and must therefore be considered with it. Most of the evolution must have taken place in a comparatively short time, perhaps about 200 million years, and since then the viscosity must have been too great for much change to take place; so that now the mean distance of the moon must be nearly constant. The most rapid part of the change must have occurred at a fairly high temperature, probably well above the boiling-point of water, but after solidification. It is thus to be placed before the oldest sedimentary rocks, and the consequent change in the ellipticity can have had little direct effect in the formation of the present mountain ranges, which have for the most part been elevated since the Carboniferous period. In one way, however, it may be important. The ellipticity of the earth would decrease as the velocity of rotation decreased, and the consequent shortening of the equator would form pre-Cambrian mountain ranges with a north and south alignment. On account of the plasticity of the rocks, however, which then extended nearly to the outer surface, these might soon cause fracture below them and disappear. In any case their existence might, however, leave a record in the form of definite lines of weakness, which would tend to localise subsequent folding caused by contraction. The tendency of folding to occur repeatedly along the same line in different geological periods may be regarded as confirming this suggestion.

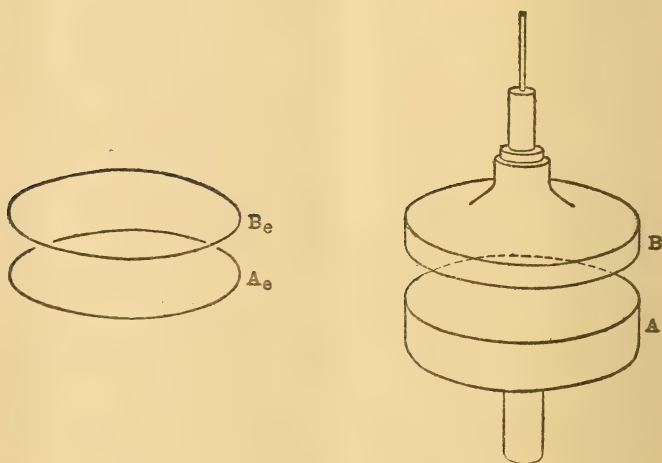
In conclusion, the author wishes to express his indebtedness to Mr. Arthur Holmes, whose helpful criticism from the geological point of view has been invaluable.

LXIV. *An Inquiry into the Possible Existence of Mutual Induction between Masses.* By MILES WALKER and W. WITCOMB STAINER *.

WE have for long been familiar with the analogy in the behaviour of matter in motion and electricity in motion. The mathematical expressions which state the relations between mechanical force, mass, and velocity are identical with those which express the corresponding electrical relations.

The object of the experiments described in this paper was to ascertain whether there is any measurable action between masses which would correspond to mutual induction between electric circuits.

Fig. 1.



If L is the inductance of the simple electric circuit A_e , of negligible resistance (fig. 1), we know that an electric impulse,

$$\int E dt = \int L \frac{di}{dt} dt = Li,$$

will start a current, i , flowing in the circuit. If there is no resistance, the current will go on flowing and the circuit will contain a store of energy, $\frac{1}{2} Li^2$. The current can be stopped by applying a back E.M.F. and the circuit made to yield up

* Communicated by the Authors, being an abstract of a paper read before the British Association at the 1915 meeting.

its energy. We shall then receive back the E.M.F. time-integral $\int E dt$.

Similarly, if M is the moment of inertia of the flywheel, A in fig. 1, a rotational impulse

$$\int T dt = \int M \frac{d\dot{\alpha}}{dt} dt = M\dot{\alpha}$$

will give to the flywheel an angular velocity, $\dot{\alpha}$ (where T stands for the turning moment necessary to give the angular acceleration $d\dot{\alpha}/dt$). If there is no friction, the flywheel will go on spinning, and it will contain a store of energy, $\frac{1}{2}M\dot{\alpha}^2$. The wheel can be stopped by applying an opposite turning moment, and can be made to yield up its energy. We shall then receive back the turning-moment time-integral $\int T dt$. So far the analogy is complete. But now observe the effect of the electric circuit A_e upon the adjacent circuit B_e .

When di/dt has some value generating the E.M.F. E in the circuit A_e there exists also an E.M.F. in the circuit B_e , the amount of which will depend upon the closeness of the magnetic coupling to the two circuits. Any change in the current in A_e will be accompanied by a tendency for current to flow in B_e .

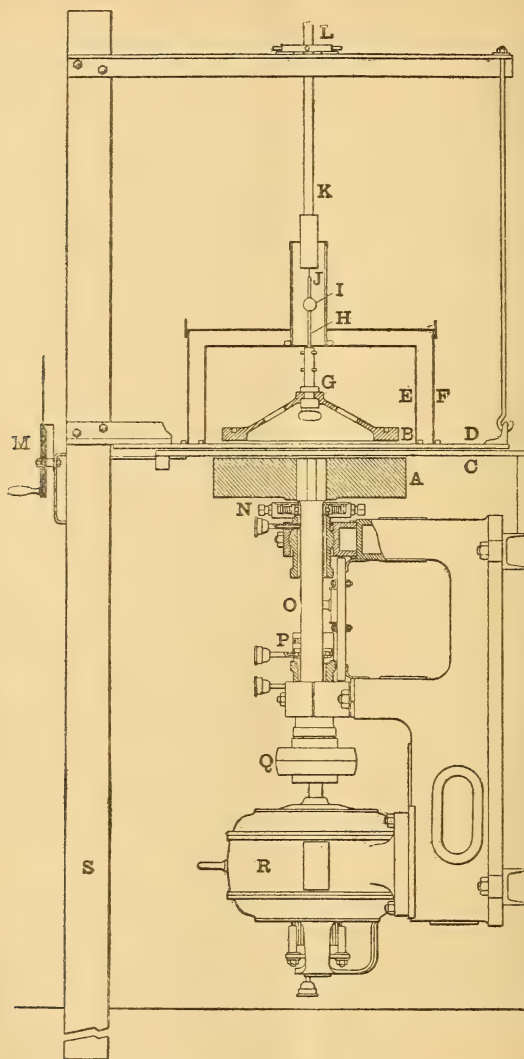
The question naturally arises, "Is there any action between masses analogous to this mutual induction between electric circuits?" If we accelerate the flywheel A (fig. 1), does it produce any force upon the suspended disk B ? It is quite possible that a small force of the kind might pass unnoticed if not specially looked for, just as the gravitational attraction between two movable objects would ordinarily escape observation.

Einstein and Grossman, in their mathematical inquiry into the theory * of gravitation, deduce the existence of such an effect, the order of magnitude of which is so small that it cannot be observed by any known apparatus.

In 1912 the authors constructed at the Manchester School of Technology the apparatus shown in fig. 2. There were special facilities existing at the school for carrying out this work. A heavy steel flywheel and an electric motor for driving it had already been constructed for other work; moreover, the School building afforded a very rigid support for the suspension, as it forms a hollow rectangle 90 metres by 60 metres, not very easily deflected by a torsional stress. The well of a lift provided a very suitable place to hang a long delicate suspension.

* See Fokker, *Phil. Mag.* [6] xxix. pp. 77-96.

Fig. 2.



- | | |
|--|---|
| A. Flywheel of cast steel. | L. Screws for adjusting K. |
| B. Suspended disk of porcelain. | M. Wheel and wire band for turning torsion head at the top of the building. |
| C. Lower windage screen of wood. | N. Spring-retained ball-bearing. |
| D. Upper windage screen of wood. | O. Over-speed cut-out. |
| E. Inner screen of sheet steel. | P. Ball-thrust bearing taking main weight. |
| F. Outer screen of sheet steel. | Q. Coupling. |
| G. Spindle of fibre carrying porcelain disk. | R. Motor. |
| H. German-silver rod for adjusting suspension to centre. | S. Steel stanchion fixed in ground and stayed to distant walls. |
| I. Mirror attached to disk. | |
| J. Bifilar suspension. | |
| K. Iron tube protecting suspension. | |

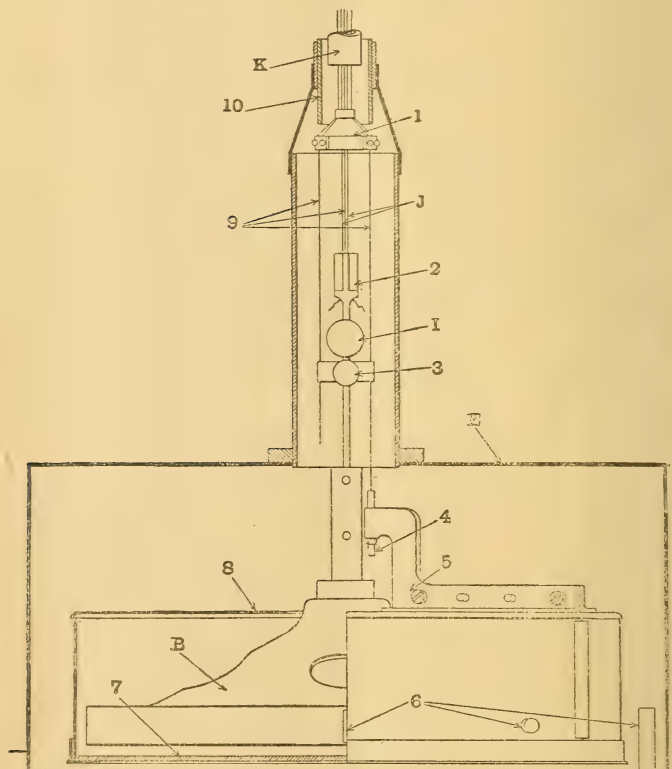
A steel flywheel A (fig. 2), 56 cm. in diameter and 11 cm. thick, is mounted on a vertical shaft, O, supported by ball-thrust bearing, P, and driven by an electric motor, R. Above the flywheel is suspended a disk, B, 51 cm. in diameter, made of very pure porcelain, weighing about 10 kg. The first suspension was made of very thin steel strip, but the final suspension is made of two round steel wires, each 0.025 cm. in diameter. The length of the suspension is 21 metres. The distance between the biflars is about 0.15 cm. The torsional control on B is extremely small, amounting to only 28 dyne-cm. for a deflexion of 1 radian. The angular swing of B upon its principal axis has a natural period of 2460 seconds. A mirror I (fig. 2) is mounted on a stiff wire attached to the disk B, which enables the movement to be accurately observed on a scale at 6 metres distance. By means of a telescope a movement of 0.01 cm. corresponding to a deflexion of $1/12,000$ of a radian, can be estimated, so that one can observe the effect of acting on the edge of the disk with a force amounting to only 10^{-11} of the weight of the disk. The chief constants for this apparatus are given in Appendix I.

The experiment consists in rapidly accelerating the flywheel A and observing the effect (if any) on B. In order to isolate B from the effects of any air currents and vibration, two screens, C and D, are interposed between A and B; the screen C is supported on a pillar of masonry to which the bedplate of the motor is attached, and the screen D is supported by a steel girder, S, fixed in the ground and stayed by means of wires from distant walls, which were not appreciably affected by vibration. Two sheet-iron covers, E and F, are placed over B to keep off currents of air. The outer cover F is to protect the inner cover from currents of air which might cause a difference in temperature in parts of the inner cover.

As might be expected, it was found to be very difficult to screen B from accidental disturbances when the flywheel was run up to speed. Many months were expended in finding out the source of the disturbances and reducing their magnitude. One cause of trouble was a very slight swirling action of the screen D, which communicated a slight swirl to the air inside, which caused B to rotate. To eliminate this effect in the later experiments, B was surrounded by an inner screen, shown at 8 in fig. 3. This inner screen was completely suspended from the top of the building by means of three wires. The torsional control upon it was 185 times greater than the control on B, so that it was hardly

affected by the swirling action of the air, and it did not communicate much force to B, because the natural period time was $1/32$ of the periodic time of B.

Fig. 3.



- | | |
|------------------------------------|---|
| B. Suspended disk of porcelain. | 3. Mirror attached to screen. |
| E. Sheet-iron cover. | 1. Brass frame to change distance between trifilar. |
| I. Mirror attached to disk. | 4. Adjusting screws for trifilar. |
| J. Bifilar suspension. | 5. Bracket enabling the screen to be built around the disk. |
| 2. Clamp for bifilar suspension. | 6. Limiting stops. |
| 8. Suspended screen. | 10. Iron tube protecting the suspension. |
| 7. Glass bottom of screen. | |
| 9. Trifilar suspension for screen. | |

The procedure in making the experiment was as follows : The disk B was brought as nearly as possible to rest. The residual motion was usually a slow swinging motion which followed strictly a sine law over one or two scale-divisions.

This was observed for several hours, so that it could be accurately plotted and the periodic time noted.

The flywheel A was then rapidly accelerated (say anti-clockwise) and run up to a speed of 2700 revs. per min., so as to give an impulse to B, if that were possible. The speed was then maintained constant for one-half the natural period of the swing of B. The flywheel was then rapidly slowed down and the direction of rotation reversed, and the speed increased to 2700 revs. per min. (in a clockwise direction). This process was repeated a number of times so as to induce resonance in B. It will be seen that any action on B due to acceleration of A was in phase with B, while any action due to velocity of A was 90 deg. out of phase. As the time taken to reverse the flywheel occupied only $1\frac{3}{4}$ minutes (a time small in comparison with the half-period of swing), the time-integral of the forces acting on B (if any) might be regarded in the nature of an impulse acting on B when it was near the centre of its swing. Any change in the velocity of B was most easily calculated from the amplitude of the swing. From the change in velocity of B we can calculate the change in angular momentum $B_{\text{mom.}}$ of B. This was expressed as a fraction of the total change of the angular momentum $A_{\text{mom.}}$ of the flywheel A.

In the early experiments made in 1913, it was found that if there was any effect of the kind looked for it was of an exceedingly small order, and that the observed movements of B were mainly due to accidental disturbing forces. At this time it was possible to assert that the ratio $\frac{B_{\text{mom.}}}{A_{\text{mom.}}}$ was certainly less than 2.3×10^{-8} .

In the later experiments the chief aim was to diminish the disturbing forces as far as possible, so that the negative result might be stated with the smallest possible limit of error. The introduction of the suspended screen and other refinements greatly improved the steadiness of B when A was running.

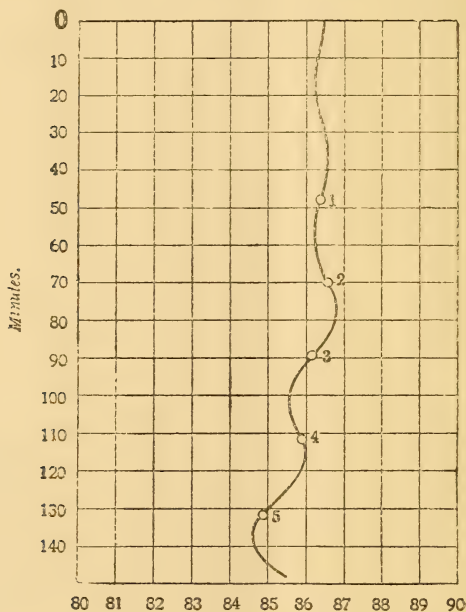
Fig. 4 gives the result of a typical experiment. The horizontal scale gives the deflexion in centimetres as observed on a scale at a distance of 6 metres. The vertical scale (read from top to bottom) gives the time in minutes. The actual deflexion in radians is obtained by multiplying the readings by 8.33×10^{-4} .

The method of calculating the gain in angular momentum by B is as follows: The amplitude of the swing of B before the acceleration of A is noted. Let this be θ_1 . Then, after

A has been run up to speed and reversed several times, it is found that B is swinging through a bigger angle. Let this be θ_2 . Then the gain in angular velocity is

$$2\pi n(\theta_2 - \theta_1).$$

Fig. 4.



Centimetres deflexion on a scale at 6 metres.

This multiplied by the moment of inertia of B, 4.2×10^6 , gives the gain in angular momentum.

In Fig. 4, $\theta_1 = 0.17 \times 8.33 \times 10^{-4}$ radian

and $\theta_2 = 0.7 \times 8.33 \times 10^{-4}$.

Therefore

$$B_{\text{mom.}} = 0.53 \times 8.33 \times 10^{-4} \times 4.2 \times 10^6 \times \frac{2\pi}{2460} = 4.8 \text{ gram. cm./sec.}$$

The flywheel A has undergone the following changes in angular momentum :—

At (1) from	0 to $+Mv$
(2) „	$+Mv$ to $-Mv$
(3) „	$-Mv$ to $+Mv$
(4) „	$+Mv$ to $-Mv$
(5) „	$-Mv$ to $+Mv$

Therefore the value of $A_{\text{mom.}} = 9 \times 1.24 \times 10^9$ gram. cm./sec.
So that the ratio

$$\frac{B_{\text{mom.}}}{A_{\text{mom.}}} = 4.8/9 \times 1.24 \times 10^9 = 4.3 \times 10^{-10}.$$

Thus, by the introduction of the suspended screen and other refinements, the steadiness of B while A was running was so much improved that we can now state that for the apparatus shown in fig. 2 the ratio of $\frac{B_{\text{mom.}}}{A_{\text{mom.}}}$ is less than 5×10^{-10} .

It will be seen from fig. 4 that the swinging of B is getting gradually out of phase with the impulses 3, 4, and 5. This is evidence that the increased motion of B is partly due to some cause in phase with the *velocity* of A, while the effect looked for should be in phase with the *acceleration* of A.

APPENDIX I.

Steel Flywheel (A) (shown in Fig. 2).—The flywheel takes the form of a solid disk of cast steel 22 in. in diameter and $4\frac{1}{2}$ in. thick.

The speed of revolution is 2700 revs. per min.

Angular velocity, $2700 \times 2\pi/60$ radians per second
 $= 2.83 \times 10^2$ radians per second.

Radius of gyration $= r\sqrt{2/2}$.
 $= 11 \times 2.54 \times 1.414/2$ cm.
 $= 19.9$ cm.

Velocity of a point at the radius of gyration
 $= 19.9 \times 2.83 \times 10^2$ cm. per sec.
 $= 5.65 \times 10^3$ cms. per sec.

Weight of flywheel $= 480$ lb. $= 220$ kg.

Angular momentum $= 5.65 \times 10^3 \times 2.20 \times 10^5$ gram. per sec.
 $= 1.24 \times 10^9$ units.

The kinetic energy $= 0.5 \times 220 \times 10^3 \times (5.65 \times 10^3)^2$ gram.
 cm. units.
 $= 3.5 \times 10^{12}$ gram. cm. units.

Porcelain Disk (B) (Fig. 3).

Weight $= 10.5$ kg. (actual measurement).

Overall diameter $= 51$ cm.

Rim section: in a radial direction = 7 cm.
in an axial direction = 4 cm.

Volume of rim $= \pi \times 4 \times [(51/2)^2 - (37/2)^2]$
 $= 3.84 \times 10^3$ cubic cm.

Weight of rim (taking specific gravity of porcelain at 2.4).
 $= 9.2 \times 10^3$ grm.

Moment of inertia of disk about principal axis
 $= 4.26 \times 10^6$ grm. cm.².

LXV. *Notices respecting New Books.*

Oliver Heaviside. Electromagnetic Theory. Volume III. London: 'The Electrician' Printing & Pub. Co., Ltd. Pp. ix + 519.

THIS is the third Volume, impatiently expected by all true admirers of Oliver Heaviside, of his unparalleled work on Electromagnetic Theory. The present volume has been delayed by "not favourable circumstances" for eight years (1904-12). Still it was published as long ago as September 1912, and the reviewer profoundly regrets that, again through "circumstances," the present Notice has been delayed over four years. But the consolation is that the reviewed volume—as, in fact, every work of Heaviside—is never too old. It has the freshness and the life of originality, and is, both as regards contents and form, stimulating beyond saying. In a very short Preface the author explains that he has "excluded parts of the third volume and included parts of the fourth." This confession is regrettable, inasmuch as it seems to imply that the plan of publishing a fourth volume has, for the time at least, been abandoned.

The present volume is inscribed:

IN MEMORY OF

George Francis FitzGerald, F.R.S.

"We needs must love the highest when we" know him.

A large portion of the substance has been reprinted from 'The Electrician' and 'Nature,' one section from 'Ency. Brit.' 10th Ed., and another from Perry's Book, 1901; but many pages of the volume are now first published. The whole volume consists of: Chapter IX., treating of *Waves from Moving Sources*, Appendices J and K, and Chapter X., entitled *Waves in the Ether*, but containing such (non-etheral, but equally interesting) matter as "The Teaching of Mathematics" and "Scientific Limitations on Human Knowledge," and, on the other hand, "Rotation of a Rigid Body" and "Deep Water Waves." In short, it is what one calls *Miscellanea*. The better so for the intelligent reader.

In Chapter IX. we have, first of all, a number of remarkably elegant solutions of problems concerning the "waves from sources," electrified points, lines, strips, and planes travelling at any speed. Then (pp. 43-61) there is a very natural digression into the "Drag of Matter upon Ether" and associated questions. The author tries several modifications of the "circuital equations" and discusses also Lorentz's equations. Here he exclaims: "To find Lorentz. After profound research I succeeded in discovering. . . Lorentz's *Versuch einer Theorie*, etc. This important application of Maxwell's theory. . . ought to have been done in English at once to save repetitional labour. Though sad, it is a fact that few Britons have any linguistic talent. . . Foreigners, on the other hand, seem to be gifted linguists quite naturally." Heaviside forgets that they also have to learn the foreign languages, English included, and that it is not an easy task in many cases. Then the author proceeds humorously: "Very well; I would say, let them give us poor islanders the benefit of their skill by doing all their best work into English. And why not make English the international scientific language? It would be all the same to the foreigners." This presupposes the absence, in all other nations, of any attachment to their traditional means of expression. It is easier to translate such classical things as Lorentz's 'Versuch' into English—and, as we know, an English translation of this by no means voluminous work is being prepared by Dr. Andrade. "On examination," Heaviside finds "that Lorentz's equations do lead to the Fresnel wave-speed," a property universally known these twenty years. Notwithstanding this, many readers will be glad to see (pp. 53-56) the way in which Fresnel's coefficient does follow from Lorentz's equations. The next section is dedicated to criticism of Larmor's equations, and contains some just remarks on the lack of clearness in 'Ether and Matter.' The next sections are dedicated to the theory of moving electrified cones and the force acting upon them and upon an electrified line, of a transversely moving electrified line segment, and of moving electrified hyperboloids. Further, the waves sent out from a growing plane source of induction and from a plane strip suddenly started are investigated. The problem of electrification moving along a straight line gives the author a good opportunity to develop his valuable ideas on operational solutions and on their algebrization. Here also a very interesting construction of simply periodic wave trains from certain electronic steady solutions is given (§ 486). §§ 488-500 treat chiefly of the interesting problems of sudden motions of an electron in connexion with Röntgen rays, and contain illuminating remarks on the peculiarities of the speed of light and an important investigation of the energy wasted in the pulse from a jerked electron and the energy left behind. The chapter closes with an investigation of the motion of a charged spheroid along its axis.

The short Appendix J contains a note on the size and inertia of electrons, now of historical interest only.

Appendix K, on Vector Analysis, was written as a review of Gibbs-Wilson's well-known treatise.

Chapter X. contains under the title of *Waves in the Ether* such a multitude of topics, masterly treated and suggestive of new problems, that even a mere enumeration of them would exceed the limited space of a Philosophical Magazine review. We shall, therefore, content ourselves with pointing out that the reader will find in that chapter Heaviside's most important investigations concerning the radiation from a moving electron, viz. "Elliptic or any other Orbit" in § 513 and "Theory of an Electric Charge in Variable Motion" in § 514, pp. 432-491, the remainder of this section (pp. 491-498) being dedicated to the interesting problems connected with spherical sources of energy, *i. e.* seats of oscillating impressed forces.

A System of Physical Chemistry. By Professor W. C. M^cC. LEWIS, Professor of Physical Chemistry in the University of Liverpool. Two volumes. Text Books of Physical Chemistry. Edited by Sir WILLIAM RAMSAY, K.C.B., F.R.S. London: Longmans, Green & Co. 1916. Price 9s. net each.

THERE is probably no other subject which is so progressive at the present time as that of physical chemistry. The seed set by vau't Hoff and by Willard Gibbs developed and grew, and the plant has begun to bear fruit on its many branches. The works of Gibbs were of rather too abstruse a character to be easily assimilated. Those of Nernst are a little too lax in their treatment to be quite satisfactory, though they represent the activities of a very brilliant investigator. Small books there are in plenty which deal more or less satisfactorily with the elementary side of the subject. But there has up to the present been no book which deals with the matter in an absolutely satisfactory way. The present two volumes in a well-known series promise to fill the gap. Professor Lewis has endeavoured to give a sufficiently complete account both on the experimental and the thermodynamic side, and we do not hesitate to say that he has succeeded.

The first volume deals with the kinetic theory with chief reference to the laws of chemical equilibrium and reaction velocity including catalysis (positive and negative). This is preceded, amongst other things, by an exceptionally good account of the various characteristic equations which have been suggested for fluids.

It is in the second volume that the unique value of the book will be perceived. The theorems of thermodynamics are not all simple, and in chemical books in particular they have often been badly slurred over. But here it is otherwise. In one or two particulars we might have put the matter otherwise. For example, does not the first law of thermodynamics state more than that "there is always a definite quantitative relationship between the heat that has disappeared as such and the work which has been

done"? As understood at the present day it asserts what this quantitative relationship is.

Professor Lewis obviously understands the mathematics which he employs. His exposition of the meaning of a certain partial differential coefficient in the footnote on p. 67 is a model of lucidity, and will be of great use to those who are less familiar with the matter than he is himself.

The osmotic pressure of concentrated solutions is very fully considered in its relation to other physical constants in Chap. VIII., while in Chap. X. there is a full account of chemical equilibrium (from the thermodynamic standpoint) when capillary or electrical effects are of importance; including adsorption and Donnan's theory of membrane equilibrium. There is still a great deal more to be said about these latter subjects, but the account given is an excellent digest of what is at present known.

In Chapter XII. Nernst's heat theorem is considered. Part III. is devoted to considerations based upon thermodynamics and statistical mechanics, including applications of the unitary theory of energy (energy quanta) to physical and chemical problems. This part is of very great interest.

We have noticed no mistakes in typography. The references are very complete. The illustrations are good. There is a subject and an authors index which are well compiled. Professor Lewis is to be congratulated in all respects.

LXVI. *Intelligence and Miscellaneous Articles.*

REFRACTION OF X-RADIATION.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

THE experiments carried out by Professor Barkla to detect the refraction of X-rays (Phil. Mag. April 1916) have led to results which show that the refractive index of potassium bromide for X-radiation ($\lambda = 5 \times 10^{-8}$ cm.) is between .999995 and 1.000005.

In the year 1914 I carried out a series of experiments in the Physical Laboratory of the University of Birmingham, in which it was attempted to refract a beam of X-radiation by passing it through a system of prisms. The refracting materials used for constructing the prisms were lead, aluminium, and sulphur. The two former were carefully shaped on a milling machine from solid blocks of material, while the sulphur prisms were prepared by casting, using the metallic prisms as moulds. No attempt was made to use optically worked surfaces. Owing to the acquisition of the University buildings by the War Office for the purposes of a military hospital, these experiments were discontinued on the

outbreak of war. Up to that time I had not obtained any evidence of refraction. It may be of interest to indicate briefly the method of procedure adopted in these experiments, and also the conclusions which were drawn regarding the order of magnitude of the possible refractive index.

The general arrangement of apparatus was very similar to that adopted by Professor Barkla. By an arrangement of lead slits an exceeding narrow beam of X-radiation gave rise to a fine horizontal line about two or three centimetres long on a photographic plate placed at about one metre beyond the levelling-table carrying the prisms. In the path of the beam was placed a train of eight prisms each of 90° refracting angle with their refracting edges horizontal and perpendicular to the beam. By the side of these prisms was placed a similar set with their refracting angles reversed. Any evidence of refraction would be recorded by a lack of alignment of the two halves of the line image obtained on the plate after several hours' exposure. No such evidence was obtained. A simple calculation showed that the refraction of X-radiation, if any, in these materials would correspond to a refractive index lying between 1.000005 and .999995.

University of Birmingham,
Nov. 11, 1916.

H. B. KEENE, D.Sc.,
Assistant Lecturer in Physics.

ADDITIONS TO PROF H. S. CARSLAW'S PAPER ON
NAPIER'S LOGARITHMS IN THE NOVEMBER NUMBER.

The following footnotes should be added:—

Page 479. To the sentence forming the third and second lines from the bottom of the page: When $r=7$, $nl_7 1 = 161180896.38$ (cf. *Constructio*, Section 53): add the following footnote:—

The error in Napier's Second Table affects the accuracy of his Canon, and this number should be 161180956.51. The correction can be made from the corrected result given by Macdonald in his English translation of the *Constructio*, pp. 94-5, for it is not difficult to show that $nl 1 = 7nl_7 10^6$.

Page 480. To the sixth line: and $nl_7 1 - nl_7 10 = 23025842.34$ (cf. *Constructio*, Section 53): add the following footnote:—

This number should be 23025850.93, since it is easy to show that $nl_7 1 - nl_7 10 = nl_7 10^6$, and Macdonald gives the corrected logarithm of 10^6 (*loc. cit.* pp. 94-5).

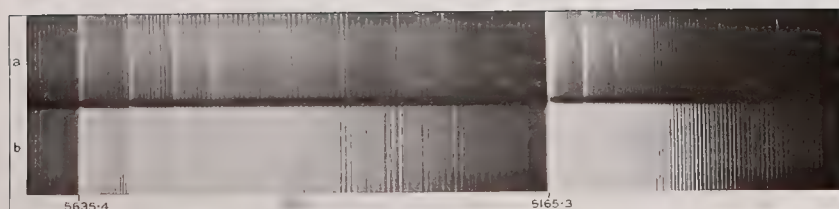


FIG. 1.—Yellow and Green Carbon Bands.

(a) Carbon arc in air.
(b) Carbon arc in a flame of coal-gas in air.

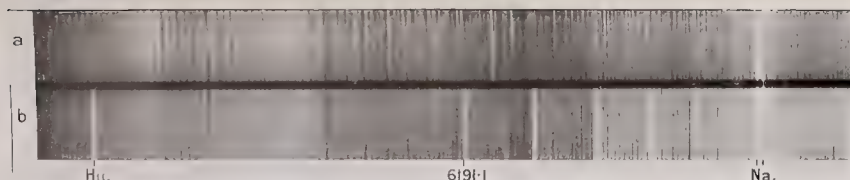


FIG. 2.—Red-orange Carbon Bands.

(a) Carbon arc in air.
(b) Carbon arc in a flame of coal-gas in air.



FIG. 3.—Hydrocarbon "f" Band of the Flame Spectrum.

(a) Fe: in carbon arc (with impurities). (b) Hydrocarbon Band in the spectrum of a Mecker coal-gas flame.



FIG. 4.—Carbon arc in a flame of coal-gas in air.

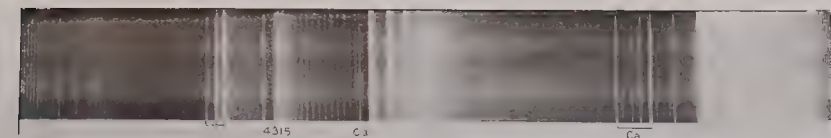


FIG. 5.—Spark between Carbon electrodes in an atmosphere of coal-gas (self-induction in circuit).



FIG. 6.—Spark in absolute alcohol (no condenser or self-induction). (Enlarged with cylindrical lens.)

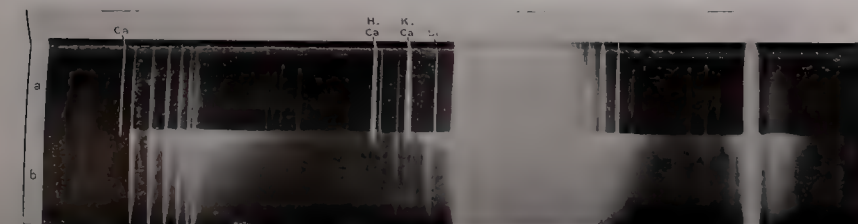


FIG. 7.—(a) Carbon arc in air showing impurity lines of Ca, Si, Al, etc.
(b) The same arc in a flame of coal-gas.

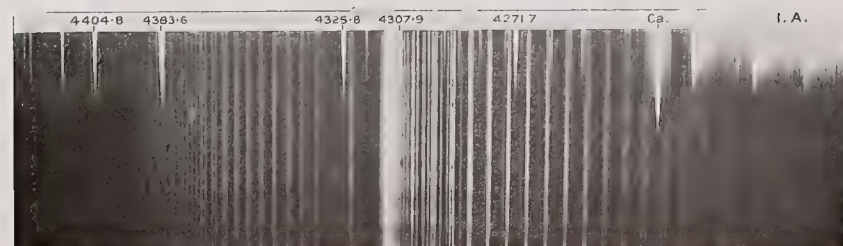


FIG. 8.—Head of the Hydrocarbon Band, Mecker coal-gas flame. (Comparison spectrum of Fe: in carbon arc.)

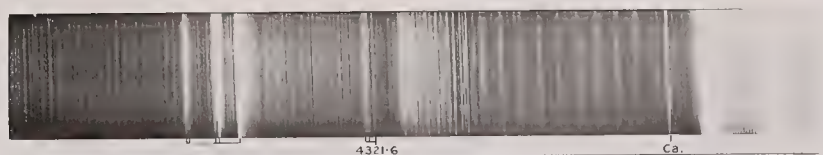


FIG. 9.—The three " and the Hydrocarbon Band of the arc-in-flame spectrum.

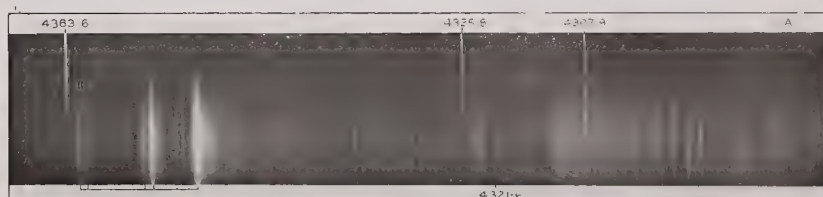


FIG. 10.—Showing the central region of Fig. 9, photographed in the 2nd order spectrum of the grating.

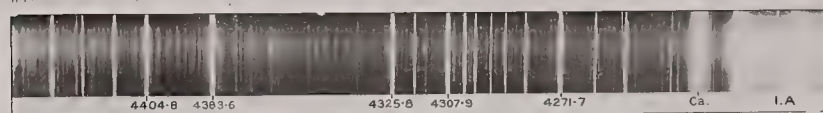


FIG. 11.—Carbon arc in air. Long exposure to show the extension of the Cy: bands and reveal the faint lines normally present in this region. Impurity lines due to Fe: Ca: etc.

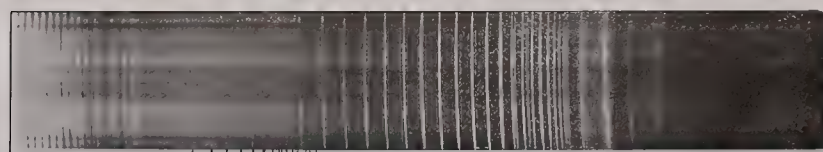


FIG. 12.—The more refrangible portion of the Hydrocarbon Band in the spectrum of the Mecker flame, showing the faint new series. Photographed with the prismatic spectrograph.

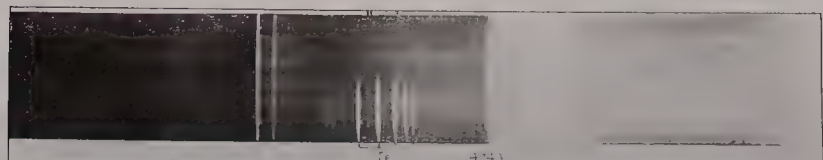


FIG. 13.—The region between the green and blue Carbon Bands in the Mecker flame spectrum, showing new bands of unknown origin and the line λ 4743. Superposed comparison spectrum Fe: spark with self-induction.

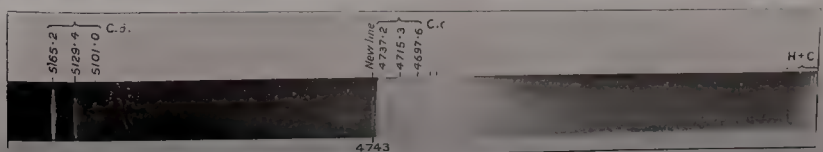


FIG. 14.—Mecker flame spectrum showing the line λ 4743. Prismatic spectrograph.

FIG. 4.

Pressure
lbs/sq.inch

3" Seamless Cold Drawn Steel Tube

3200

P-L Curves

2800

Curve I	$\frac{t}{p}$	·0245
" II	"	·0225
" III	"	·0190
" IV	"	·0150
" V	"	·0100

o

FIG. 1.

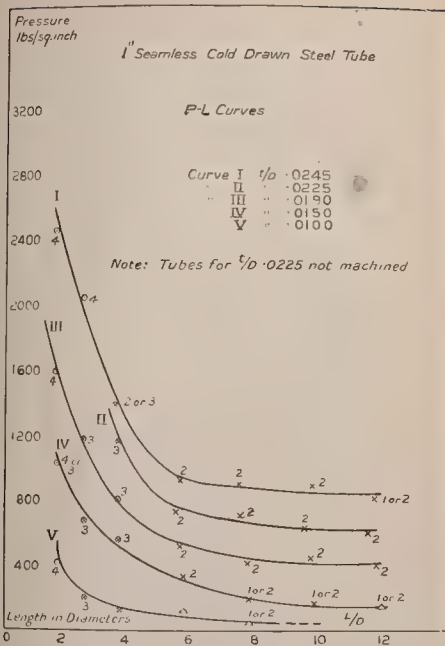


FIG. 2.

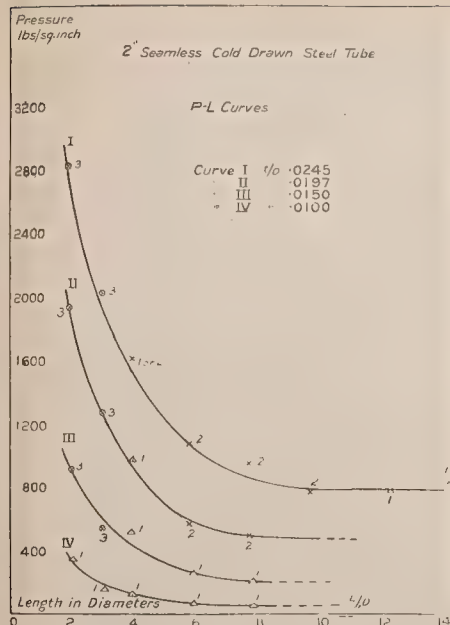


FIG. 3.

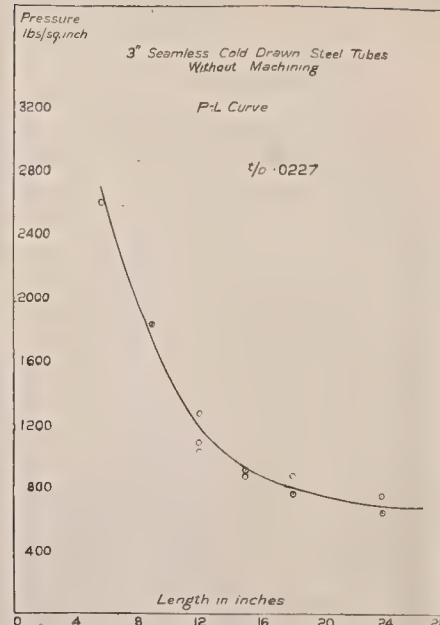


FIG. 4.

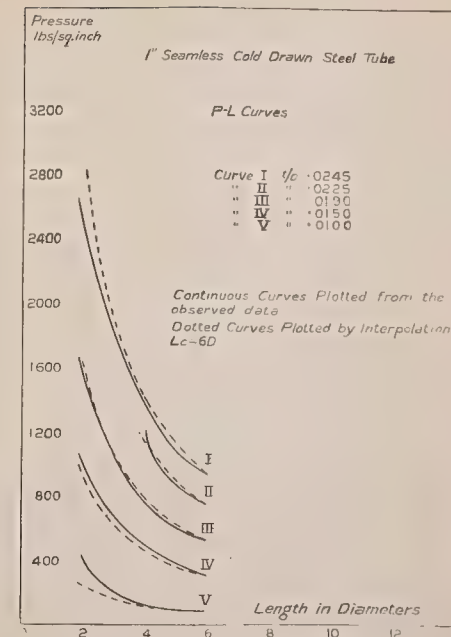


FIG. 5.

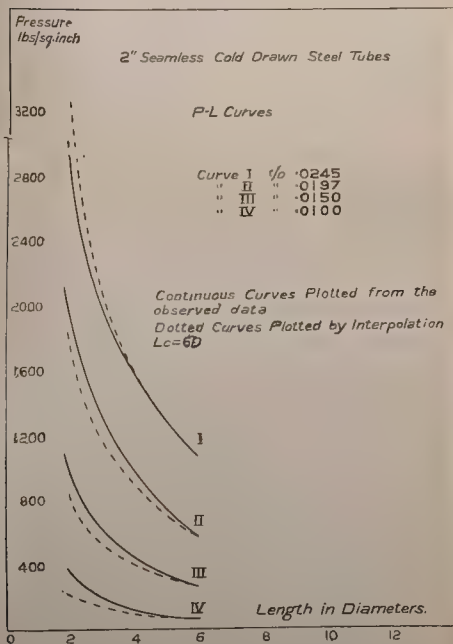


FIG. 6.

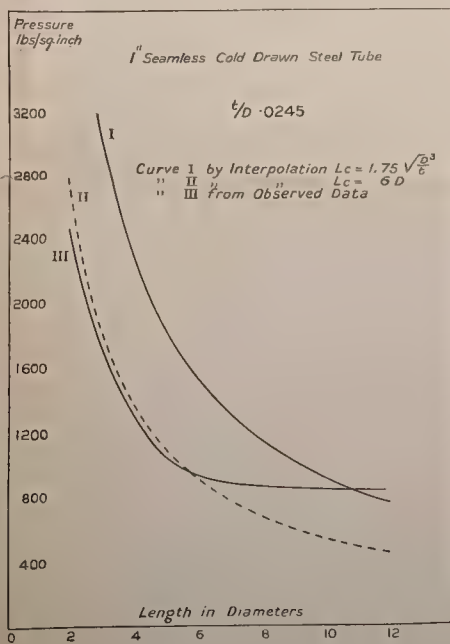
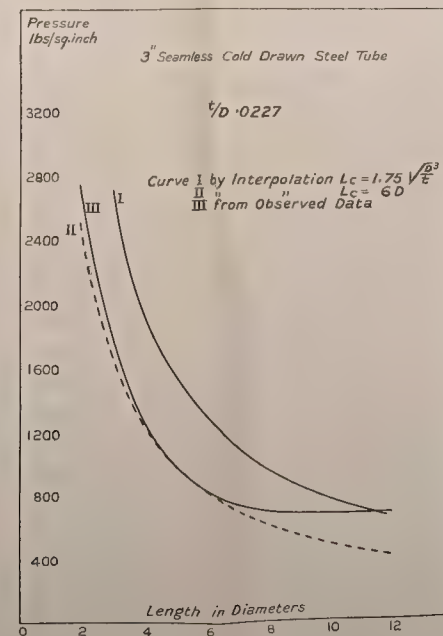


FIG. 7.





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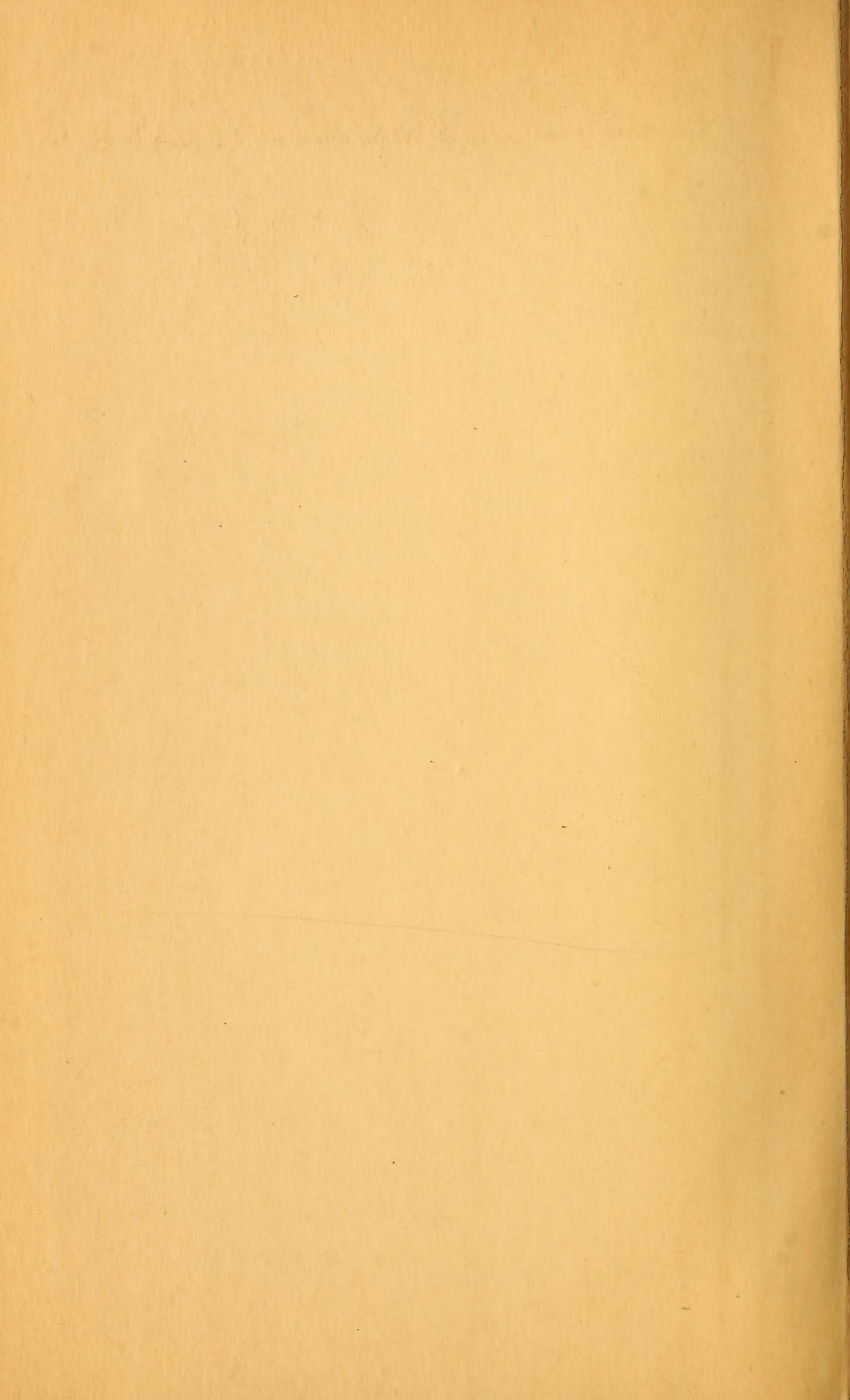
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